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Statistical inference of $P(X < Y)$ for the Burr Type XII distribution based on records

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Abstract

In this paper, the maximum likelihood and Bayesian approaches have been used to obtain the estimates of the stress-strength reliability $R = P(X < Y)$ based on upper record values for the two-parameter Burr Type XII distribution. A necessary and sufficient condition is studied for the existence and uniqueness of the maximum likelihood estimates of the parameters. When the first shape parameter of X and Y is common and unknown, the maximum likelihood (ML) estimate and asymptotic confidence interval of R are obtained. In this case, the Bayes estimate of R has been developed by using Lindley's approximation and the Markov Chain Monte Carlo (MCMC) method due to lack of explicit forms under the squared error (SE) and linear-exponential (LINEX) loss functions for informative prior. The MCMC method has been also used to construct the highest posterior density (HPD) credible interval. When the first shape parameter of X and Y is common and known, the ML, uniformly minimum variance unbiased (UMVU) and Bayes estimates, Bayesian and HPD credible as well as exact and approximate intervals of R are obtained. The comparison of the derived estimates is carried out by using Monte Carlo simulations. Two real life data sets are analysed for the illustration purposes.

Keywords: Burr Type XII distribution, Stres-strength model, Record values, Bayes estimation.

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1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of continuous random variables. X_k is an upper record value if its value is greater than all preceding values $X_1, X_2, \ldots, X_{k-1}$. By definition, X_1 is an upper record value. An analogous definition can be provided for lower record values. The theory of record values was first introduced by Chandler [17] and it has been extensively studied in the literature since then. More details and references may be found in Ahsanullah [2], Arnold et al. [5] and Nevzorov [38].

Record values and the associated statistics are of interest in many real life applications, such as weather, sports, economics, life-tests and so on. For example, in the manufacturing industry, it might be interesting to a researcher to determine the minimum failure stress of the products sequentially, while the amount of the rainfall that is grater (smaller) than the previous once is of importance to climatologists and hydrologists. In some experiments, an observation is stored only if it is an upper (lower) record value because the measurement saving can be important especially when the sample size is very big, costly or all (some portion) of the data is destroyed. For specific examples, see Gulati and Padgett [23].

In the reliability context, the stress-strength model can be described as an assessment of reliability of a system in terms of random variables X representing stress experienced by the system and Y representing the strength of the system available to overcome the stress. If the stress exceeds the strength, then the system will fail. Thus $R = P(X < Y)$ is a reliability of a system. The main idea was introduced by Birnbaum [13] and developed by Birnbaum and McCarty [14]. A comprehensive account of this topic is presented by Kotz et al. [24]. It provides an excellent review of the development of the stress-strength up to the year 2003.

In the literature, many papers are available for an estimate of the reliability based on a random sample or record values. When the X and Y are independent and follow the Burr Type III, X and XII, generalized exponential, Weibull, Gompertz, Kumaraswamy and Levy distributions, the estimation of R based on a random sample were studied by Mokhlis [31], Ahmad et al. [1], Awad and Gharraf [9], Kundu Gupta [25, 26], Saraçoğlu et al. [41], Nadar et al. [32] and Najarzadegan et al. [36], respectively. When the X and Y are independent and follow one and two parameters generalized exponential, Weibull, exponentiated gumbel, Kumaraswamy, one and two parameters exponential and Burr Type X distributions, the classical and Bayesian estimates of R based on records were considered by Baklizi [10], Asgharzadeh et al. [7], Baklizi [11], Tavirdizade [43], Nadar and Kızılaslan [33], Baklizi [12] and Tavirdizade and Garehchobogh [44], respectively.

The Burr Type XII distribution was introduced by Burr [16]. If a random variable X follows a Burr Type XII distribution, denoted by $X \sim Burr(\alpha, \beta)$, then the cumulative distribution function (cdf) and the probability density function (pdf) are given by, respectively,

$$F(x; \alpha, \beta) = 1 - (1 + x^{\alpha})^{-\beta}, \; x > 0, \; \alpha > 0, \; \beta > 0, \tag{1.1}$$

$$f(x; \alpha, \beta) = \alpha\beta x^{\alpha-1}(1 + x^{\alpha})^{-(\beta+1)}, \; x > 0. \tag{1.2}$$

Here $\alpha > 0$ and $\beta > 0$ are the two shape parameters. This distribution has been studied by the several authors; see, for example, Al-Hussaini and Jaheen [3, 4], Ghitany and Al-Awadhi [21], Nadar and Papadopoulos [35], Nadar and Kızılaslan [34] and Rao et al. [40].

The main purpose of this paper is to improve the inference procedures for the stress-strength reliability based on upper record values while the measurements follow the two-parameter Burr Type XII distribution when the first shape parameters are common. When the first shape parameter α is unknown, the ML and Bayes estimates, as well as asymptotic confidence and HPD credible intervals are derived. When α is known, different estimates, namely ML, UMVU, Bayes and empirical Bayes estimates, are obtained. The Bayes estimates of R under the SE and LINEX loss functions are derived in closed forms for informative and non informative prior cases. It is also obtained by using Lindley's approximation and MCMC method. The exact and other Bayes estimates are compared in terms of estimated risk (ER) by the Monte Carlo simulations. Also, the exact and asymptotic confidence intervals, as well as Bayesian, empirical Bayesian and HPD credible intervals are constructed for R.

The rest of the paper is organized as follows. In Section 2, a necessary and sufficient condition for the existence and uniqueness of the ML estimates of the parameters is established when α is unknown. The ML and Bayesian estimates as well as the asymptotic confidence and HPD credible intervals of R are obtained. In Section 3, the ML and UMVU estimates, as well as exact and asymptotic confidence intervals are obtained for R when α is known. The Bayes estimates are derived analytically and also obtained by using Lindley's approximation and MCMC method for informative and non informative prior cases. Moreover, Bayesian, empirical Bayesian and HPD credible intervals of R are constructed. In Section 4, the different proposed methods have been compared by using Monte Carlo simulations and the findings are illustrated by tables and plots. Furthermore, two real data sets analysis are presented. Finally, we conclude the paper in Section 5.

2. Estimation of R when the first shape parameter α is common

In this section, we investigate the properties of $R = P(X < Y)$, when the first shape parameter α is common for the distributions of X and Y. The ML estimates, its existence and uniqueness, asymptotic confidence intervals, as well as Bayes estimates and HPD credible interval for R are obtained.

2.1. MLE of R. Let $X \sim Burr(\alpha, \beta_1)$ and $Y \sim Burr(\alpha, \beta_2)$ are independent random variables. Then, the reliability $R = P(X < Y)$ is

$$\begin{aligned} R &= P(X<Y) = \int_0^\infty f_Y(y) P(X < Y \mid Y = y) dy \\ &= \frac{\beta_1}{\beta_1 + \beta_2}. \end{aligned} \tag{2.1}$$

The estimate of R are considered based on upper record data on both variables. Let $R_1, \ldots, R_n$ be a set of upper records from $Burr(\alpha, \beta_1)$ and $S_1, \ldots, S_m$ be a set of upper records from $Burr(\alpha, \beta_2)$ independently from the first sample. The likelihood functions based on records are given by, see Arnold et al. [5],

$$\begin{aligned} L_1(\beta_1, \alpha \,|\, \underline{r}) &= f(r_n; \alpha, \beta_1) \prod_{i=1}^{n-1} \frac{f(r_i; \alpha, \beta_1)}{1 - F(r_i; \alpha, \beta_1)}, \ 0 < r_1 < \ldots < r_n < \infty, \\ L_2(\beta_2, \alpha \,|\, \underline{s}) &= g(s_m; \alpha, \beta_2) \prod_{j=1}^{m-1} \frac{g(s_j; \alpha, \beta_2)}{1 - G(s_j; \alpha, \beta_2)}, \ 0 < s_1 < \ldots < s_m < \infty, \end{aligned}$$

where $\underline{r} = (r_1, \ldots, r_n)$, $\underline{s} = (s_1, \ldots, s_m)$, f and F are the pdf and cdf of X follows $Burr(\alpha, \beta_1)$, respectively and g and G are the pdf and cdf of Y follows $Burr(\alpha, \beta_2)$, respectively. Then, the joint likelihood function of $(\beta_1, \beta_2, \alpha)$ given $(\underline{r}, \underline{s})$ is given by

$$L(\beta_1, \beta_2, \alpha \,|\underline{r}, \underline{s}\,) = h_1(\underline{r}; \alpha) h_2(\underline{s}; \alpha) \alpha^{n+m} \beta_1^n \beta_2^m e^{-\beta_1 T_1(r_n;\alpha)} e^{-\beta_2 T_2(s_m;\alpha)}, \tag{2.2}$$

where

$$h_1(\underline{r}; \alpha) = \prod_{i=1}^{n} \frac{r_i^{\alpha-1}}{1+r_i^{\alpha}},\ \ h_2(\underline{s}; \alpha) = \prod_{j=1}^{m} \frac{s_j^{\alpha-1}}{1+s_j^{\alpha}}, \tag{2.3}$$

$$T_1(r_n; \alpha) = \ln(1 + r_n^{\alpha}), T_2(s_m; \alpha) = \ln(1 + s_m^{\alpha}). \tag{2.4}$$

The joint log-likelihood function is

$$\begin{aligned} l(\beta_1, \beta_2, \alpha \,|\underline{r}, \underline{s}\,) &= \ln h_1(\underline{r}; \alpha) + \ln h_2(\underline{s}; \alpha) + (n+m)\ln\alpha + n\ln\beta_1 \\ &\quad + m\ln\beta_2 - \beta_1 T_1(r_n; \alpha) - \beta_2 T_2(s_m; \alpha). \end{aligned} \tag{2.5}$$

The ML estimates of β_1, β_2 and α are given by

$$\widehat{\beta}_1 = \frac{n}{T_1(r_n; \widehat{\alpha})}, \tag{2.6}$$

$$\widehat{\beta}_2 = \frac{m}{T_2(s_m; \widehat{\alpha})}, \tag{2.7}$$

and $\widehat{\alpha}$ is the solution of the following non-linear equation

$$\begin{aligned} &\frac{n+m}{\alpha} + \sum_{i=1}^{n} \frac{\ln r_i}{1+r_i^{\alpha}} + \sum_{j=1}^{m} \frac{\ln s_j}{1+s_j^{\alpha}} - \left(\frac{n}{\ln(1+r_n^{\alpha})}\right) \frac{r_n^{\alpha} \ln r_n}{1+r_n^{a}} \\ &- \left(\frac{m}{\ln(1+s_m^{\alpha})}\right) \frac{s_m^{\alpha} \ln s_m}{1+s_m^{\alpha}} = 0. \end{aligned}$$

Therefore, $\widehat{\alpha}$ can be obtained as a solution of the non-linear equation of the form $h(\alpha) = \alpha$ where

$$\begin{aligned} h(a) = -(n+m) &\left[\sum_{i=1}^{n} \frac{\ln r_i}{1+r_i^{\alpha}} + \sum_{j=1}^{m} \frac{\ln s_j}{1+s_j^{\alpha}} - \left(\frac{n}{\ln(1+r_n^{\alpha})}\right) \frac{r_n^{\alpha} \ln r_n}{1+r_n^{a}} \right. \\ &\left. - \left(\frac{m}{\ln(1+s_m^{\alpha})}\right) \frac{s_m^{\alpha} \ln s_m}{1+s_m^{\alpha}} \right]^{-1}. \end{aligned} \tag{2.8}$$

Since, $\widehat{\alpha}$ is a fixed point solution of the non-linear Equation (2.8), its value can be obtained using an iterative scheme as: $h(\alpha_{(j)}) = \alpha_{(j+1)}$, where $\alpha_{(j)}$ is the j^{th} iterate of $\widehat{\alpha}$. The iteration procedure should be stopped when $\left|\alpha_{(j+1)} - \alpha_{(j)}\right|$ is sufficiently small. After $\widehat{\alpha}$ is obtained, $\widehat{\beta}_1$ and $\widehat{\beta}_2$ can be obtained from (2.6) and (2.7), respectively. Therefore, the MLE of R, say $\widehat{R}$, is given as

$$\widehat{R} = \frac{\widehat{\beta}_1}{\widehat{\beta}_1 + \widehat{\beta}_2}. \tag{2.9}$$

2.2. Existence and uniqueness of the ML estimates. We establish the existence and uniqueness of the ML estimates of the parameters β_1, β_2 and α. We present the following lemma that will be used in proof of 2.2 Theorem.

2.1. Lemma. *Let*

$$w(x) = [\ln(1+x)]^2 + \xi^2(x) \left[\frac{\ln(1+x)}{x} - 1\right],$$

where $\xi(x) = x\ln(x)/(1+x)$. *Then* $w(x) \geq 0$ *for* $x \geq 0$.

Proof. For a proof, one may refer to Ghitany and Al-Awadhi [21]. □

2.2. Theorem. *The ML estimates of the parameters* β_1, β_2 *and* α *are unique, with* $\widehat{\beta}_1 = n/T_1(r_n;\widehat{\alpha})$, $\widehat{\beta}_2 = m/T_2(s_m;\widehat{\alpha})$ *where* $\widehat{\alpha}$ *is the solution of the non-linear equation*

$$G(\alpha) \equiv \frac{n+m}{\alpha} + \sum_{i=1}^{n} \frac{\ln r_i}{1+r_i^\alpha} + \sum_{j=1}^{m} \frac{\ln s_j}{1+s_j^\alpha} - \left(\frac{n}{\ln(1+r_n^\alpha)}\right)\frac{r_n^\alpha \ln r_n}{1+r_n^\alpha}$$

$$-\left(\frac{m}{\ln(1+s_m^\alpha)}\right)\frac{s_m^\alpha \ln s_m}{1+s_m^\alpha} = 0,$$

if at least one of the r_i, $i = 1, ..., n$ *(or* s_j, $j = 1, ..., m$*) is less than unity.*

Proof. We have

$$G(0) \equiv \lim_{\alpha\to 0} G(\alpha) = \lim_{\alpha\to 0}\left(\frac{n+m}{\alpha}\right) + \frac{1}{2}\sum_{i=1}^{n}\ln r_i + \frac{1}{2}\sum_{j=1}^{m}\ln s_j - \frac{n\ln r_n}{2\ln 2}$$

$$-\frac{m\ln s_m}{2\ln 2} = \infty.$$

Let

$$G_1(\alpha;\underline{r}) = \frac{n}{\alpha} + \sum_{i=1}^{n}\frac{\ln r_i}{1+r_i^\alpha} - \left(\frac{n}{\ln(1+r_n^\alpha)}\right)\frac{r_n^\alpha \ln r_n}{1+r_n^a},$$

and

$$G_2(\alpha;\underline{s}) = \frac{m}{\alpha} + \sum_{j=1}^{m}\frac{\ln s_j}{1+s_j^\alpha} - \left(\frac{m}{\ln(1+s_m^\alpha)}\right)\frac{s_m^\alpha \ln s_m}{1+s_m^a}.$$

Then, $G(\alpha) = G_1(\alpha;\underline{r}) + G_2(\alpha;\underline{s})$. Firstly, we consider the limit of $G_1(\alpha;\underline{r})$ as $\alpha \to \infty$.

(i) If r_n is less than unity, that is $r_i < 1$, $i = 1, ..., n$, then

$$\begin{aligned} G_1(\infty;\underline{r}) &\equiv \lim_{\alpha\to\infty} G_1(\alpha;\underline{r}) = \lim_{\alpha\to\infty}\left(\frac{n}{\alpha} + \sum_{i=1}^{n}\frac{\ln r_i}{1+r_i^\alpha} - n\frac{\ln r_n/(1+r_n^\alpha)}{\ln(1+r_n^\alpha)/r_n^\alpha}\right) \\ &= \sum_{i=1}^{n}(\ln r_i - \ln r_n) < 0. \end{aligned}$$

(ii) If only r_n is greater than or equal to unity, that is $r_n \geq 1$ and $r_i < 1$, $i = 1, ..., n-1$, then

$$\begin{aligned} G_1(\infty;\underline{r}) &= \lim_{\alpha\to\infty}\left(\frac{n}{\alpha} + \sum_{i=1}^{n-1}\frac{\ln r_i}{1+r_i^\alpha} + \frac{\ln r_n}{1+r_n^\alpha} - n\frac{r_n^\alpha \ln r_n/(1+r_n^\alpha)}{\ln(1+r_n^\alpha)}\right) \\ &= \sum_{i=1}^{n-1}\ln r_i < 0. \end{aligned}$$

(iii) If r_n and some r_i record values are greater than unity and some r_i record values are less than unity, that is $r_n > 1$ and $r_i > 1$, $i = p, ..., t$, $1 < p \leq t < n$, then

$$\begin{aligned} G_1(\infty;\underline{r}) &= \lim_{\alpha\to\infty}\left(\frac{n}{\alpha} + \sum_{i=1(r_i<1)}^{n}\frac{\ln r_i}{1+r_i^\alpha} + \sum_{i=1(r_i>1)}^{n}\frac{\ln r_i}{1+r_i^\alpha} - n\frac{r_n^\alpha \ln r_n/(1+r_n^\alpha)}{\ln(1+r_n^\alpha)}\right) \\ &= \sum_{i=1(r_i<1)}^{n}\ln r_i < 0. \end{aligned}$$

When the conditions given in (i)-(iii) holds for s_j, $j = 1, ..., m$, $G_2(\alpha;\underline{s}) < 0$ as $\alpha \to \infty$. So that, the limit of $G(\alpha) = G_1(\alpha;\underline{r}) + G_2(\alpha;\underline{s}) < 0$ as $\alpha \to \infty$ when r_i, $i = 1, ..., n$ and s_j, $j = 1, ..., m$ satisfy any of the conditions given in (i)-(iii).

Next, we need to show the limit of $G(\alpha) < 0$ as $\alpha \to \infty$ for $s_j > 1$, $j = 1, ..., m$ and when the conditions given (i)-(iii) holds for r_i, $i = 1, ..., n$ (or $r_i > 1, i = 1, ..., n$ and when the conditions given (i)-(iii) holds for s_j, $j = 1, ..., m$). In particular, when $s_j > 1$, $j = 1, ..., m$ and the conditions given (i) holds for r_i, $i = 1, ..., n$, we can take α large enough, such that $G_2(\alpha; \underline{s}) \to 0^+$ and $G_1(\alpha; \underline{r}) + G_2(\alpha; \underline{s}) < 0$ as $\alpha \to \infty$. Other cases can be obtained similarly.

Finally, we need to show that there is no solution if all records are greater than unity, that is $r_i > 1$, $i = 1, ..., n$ and $s_j > 1$, $j = 1, ..., m$. If $r_i > 1$, $i = 1, ..., n$, then

$$G_1(\alpha; \underline{r}) < \frac{n}{\alpha} + n \ln r_n \left[\frac{1}{1 + r_1^\alpha} - \frac{r_n^\alpha}{(1 + r_n^\alpha)^2} \right] \to 0^+ \text{ as } \alpha \to \infty.$$

Similarly, $G_2(\alpha; \underline{s}) \to 0^+$ as $\alpha \to \infty$. Therefore, $G(\alpha) \to 0^+$ as $\alpha \to \infty$.

Except all records are greater than unity, we obtain that $\lim_{\alpha \to 0} G(\alpha) = \infty$ and $\lim_{\alpha \to \infty} G(\alpha) < 0$. By the intermediate value theorem $G(\alpha)$ has at least one root in $(0, \infty)$. If it can be shown that $G(\alpha)$ is decreasing, then the proof will be completed. It is easily obtained that

$$\begin{aligned} \frac{dG_1(\alpha; \underline{r})}{d\alpha} &= -\frac{1}{\alpha^2} \left[n + \sum_{i=1}^n \frac{\xi^2(r_i^\alpha)}{r_i^\alpha} + \frac{n\xi^2(r_n^\alpha)}{\ln(1 + r_n^\alpha)} \left(\frac{1}{r_n^\alpha} - \frac{1}{\ln(1 + r_n^\alpha)} \right) \right] \\ &= -\frac{1}{\alpha^2} \left[\sum_{i=1}^n \frac{\xi^2(r_i^\alpha)}{r_i^\alpha} + \frac{n}{(\ln(1 + r_n^\alpha))^2} w(r_n^\alpha) \right]. \end{aligned}$$

Similarly,

$$\frac{dG_2(\alpha; \underline{s})}{d\alpha} = -\frac{1}{\alpha^2} \left[\sum_{j=1}^m \frac{\xi^2(s_j^\alpha)}{s_j^\alpha} + \frac{n}{(\ln(1 + s_m^\alpha))^2} w(s_m^\alpha) \right].$$

It is clear that $dG_1(\alpha; \underline{r})/d\alpha < 0$ and $dG_2(\alpha; \underline{s})/d\alpha < 0$ by using 2.1 Lemma. Therefore, $dG(\alpha)/d\alpha < 0$.

Finally, we will show that the ML estimates of $(\beta_1, \beta_2, \alpha)$ maximizes the log-likelihood function $l(\beta_1, \beta_2, \alpha \,|\underline{r}, \underline{s})$. Let $H(\beta_1, \beta_2, \alpha)$ be the Hessian matrix of $l(\beta_1, \beta_2, \alpha \,|\underline{r}, \underline{s})$ at $(\beta_1, \beta_2, \alpha)$. It is clear that if $\det(H) \neq 0$ for the critical point $(\beta_1, \beta_2, \alpha)$ and $\det(H_1) < 0$, $\det(H_2) > 0$ and $\det(H_3) < 0$ at $(\beta_1, \beta_2, \alpha)$ then it is a local maximum of $l(\beta_1, \beta_2, \alpha \,|\underline{r}, \underline{s})$, where

$$H_1 = \frac{\partial^2 l}{\partial \beta_1^2}, \; H_2 = \begin{pmatrix} \frac{\partial^2 l}{\partial \beta_1^2} & \frac{\partial^2 l}{\partial \beta_1 \partial \beta_2} \\ \frac{\partial^2 l}{\partial \beta_2 \partial \beta_1} & \frac{\partial^2 l}{\partial \beta_2^2} \end{pmatrix}, \; H_3 = H \text{ and } l = l(\beta_1, \beta_2, \alpha \,|\underline{r}, \underline{s}).$$

It can be easily seen that

$$\det(H_1(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha})) = -\frac{\left(\ln(1 + r_n^{\widehat{\alpha}})\right)^2}{n} < 0,$$

$$\det(H_2(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha})) = \frac{\left(\ln(1 + r_n^{\widehat{\alpha}})\right)^2}{n} \frac{\left(\ln(1 + s_m^{\widehat{\alpha}})\right)^2}{m} > 0,$$

and

$$\det(H_2(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha})) = \frac{\partial G(\widehat{\alpha})}{\partial \alpha} \frac{\left(\ln(1 + r_n^{\widehat{\alpha}})\right)^2}{n} \frac{\left(\ln(1 + s_m^{\widehat{\alpha}})\right)^2}{m} < 0.$$

Hence, $(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha})$ is the local maximum of $l(\beta_1, \beta_2, \alpha \,|\underline{r}, \underline{s})$. Since there is no singular point of $l(\beta_1, \beta_2, \alpha \,|\underline{r}, \underline{s})$ and it has a single critical point then, it is enough to show that the absolute maximum of the function is indeed the local maximum. Assume that there exist an $\widehat{\alpha}_0$ in the domain in which $l^*(\widehat{\alpha}_0) > l^*(\widehat{\alpha})$, where $l^*(\widehat{\alpha}) = l(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha} \,|\underline{r}, \underline{s})$. Since $\widehat{\alpha}$ is the local maximum there should be some point α_1 in the neighborhood of $\widehat{\alpha}$ such that $l^*(\widehat{\alpha}) > l^*(\alpha_1)$. Let $k(\alpha) = l^*(\alpha) - l^*(\widehat{\alpha})$ then $k(\widehat{\alpha}_0) > 0$, $k(\alpha_1) < 0$ and $k(\widehat{\alpha}) = 0$. This implies that α_1 is a local minimum of the $l^*(\alpha)$, but $\widehat{\alpha}$ is the only critical point so it is a contradiction. Therefore, $(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha})$ is the absolute maximum of $l(\beta_1, \beta_2, \alpha \,|\underline{r}, \underline{s})$. $\square$

2.3. Remark. In case all records are greater than one, we can still get a unique solution of the parameters when we divide the record values, say by r_n (or by s_m or divide r_i by r_n and divide s_j by s_m) as long as the transformed observations follow from Burr Type XII.

2.3. Asymptotic distribution and confidence intervals for R. The Fisher information matrix of $I \equiv I(\beta_1, \beta_2, \alpha)$ is given by

$$I = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \beta_1^2}\right) & E\left(\frac{\partial^2 l}{\partial \beta_1 \partial \beta_2}\right) & E\left(\frac{\partial^2 l}{\partial \beta_1 \partial \alpha}\right) \\ E\left(\frac{\partial^2 l}{\partial \beta_2 \partial \beta_1}\right) & E\left(\frac{\partial^2 l}{\partial \beta_2^2}\right) & E\left(\frac{\partial^2 l}{\partial \beta_2 \partial \alpha}\right) \\ E\left(\frac{\partial^2 l}{\partial \alpha \partial \beta_1}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \beta_2}\right) & E\left(\frac{\partial^2 l}{\partial \alpha^2}\right) \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix},$$

where $I_{11} = n/\beta_1^2$, $I_{22} = m/\beta_2^2$,

$$I_{13} = E\left(\frac{R_n \ln R_n}{1 + R_n^{\alpha}}\right) = \frac{\beta_1^n}{\alpha\Gamma(n)}\psi_1(n, \beta_1),$$

$$I_{23} = \frac{\beta_2^m}{\alpha\Gamma(m)}\psi_1(m, \beta_2), \;\; \psi_1(a, b) = \int_0^{\infty} \frac{x \ln x (\ln(1+x))^{a-1}}{(1+x)^{b+2}} dx,$$

$$I_{33} = \frac{n+m}{\alpha^2} + \sum_{i=1}^{n} \frac{\beta_1^i \psi_2(i, \beta_1)}{\alpha^2 \Gamma(i)} + \sum_{j=1}^{m} \frac{\beta_2^j \psi_2(j, \beta_2)}{\alpha^2 \Gamma(j)} + \frac{\beta_1^{n+1} \psi_2(n, \beta_1)}{\alpha^2 \Gamma(n)}$$

$$+ \frac{\beta_2^{m+1} \psi_2(m, \beta_2)}{\alpha^2 \Gamma(m)}, \;\; \psi_2(a, b) = \int_0^{\infty} \frac{x \,(\ln x)^2 \,(\ln(1+x))^{a-1}}{(1+x)^{b+3}} dx.$$

By the asymptotic properties of the MLE, $\widehat{R}$ is asymptotically normal with mean R and asymptotic variance

$$\sigma_R^2 = \sum_{j=1}^{3} \sum_{i=1}^{3} \frac{\partial R}{\partial \beta_i} \frac{\partial R}{\partial \beta_j} I_{ij}^{-1},$$

where $\beta_3 \equiv \alpha$ and I_{ij}^{-1} is the (i, j)th element of the inverse of the $I(\beta_1, \beta_2, \alpha)$, see Rao [39]. Then,

$$\sigma_R^2 = \left(\frac{\partial R}{\partial \beta_1}\right)^2 I_{11}^{-1} + 2 \frac{\partial R}{\partial \beta_1} \frac{\partial R}{\partial \beta_2} I_{12}^{-1} + \left(\frac{\partial R}{\partial \beta_2}\right)^2 I_{22}^{-1}, \tag{2.10}$$

where

$$\frac{\partial R}{\partial \beta_1} = \frac{\beta_2}{(\beta_1 + \beta_2)^2}, \;\; \frac{\partial R}{\partial \beta_2} = \frac{-\beta_1}{(\beta_1 + \beta_2)^2}.$$

Therefore, an asymptotic $100(1-\gamma)\%$ confidence interval of R is

$$\left(\widehat{R} - z_{\gamma/2}\widehat{\sigma}_R, \widehat{R} + z_{\gamma/2}\widehat{\sigma}_R\right), \tag{2.11}$$

where z_{γ} is the upper γth quantile of the standard normal distribution and $\widehat{\sigma}_R$ is the value of σ_R at the MLE of the parameters.

If the likelihood equations have a unique solution $\widehat{\theta}_n$, then $\widehat{\theta}_n$ is consistent, asymptotically normal and efficient (see Lehmann and Casella [28]). When the likelihood equations have a unique solution, the observed information matrix $J_m(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha})/m$ is a consistent estimator for $I_m(\beta_1, \beta_2, \alpha)/m$ (see Appendix C in Lawless [27]). The observed information matrix $J(\beta_1, \beta_2, \alpha)$ is given by

$$J(\beta_1, \beta_2, \alpha) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \beta_1^2} & \frac{\partial^2 l}{\partial \beta_1 \partial \beta_2} & \frac{\partial^2 l}{\partial \beta_1 \partial \alpha} \\ \frac{\partial^2 l}{\partial \beta_2 \partial \beta_1} & \frac{\partial^2 l}{\partial \beta_2^2} & \frac{\partial^2 l}{\partial \beta_2 \partial \alpha} \\ \frac{\partial^2 l}{\partial \alpha \partial \beta_1} & \frac{\partial^2 l}{\partial \alpha \partial \beta_2} & \frac{\partial^2 l}{\partial \alpha^2} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix},$$

where

$$J_{11} = \frac{n}{\beta_1^2},\ J_{12} = J_{21} = \frac{r_n^\alpha \ln r_n}{1 + r_n^a},\ J_{22} = \frac{m}{\beta_2^2},\ J_{23} = J_{32} = \frac{s_m^\alpha \ln s_m}{1 + s_m^\alpha},$$

$$J_{33} = \frac{n+m}{\alpha^2} + \sum_{i=1}^{n} r_i^\alpha \left(\frac{\ln r_i}{1 + r_i^\alpha}\right)^2 + \sum_{j=1}^{m} s_j^\alpha \left(\frac{\ln s_j}{1 + s_j^\alpha}\right)^2 + \beta_1 r_n^\alpha \left(\frac{\ln r_n}{1 + r_n^a}\right)^2$$
$$+\beta_2 s_m^\alpha \left(\frac{\ln s_m}{1 + s_m^\alpha}\right)^2.$$

Therefore, an asymptotic $100(1-\gamma)\%$ confidence interval of R can be obtained following from Equation (2.11) by replacing I with J in Equation (2.10).

2.4. Bayes estimation of R. Bayesian approach has a number of advantages over the conventional frequentist approach. Bayes theorem is a consistent way to modify our beliefs about the parameters given the data that actually occurred (see Bolstad [15]). In this subsection, we consider the Bayes estimates of the stress-strength reliability for Burr Type XII distribution under different loss functions.

In the Bayesian inference, the most commonly used loss function is the squared error (SE) loss, $L(\theta^*, \theta) = (\theta^* - \theta)^2$, where θ^* is an estimate of θ. This loss function is symmetrical and gives equal weight to overestimation as well as underestimation. It is well known that the use of symmetric loss functions may be inappropriate in many circumstances, particularly when positive and negative errors have different consequences. A useful asymmetric loss function is the linear-exponential (LINEX) loss, $L(\theta^*, \theta) = e^{v(\theta^* - \theta)} - v(\theta^* - \theta) - 1,\ v \neq 0$, introduced by Varian [46]. The sign and magnitude of v represents the direction and degree of asymmetry, respectively. For v close to zero, the LINEX loss is approximately equal to the SE loss and therefore almost symmetric.

We assume that all parameters β_1, β_2 and α are unknown and have independent gamma prior distributions with parameters (a_i, b_i), $i = 1, 2, 3$, respectively. The density function of a gamma random variable X with parameters (a, b) is

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-xb},\ x > 0,\ a, b > 0.$$

Then, the joint posterior density function of β_1, β_2 and α is

$$\begin{aligned} \pi\left(\beta_1, \beta_2, \alpha \,|\underline{r}, \underline{s}\right) &= I(\underline{r}, \underline{s}) h_1(\underline{r}; \alpha) h_2(\underline{s}; \alpha) \alpha^{n+m+a_3-1} \beta_1^{n+a_1-1} \beta_2^{m+a_2-1} \\ &\quad \exp\left\{-\alpha b_3 - \beta_1\left(b_1 + T_1(r_n; \alpha)\right) - \beta_2\left(b_2 + T_2(s_m; \alpha)\right)\right\}, \end{aligned} \tag{2.12}$$

where

$$\frac{[I(\underline{r}, \underline{s})]^{-1}}{\Gamma(n + a_1)\Gamma(m + a_2)} = \int_0^\infty \frac{h_1(\underline{r}; \alpha) h_2(\underline{s}; \alpha) \alpha^{n+m+a_3-1} e^{-\alpha b_3}}{\left(b_1 + T_1(r_n; \alpha)\right)^{n+a_1} \left(b_2 + T_2(s_m; \alpha)\right)^{m+a_2}} d\alpha.$$

Then, the Bayes estimate of a given measurable function of β_1, β_2 and α, say $u(\beta_1, \beta_2, \alpha)$ under the SE loss function is

$$\widehat{u}_B = \int_0^\infty \int_0^\infty \int_0^\infty u(\beta_1, \beta_2, \alpha) \pi(\beta_1, \beta_2, \alpha \,|\underline{r}, \underline{s}) d\beta_1 d\beta_2 d\alpha. \tag{2.13}$$

It is not possible to compute Equation (2.13) analytically. Two approaches can be applied to approximate Equation (2.13), namely, Lindley's approximation and MCMC method.

2.4.1. *Lindley's approximation.* Lindley proposed a method to approximate the ratio of two integrals such as Equation (2.13) in [30]. This procedure are also employed to the posterior expectation of the function $U(\lambda)$, for given $\mathbf{x}$, is

$$E(u(\lambda)\,|\mathbf{x}) = \frac{\int u(\lambda)e^{Q(\lambda)}d\lambda}{\int e^{Q(\lambda)}d\lambda},$$

where $Q(\lambda) = l(\lambda) + \rho(\lambda)$, $l(\lambda)$ is the logarithm of the likelihood function and $\rho(\lambda)$ is the logarithm of the prior density of λ. Using Lindley's approximation, $E(u(\lambda)\,|\mathbf{x})$ is approximately estimated by

$$\begin{aligned} E(u(\lambda)\,|\mathbf{x}) &= \left[u + \frac{1}{2}\sum_i\sum_j (u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l\right]_{\widehat{\lambda}} \\ &\quad + \text{terms of order } n^{-2} \text{ or smaller,} \end{aligned}$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$, $i, j, k, l = 1, ..., m$, $\widehat{\lambda}$ is the MLE of λ, $u = u(\lambda)$, $u_i = \partial u/\partial\lambda_i$, $u_{ij} = \partial^2 u/\partial\lambda_i\partial\lambda_j$, $L_{ijk} = \partial^3 l/\partial\lambda_i\partial\lambda_j\partial\lambda_k$, $\rho_j = \partial\rho/\partial\lambda_j$, and $\sigma_{ij} = (i,j)$th element in the inverse of the matrix $\{-L_{ij}\}$ all evaluated at the MLE of the parameters.

For the three parameter case $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, Lindley's approximation leads to

$$\begin{aligned} \widehat{u}_B &= E(u(\lambda)\,|\mathbf{x}) = u + (u_1c_1 + u_2c_2 + u_3c_3 + c_4 + c_5) + \frac{1}{2}\,[A(u_1\sigma_{11} \\ &+ u_2\sigma_{12} + u_3\sigma_{13}) + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})]\,, \end{aligned}$$

evaluated at $\widehat{\lambda} = (\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3)$, where

$$\begin{aligned} c_i &= \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, \; i = 1,2,3, \\ c_4 &= u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23}, \\ c_5 &= \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}), \\ A &= \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{23}L_{231} + \sigma_{22}L_{221} + \sigma_{33}L_{331}, \\ B &= \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{23}L_{232} + \sigma_{22}L_{222} + \sigma_{33}L_{332}, \\ C &= \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{22}L_{223} + \sigma_{33}L_{333}. \end{aligned}$$

In our case, $(\lambda_1, \lambda_2, \lambda_3) \equiv (\beta_1, \beta_2, \alpha)$ and

$$\begin{aligned} \rho_1 &= \frac{(a_1-1)}{\beta_1} - b_1, \; \rho_2 = \frac{(a_2-1)}{\beta_2} - b_2, \; \rho_3 = \frac{(a_3-1)}{\alpha} - b_3, \\ L_{11} &= -\frac{n}{\beta_1^2}, \; L_{22} = -\frac{m}{\beta_2^2}, \\ L_{13} &= L_{31} = -\frac{r_n^\alpha \ln r_n}{1 + r_n^a}, \; L_{23} = L_{32} = -\frac{s_m^\alpha \ln s_m}{1 + s_m^\alpha}, \\ L_{33} &= -\frac{n+m}{\alpha^2} - \sum_{i=1}^n r_i^\alpha\left(\frac{\ln r_i}{1 + r_i^\alpha}\right)^2 - \sum_{j=1}^m s_j^\alpha\left(\frac{\ln s_j}{1 + s_j^\alpha}\right)^2 \\ &- \beta_1 r_n^\alpha\left(\frac{\ln r_n}{1 + r_n^a}\right)^2 - \beta_2 s_m^\alpha\left(\frac{\ln s_m}{1 + s_m^\alpha}\right)^2, \end{aligned}$$

σ_{ij}, $i, j = 1,2,3$ are obtained by using L_{ij}, $i, j = 1,2,3$ and

$$L_{111} = \frac{2}{\beta_1^3}, \; L_{222} = \frac{2m}{\beta_2^3},$$

$$L_{133} = L_{331} = -r_n^{\alpha}\left(\frac{\ln r_n}{1+r_n^{\alpha}}\right)^2,\ L_{233} = L_{322} = -s_m^{\alpha}\left(\frac{\ln s_m}{1+s_m^{\alpha}}\right)^2,$$

$$\begin{aligned} L_{333} &= \frac{2(n+m)}{\alpha^3} - \sum_{i=1}^{n} r_i^{\alpha}(1-r_i^{\alpha})\left(\frac{\ln r_i}{1+r_i^{\alpha}}\right)^3 - \sum_{j=1}^{m} s_j^{\alpha}(1-s_j^{\alpha})\left(\frac{\ln s_j}{1+s_j^{\alpha}}\right)^3 \\ &\quad -\beta_1 r_n^{\alpha}(1-r_n^{\alpha})\left(\frac{\ln r_n}{1+r_n^{\alpha}}\right)^2 - \beta_2 s_m^{\alpha}(1-s_m^{\alpha})\left(\frac{\ln s_m}{1+s_m^{\alpha}}\right)^2. \end{aligned}$$

Moreover, $A = \sigma_{11}L_{111} + \sigma_{33}L_{331}$, $B = \sigma_{22}L_{222} + \sigma_{33}L_{332}$ and $C = 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{33}L_{333}$. To obtain the Bayes estimate of R under the SE loss function, we take $u(\beta_1, \beta_2, \alpha) = R = \beta_1/(\beta_1 + \beta_2)$. Then, $u_3 = u_{13} = u_{23} = u_{33} = 0$,

$$u_1 = \frac{\partial R}{\partial \beta_1} = \frac{\beta_2}{(\beta_1+\beta_2)^2},\ u_2 = \frac{\partial R}{\partial \beta_2} = \frac{-\beta_1}{(\beta_1+\beta_2)^2},\ u_{12} = u_{21} = \frac{\beta_1 - \beta_2}{(\beta_1+\beta_2)^3},$$

$$u_{11} = \frac{\partial^2 R}{\partial \beta_1^2} = \frac{-2\beta_2}{(\beta_1+\beta_2)^3},\ u_{22} = \frac{\partial^2 R}{\partial \beta_2^2} = \frac{2\beta_1}{(\beta_1+\beta_2)^3},$$

and

$$c_4 = u_{12}\sigma_{12},\ c_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22}).$$

Hence, the Bayes estimate of R under the SE loss function is given as

$$\begin{aligned} \widehat{R}_{BS,Lindley} &= R + [u_1c_1 + u_2c_2 + c_4 + c_5] \\ &\quad + \frac{1}{2}\left\{A\left[u_1\sigma_{11} + u_2\sigma_{12}\right] + B\left[u_1\sigma_{21} + u_2\sigma_{22}\right] + C\left[u_1\sigma_{31} + u_2\sigma_{32}\right]\right\}. \end{aligned} \tag{2.14}$$

Notice that all parameters are evaluated at $(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha})$.

For the Bayes estimate of R under the LINEX loss function, we take $u(\beta_1, \beta_2, \alpha) = e^{-vR}$. Then, $u_3^* = u_{13}^* = u_{23}^* = u_{33}^* = 0$,

$$u_1^* = \frac{-v\beta_2 e^{-vR}}{(\beta_1+\beta_2)^2},\ u_{11}^* = \frac{ve^{-vR}(v\beta_2^2 + 2\beta_1\beta_2 + 2\beta_2^2)}{(\beta_1+\beta_2)^4},$$

$$u_2^* = \frac{v\beta_1 e^{-vR}}{(\beta_1+\beta_2)^2},\ u_{22}^* = \frac{ve^{-vR}(v\beta_1^2 - 2\beta_1\beta_2 - 2\beta_1^2)}{(\beta_1+\beta_2)^4},$$

$$u_{12}^* = -ve^{-vR}\left(\frac{v\beta_1\beta_2}{(\beta_1+\beta_2)^4} + \frac{\beta_1 - \beta_2}{(\beta_1+\beta_2)^3}\right).$$

and $c_4^* = u_{12}\sigma_{12}$, $c_5^* = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22})$. Then, the Bayes estimate of R under the LINEX loss function is given as

$$\widehat{R}_{BL,Lindley} = -\frac{1}{v}\ln E(e^{-vR}), \tag{2.15}$$

where

$$\begin{aligned} E(e^{-vR}) &= e^{-vR} + [u_1^*c_1 + u_2^*c_2 + c_4^* + c_5^*] \\ &\quad + \frac{1}{2}\left\{A\left[u_1^*\sigma_{11} + u_2^*\sigma_{12}\right] + B\left[u_1^*\sigma_{21} + u_2^*\sigma_{22}\right] + C\left[u_1^*\sigma_{31} + u_2^*\sigma_{32}\right]\right\}. \end{aligned} \tag{2.16}$$

Notice that all parameters are evaluated at $(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\alpha})$.

2.4.2. *MCMC method.* In the previous subsection, the Bayes estimate of R are obtained by using Lindley's approximation under the SE and the LINEX loss functions. Since the exact probability distribution of R is not known, it is difficult to evaluate Bayesian credible interval of R. For this reason, we use the MCMC method to compute the Bayes estimate R under the SE and the LINEX loss functions as well as the HPD credible interval.

We consider the MCMC method to generate samples from the posterior distributions and then compute the Bayes estimate of R under the SE and the LINEX loss functions. The joint posterior density of β_1, β_2 and α is given by Equation (2.12). It is easy to see that the posterior density functions of β_1, β_2 and α are

$$\beta_1 \,|\alpha, \underline{r}, \underline{s} \sim Gamma(n + a_1, b_1 + T_1(r_n; \alpha)),$$

$$\beta_2 \,|\alpha, \underline{r}, \underline{s} \sim Gamma(m + a_2, b_2 + T_2(s_m; \alpha)),$$

and

$$\pi(\alpha \,|\beta_1, \beta_2, \underline{r}, \underline{s}) \propto \alpha^{n+m+a_3-1} \exp\left\{-\alpha b_3 - \beta_1 T_1(r_n; \alpha) - \sum_{i=1}^{n} \ln(1 + r_i^{\alpha}) -\beta_2 T_2(s_m; \alpha) + \alpha \left(\sum_{i=1}^{n} \ln r_i + \sum_{j=1}^{m} \ln s_j\right) - \sum_{j=1}^{m} \ln(1 + s_j^{\alpha})\right\}. \tag{2.17}$$

Therefore, samples of β_1 and β_2 can be generated by using the gamma distribution. However, the posterior distribution of α cannot be reduced analytically to well known distribution, therefore it is not possible to sample directly by standard methods. If the posterior density of α is unimodal and roughly symmetric then it is often convenient to approximate it by a normal distribution (see Gelman et al. [20]. Since the posterior density of α is log-concave density (so unimodal) and it is roughly symmetric (by experimentation), we use the Metropolis-Hasting algorithm with the normal proposal distribution to generate a random sample from the posterior density of α. The hybrid Metropolis-Hastings and Gibbs sampling algorithm, which will be used to solve our problem, is suggested by Tierney [45]. This algorithm combines the Metropolis-Hastings with Gibbs sampling scheme under the normal proposal distribution.

Step 1. Start with initial guess $\alpha^{(0)}$.

Step 2. Set $i = 1$.

Step 3. Generate $\beta_1^{(i)}$ from $Gamma(n + a_1, T_1(r_n; \alpha^{(i-1)}) + b_1)$.

Step 4. Generate $\beta_2^{(i)}$ from $Gamma(m + a_2, T_2(s_m; \alpha^{(i-1)}) + b_2)$.

Step 5. Generate $\alpha^{(i)}$ from $\pi(\alpha \,|\beta_1, \beta_2, \underline{r}, \underline{s})$ using the Metropolis-Hastings algorithm with the proposal distribution $q(\alpha) \equiv N(\alpha^{(i-1)}, 1)$:

(a) Let $v = \alpha^{(i-1)}$.

(b) Generate w from the proposal distribution q.

(c) Let $p(v, w) = \min\left\{1, \dfrac{\pi(w \left|\beta_1^{(i)}, \beta_2^{(i)}, \underline{r}, \underline{s}\right.)\; q(v)}{\pi(v \left|\beta_1^{(i)}, \beta_2^{(i)}, \underline{r}, \underline{s}\right.)\; q(w)}\right\}$.

(d) Generate u from $Uniform(0, 1)$. If $u \leq p(v, w)$ then accept the proposal and set $\alpha^{(i)} = w$; otherwise, set $\alpha^{(i)} = v$.

Step 6. Compute the $R^{(i)} = \beta_1^{(i)} / (\beta_1^{(i)} + \beta_2^{(i)})$.

Step 7. Set $i = i + 1$.

Step 8. Repeat Steps 2-7, N times, and obtain the posterior sample $R^{(i)}$, $i = 1, ..., N$.

This sample are used to compute the Bayes estimate and to construct the HPD credible interval for R. The Bayes estimate of R under the SE and the LINEX loss function are given as

$$\widehat{R}_{BS,MCMC} = \frac{1}{N-M}\sum_{i=M+1}^{N-M} R^{(i)},$$

$$\widehat{R}_{BL,MCMC} = -\frac{1}{v}\ln E(e^{-vR}) = -\frac{1}{v}\ln\left(\frac{1}{N-M}\sum_{i=M+1}^{N-M} e^{-vR^{(i)}}\right).$$

where M is the burn-in period.

The HPD $100(1-\gamma)\%$ credible interval of R is obtained by using the method given in Chen and Shao [18]. From MCMC, the sequence $R^{(1)},\ldots,R^{(N)}$, are obtained and ordered as $R_{(1)}<\ldots<R_{(N)}$. The credible intervals are constructed as $\left(R_{(j)}, R_{(j+[N(1-\gamma)])}\right)$ for $j=1,...,N-[N(1-\gamma)]$ where $[x]$ denotes the largest integer less than or equal to x. Then, the HPD credible interval of R is that interval which has the shortest length.

3. Estimation of R when the first shape parameter α is known

In this section, we consider the estimation of R when α is known, say $\alpha=\alpha_0$. Let $R_1,\ldots,R_n$ be a set of upper records from $Burr(\alpha_0,\beta_1)$ and $S_1,\ldots,S_m$ be a set of upper records from $Burr(\alpha_0,\beta_2)$ independently from the first sample.

3.1. MLE estimation and confidence intervals of R. Based on the samples described above, the MLE of R, say $\widehat{R}_{MLE}$, is

$$\widehat{R}_{MLE} = \frac{\widehat{\beta}_1}{\widehat{\beta}_1+\widehat{\beta}_2} = \frac{nT_2(s_m;\alpha_0)}{nT_2(s_m;\alpha_0)+mT_1(r_n;\alpha_0)}, \tag{3.1}$$

where $T_1(r_n;\alpha_0)=\ln(1+r_n^{\alpha_0})$, $T_2(s_m;\alpha_0)=\ln(1+s_m^{\alpha_0})$.

It is easy to see that $2\beta_1\ln(1+r_n^{\alpha_0})\sim\chi^2(2n)$ and $2\beta_2\ln(1+s_m^{\alpha_0})\sim\chi^2(2m)$. Therefore,

$$F^* = \left(\frac{R}{1-R}\right)\left(\frac{1-\widehat{R}_{MLE}}{\widehat{R}_{MLE}}\right)$$

is an F distributed random variable with $(2n,2m)$ degrees of freedom. The pdf of $\widehat{R}_{MLE}$ is

$$f_{\widehat{R}_{MLE}}(r) = \frac{1}{r^2B(m,n)}\left(\frac{n\beta_1}{m\beta_2}\right)^n \frac{(\frac{1-r}{r})^{n-1}}{\left(1+\frac{n\beta_1(1-r)}{m\beta_2 r}\right)^{n+m}},$$

where $0<r<1$. The $100(1-\gamma)\%$ exact confidence interval for R can be obtained as

$$\left(\frac{1}{1+F_{2m,2n;\frac{\gamma}{2}}\left(\frac{1-\widehat{R}_{MLE}}{\widehat{R}_{MLE}}\right)}, \frac{1}{1+F_{2m,2n;1-\frac{\gamma}{2}}\left(\frac{1-\widehat{R}_{MLE}}{\widehat{R}_{MLE}}\right)}\right), \tag{3.2}$$

where $F_{2m,2n;\frac{\gamma}{2}}$ and $F_{2m,2n;1-\frac{\gamma}{2}}$ are the lower and upper $\frac{\gamma}{2}$th percentile points of a F distribution with $(2m,2n)$ degrees of freedom.

On the other hand, the approximate confidence interval of R can be easily obtained by using the Fisher information matrix. The Fisher information matrix of (β_1,β_2) is

$$I = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial\beta_1^2}\right) & E\left(\frac{\partial^2 l}{\partial\beta_1\partial\beta_2}\right) \\ E\left(\frac{\partial^2 l}{\partial\beta_1\partial\beta_2}\right) & E\left(\frac{\partial^2 l}{\partial\beta_2^2}\right)\end{pmatrix} = \begin{pmatrix} n/\beta_1^2 & 0 \\ 0 & m/\beta_2^2\end{pmatrix}.$$

By the asymptotic properties of the MLE, $\widehat{R}_{MLE}$ is asymptotically normal with mean R and asymptotic variance

$$\sigma_R^2 = \sum_{j=1}^{2}\sum_{i=1}^{2}\frac{\partial R}{\partial \beta_i}\frac{\partial R}{\partial \beta_j}I_{ij}^{-1}$$

where I_{ij}^{-1} is the (i,j) th element of the inverse of the I, see Rao [39]. Then

$$\sigma_R^2 = R^2(1-R)^2\left(\frac{1}{n}+\frac{1}{m}\right) \tag{3.3}$$

Therefore, an asymptotic $100(1-\gamma)\%$ confidence interval for R is

$$\left(\widehat{R}_{MLE} - z_{\gamma/2}\widehat{\sigma}_R, \widehat{R}_{MLE} + z_{\gamma/2}\widehat{\sigma}_R\right), \tag{3.4}$$

where z_γ is the upper γth percentile points of a standard normal distribution and $\widehat{\sigma}_R$ is the value of σ_R at the MLE of the parameters.

3.2. UMVUE of R**.** In this subsection, we obtain the UMVUE of R. When the first shape parameter α is known, $(T_1(r_n;\alpha_0), T_2(s_m;\alpha_0))$ is a sufficient statistics for (β_1,β_2). It can be shown that it is also a complete sufficient statistic by using Theorem 10-9 in Arnold [6]. Let us define

$$\phi(R_1,S_1) = \begin{cases} 1 & \text{if } R_1 < S_1 \\ 0 & \text{if } R_1 \geq S_1 \end{cases}.$$

Then $E\left(\phi(R_1,S_1)\right) = R$ so it is an unbiased estimator of R. Let $P_1 = \ln(1+R_1^{\alpha_0})$ and $P_2 = \ln(1+S_1^{\alpha_0})$. The UMVUE of R, say $\widehat{R}_U$, can be obtained by using the Rao-Blackwell and the Lehmann-Scheffe's Theorems, see Arnold [6],

$$\begin{aligned}
\widehat{R}_U &= E\left(\phi(P_1,P_2) \mid (T_1,T_2)\right) \\
&= \int_{P_2}\int_{P_1}\phi(P_1,P_2)\ f(p_1,p_2\,|T_1,T_2\,)dp_1dp_2 \\
&= \int_{P_2}\int_{P_1}\phi(P_1,P_2)f_{P_1|T_1}(p_1\,|T_1\,)f_{P_2|T_2}(p_2\,|T_2\,)dp_1dp_2,
\end{aligned}$$

where $(T_1,T_2) = (T_1(r_n;\alpha_0), T_2(s_m;\alpha_0))$, $f(p_1,p_2\,|T_1,T_2\,)$ is the conditional pdf of (P_1,P_2) given (T_1,T_2). Using the joint pdf of (R_1,R_n) and (S_1,S_m) and after making a simple transformation, we obtain the $f_{P_1|T_1}(p_1\,|T_1\,)$ and $f_{P_2|T_2}(p_2\,|T_2\,)$, and are given by

$$f_{P_1|T_1}(p_1 \mid T_1) = (n-1)\frac{(t_1-p_1)^{n-2}}{t_1^{n-1}},\ 0 < p_1 < t_1,$$

$$f_{P_2|T_2}(p_2 \mid T_2) = (m-1)\frac{(t_2-p_2)^{m-2}}{t_2^{m-1}},\ 0 < p_2 < t_2.$$

Therefore,

$$\begin{aligned}
\widehat{R}_U &= \iint\limits_{P_1<P_2} f_{P_1|T_1}(p_1 \mid T_1)f_{P_2|T_2}(p_2 \mid T_2)dp_1dp_2 \\
&= \begin{cases} \int_0^{t_1}\int_{p_1}^{t_2}(n-1)(m-1)\frac{(t_1-p_1)^{n-2}}{t_1^{n-1}}\frac{(t_2-p_2)^{m-2}}{t_2^{m-1}}dp_2dp_1 & \text{if } t_2 \geq t_1 \\ \int_0^{t_2}\int_0^{p_2}(n-1)(m-1)\frac{(t_1-p_1)^{n-2}}{t_1^{n-1}}\frac{(t_2-p_2)^{m-2}}{t_2^{m-1}}dp_1dp_2 & \text{if } t_2 < t_1 \end{cases} \\
&= \begin{cases} {}_2F_1(1,1-m;n;t_1/t_2) & \text{if } t_2 \geq t_1 \\ 1 -{}_2 F_1(1,1-n;m;t_2/t_1) & \text{if } t_2 < t_1 \end{cases},
\end{aligned} \tag{3.5}$$

where ${}_2F_1(.,.;.;.)$ is Gauss hypergeometric function, see formula 3.196(1) in Gradshteyn and Ryzhik [22].

3.3. Bayes estimation of R. In this subsection, we assume that β_1 and β_2 are unknown and have independent gamma prior distributions with parameters (a_i, b_i), $i = 1, 2$, respectively. Then, the joint posterior density function of β_1 and β_2 is

$$\pi\left(\beta_1, \beta_2 \,|\alpha_0, \underline{r}, \underline{s}\,\right) = \frac{\lambda_1^{\delta_1}\lambda_2^{\delta_2}}{\Gamma(\delta_1)\Gamma(\delta_2)}\beta_1^{\delta_1-1}\beta_2^{\delta_2-1}e^{-\beta_1\lambda_1}e^{-\beta_2\lambda_2}, \tag{3.6}$$

where $\lambda_1 = b_1 + T_1(r_n; \alpha_0)$, $\lambda_2 = b_2 + T_2(s_m; \alpha_0)$, $\delta_1 = n + a_1$, $\delta_2 = m + a_2$. We can obtain the posterior pdf of R using the joint posterior density function and is given by

$$f_R(r) = \frac{\lambda_1^{\delta_1}\lambda_2^{\delta_2}}{B(\delta_1, \delta_2)}\frac{r^{\delta_1-1}(1-r)^{\delta_2-1}}{(r\lambda_1 + (1-r)\lambda_2)^{\delta_1+\delta_2}},\ 0 < r < 1. \tag{3.7}$$

The Bayes estimate of R, say $\widehat{R}_{BS}$, under the SE loss function is

$$\widehat{R}_{BS} = \int_0^1 r\ f_R(r)dr.$$

After making suitable transformations and simplifications by using formula 3.197(3) in Gradshteyn and Ryzhik [22], we get

$$\widehat{R}_{BS} = \begin{cases} \frac{\delta_1}{\delta_1+\delta_2}\left(\frac{\lambda_1}{\lambda_2}\right)^{\delta_1}\ {}_2F_1(\delta_1+\delta_2, \delta_1+1; \delta_1+\delta_2+1; 1-\frac{\lambda_1}{\lambda_2}) & \text{if } \lambda_1 < \lambda_2 \\ \frac{\delta_1}{\delta_1+\delta_2}\left(\frac{\lambda_2}{\lambda_1}\right)^{\delta_2}\ {}_2F_1(\delta_1+\delta_2, \delta_2; \delta_1+\delta_2+1; 1-\frac{\lambda_2}{\lambda_1}) & \text{if } \lambda_2 \leq \lambda_1 \end{cases}. \tag{3.8}$$

The Bayes estimator of R under the LINEX loss function, say $\widehat{R}_{BL}$, is

$$\widehat{R}_{BL} = -\frac{1}{v}\ln E_R(e^{-vR}),$$

where $E_R(.)$ denotes posterior expectation with respect to the posterior density of R. It can be easily obtained that

$$\begin{aligned} E(e^{-vR}) &= \int_0^1 e^{-vr}f_R(r)dr \\ &= \begin{cases} (\frac{\lambda_1}{\lambda_2})^{\delta_1}\ \Phi_1(\delta_1, \delta_1+\delta_2, \delta_1+\delta_2, 1-\frac{\lambda_1}{\lambda_2}, -v) & \text{if } \lambda_1 < \lambda_2 \\ (\frac{\lambda_2}{\lambda_1})^{\delta_2}\ e^{-v}\ \Phi_1(\delta_2, \delta_1+\delta_2, \delta_1+\delta_2, 1-\frac{\lambda_2}{\lambda_1}, v) & \text{if } \lambda_2 \leq \lambda_1 \end{cases}, \end{aligned}$$

where $\Phi_1(.,.,.,.,.)$ is confluent hypergeometric series of two variables, see formulas 3.385 and 9.261(1) in Gradshteyn and Ryzhik [22]. Therefore,

$$\widehat{R}_{BL} = \begin{cases} -\frac{1}{v}\left(c_1 + \ln\left[\Phi_1(\delta_1, \delta_1+\delta_2, \delta_1+\delta_2, 1-\frac{\lambda_1}{\lambda_2}, -v)\right]\right) & \text{if } \lambda_1 < \lambda_2 \\ -\frac{1}{v}\left(c_2 + \ln\left[\Phi_1(\delta_2, \delta_1+\delta_2, \delta_1+\delta_2, 1-\frac{\lambda_2}{\lambda_1}, v)\right]\right) & \text{if } \lambda_2 \leq \lambda_1 \end{cases}, \tag{3.9}$$

where $c_1 = \delta_1 \ln(\lambda_1/\lambda_2)$ and $c_2 = \delta_2 \ln(\lambda_2/\lambda_1) - v$.

If we use the Jeffrey's non informative prior, is given by $\sqrt{\det I}$, then the joint prior density function is $\pi(\beta_1, \beta_2) \propto 1/\beta_1\beta_2$. Therefore, the joint posterior density function of β_1 and β_2 is

$$\pi(\beta_1, \beta_2 \,|\alpha_0, \underline{r}, \underline{s}\,) = \frac{T_1^n {T_2}^m}{\Gamma(n)\Gamma(m)}\beta_1^{n-1}\beta_2^{m-1}e^{-\beta_1 T_1}e^{-\beta_2 T_2},$$

and the posterior pdf of R is given by

$$f_R(r) = \frac{T_1^n {T_2}^m}{B(n,m)}\frac{r^{n-1}(1-r)^{m-1}}{(rT_1 + (1-r)T_2)^{n+m}},\ 0 < r < 1,$$

where $T_1 = T_1(r_n;\alpha_0)$ and $T_2 = T_2(s_m;\alpha_0)$. The Bayes estimate of R under the SE and the LINEX loss function, say $\widehat{R}^*_{BS}$ and $\widehat{R}^*_{BL}$ respectively, are

$$(3.10)\qquad \widehat{R}^*_{BS} = \begin{cases} (\frac{T_1}{T_2})^n(\frac{n}{n+m})\ {}_2F_1(n+m, n+1; n+m+1; 1-\frac{T_1}{T_2}) & \text{if } T_1 < T_2 \\ (\frac{T_2}{T_1})^m(\frac{n}{n+m})\ {}_2F_1(n+m, m; n+m+1; 1-\frac{T_2}{T_1}) & \text{if } T_2 \le T_1 \end{cases},$$

and

$$(3.11)\qquad \widehat{R}^*_{BL} = \begin{cases} -\frac{1}{v}\left(c_3 + \ln\left[\Phi_1(n, n+m, n+m, 1-\frac{T_1}{T_2}, -v)\right]\right) & \text{if } T_1 < T_2 \\ -\frac{1}{v}\left(c_4 + \ln\left[\Phi_1(m, n+m, n+m, 1-\frac{T_2}{T_1}, v)\right]\right) & \text{if } T_2 \le T_1 \end{cases},$$

where $c_3 = n\ln(T_1/T_2)$ and $c_4 = m\ln(T_2/T_1) - v$

The Bayes estimates are not always derived in the closed forms. However, for our case the Bayes estimates are obtained in the closed form. These estimates can be obtained by using alternative methods such as Lindley's approximation and the MCMC method. The purpose of applying all these two methods is to see how good the approximate methods compared with the exact one. If these result are close, then it will be encouraging to use the approximate methods when the exact form can not be obtained as in the case of α unknown. These estimators will be compared in the simulation study section. Next, we give the Bayes estimates of R using Lindley's approximation and the MCMC method.

3.3.1. *Lindley's approximation.* The approximate Bayes estimate of R under the SE and the LINEX loss functions for the informative prior case, say $\widehat{R}_{BS,Lindley}$ and $\widehat{R}_{BL,Lindley}$ respectively, are

$$(3.12)\qquad \widehat{R}_{BS,Lindley} = \widetilde{R}\left(1 + \frac{(1-\widetilde{R})^2}{n+a_1-1} - \frac{\widetilde{R}(1-\widetilde{R})}{m+a_2-1}\right),$$

and

$$(3.13)\qquad \widehat{R}_{BL,Lindley} = \widetilde{R} - \frac{1}{v}\ln\left[1 + \frac{v\widetilde{R}(1-\widetilde{R})^2(v\widetilde{R}-2)}{2(n+a_1-1)} + \frac{v\widetilde{R}^2(1-\widetilde{R})(v-v\widetilde{R}+2)}{2(m+a_2-1)}\right],$$

where $\widetilde{R} = \frac{\widetilde{\beta}_1}{\widetilde{\beta}_1+\widetilde{\beta}_2}$, $\widetilde{\beta}_1 = \frac{n+a_1-1}{b_1+T_1(r_n;\alpha_0)}$ and $\widetilde{\beta}_2 = \frac{m+a_2-1}{b_2+T_2(s_m;\alpha_0)}$.

If we use the Jeffrey's non informative prior, the approximate Bayes estimate of R under the SE and the LINEX loss functions, say $\widehat{R}^*_{BS,Lindley}$ and $\widehat{R}^*_{BL,Lindley}$ respectively, are

$$(3.14)\qquad \widehat{R}^*_{BS,Lindley} = \widetilde{R}\left(1 + \frac{(1-\widetilde{R})^2}{n-1} - \frac{\widetilde{R}(1-\widetilde{R})}{m-1}\right),$$

and

$$(3.15)\qquad \widehat{R}^*_{BL,Lindley} = \widetilde{R} - \frac{1}{v}\ln\left[1 + \frac{v\widetilde{R}(1-\widetilde{R})^2(v\widetilde{R}-2)}{2(n-1)} + \frac{v\widetilde{R}^2(1-\widetilde{R})(v-v\widetilde{R}+2)}{2(m-1)}\right],$$

where $\widetilde{R} = \frac{\widetilde{b}_1}{\widetilde{b}_1+\widetilde{b}_2}$, $\widetilde{b}_1 = \frac{n-1}{T_1(r_n;\alpha_0)}$ and $\widetilde{b}_2 = \frac{m-1}{T_2(s_m;\alpha_0)}$.

3.3.2. *MCMC method.* It is clear from Equation (3.6) that the marginal posterior densities of β_1 and β_2 are gamma distribution with the parameters (δ_1, λ_1) and (δ_2, λ_2), respectively. We generate a samples by using Gibss sampling from these distributions. The following algorithm are used.

Step 1. Set $i = 1$.
Step 2. Generate $\beta_1^{(i)}$ from $Gamma(\delta_1, \lambda_1)$.
Step 3. Generate $\beta_2^{(i)}$ from $Gamma(\delta_2, \lambda_2)$.
Step 4. Compute the $R^{(i)} = \beta_1^{(i)}/(\beta_1^{(i)} + \beta_2^{(i)})$.

Step 5. Set $i = i + 1$.

Step 6. Repeat Steps 2-5, N times, and obtain the posterior sample $R^{(i)}$, $i = 1, ..., N$.

This sample is used to compute the Bayes estimate and to construct the HPD credible interval for R. The Bayes estimate of R under the SE and the LINEX loss functions are given as

$$\widehat{R}_{BS,MCMC} = \frac{1}{N}\sum_{i=1}^{N} R^{(i)},$$

$$\widehat{R}_{BL,MCMC} = -\frac{1}{v}\ln E(e^{-vR}) = -\frac{1}{v}\ln\left(\frac{1}{N}\sum_{i=1}^{N} e^{-vR^{(i)}}\right).$$

The HPD $100(1-\gamma)\%$ credible interval of R can be obtained by the method of Chen and Shao [18]. Its algorithm is given in Subsection 2.4.2.

3.4. Empirical Bayes estimation of R. We obtained the Bayes estimates of R using three different ways. It is clear that these estimates depend on the prior parameters. However, the Bayes estimates can be also obtained independently of the prior parameters.

These prior parameters could be estimated by means of an empirical Bayes procedure, see Lindley [29] and Awad and Gharraf [9]. Let $R_1, \ldots, R_n$ and $S_1, \ldots, S_m$ be two independent random samples from $Burr(\alpha_0, \beta_1)$ and $Burr(\alpha_0, \beta_2)$, respectively. For fixed $\underline{r}$, the function $L_1(\beta_1 | \alpha_0, \underline{r})$ of β_1 can be considered as a gamma density with parameters $(n+1, T_1(r_n; \alpha_0))$. Therefore, it is proposed to estimate the prior parameters α_1 and β_1 from the samples as $n+1$ and $T_1(r_n; \alpha_0)$, respectively. Similarly, α_2 and β_2 could be estimated from the samples as $m+1$ and $T_2(s_m; \alpha_0)$, respectively. Hence, the empirical Bayes estimate of R with respect to SE and LINEX loss functions, say $\widehat{R}_{EBS}$ and $\widehat{R}_{EBL}$, respectively, could be given as

$$\widehat{R}_{EBS} = \begin{cases} c_6 c_7 \; {}_2F_1(2n+2m+2, 2n+2; 2n+2m+3; c_9) & \text{if } T_1 < T_2 \\ c_6 c_8 \; {}_2F_1(2n+2m+2, 2m+1; 2n+2m+3; c_{10}) & \text{if } T_2 \le T_1 \end{cases}, \tag{3.16}$$

and

$$\widehat{R}_{EBL} = \begin{cases} -\frac{1}{v}\Big((2n+1)\ln(\frac{T_1}{T_2}) + \ln c_{11}\Big) & \text{if } T_1 < T_2 \\ -\frac{1}{v}\Big((2m+1)\ln(\frac{T_2}{T_1}) - v + \ln c_{12}\Big) & \text{if } T_2 \le T_1 \end{cases}. \tag{3.17}$$

where $c_6 = (2n+1)/(2n+2m+2)$, $c_7 = (T_1/T_2)^{2n+1}$, $c_8 = (T_2/T_1)^{2m+1}$, $c_9 = 1-(T_1/T_2)$, $c_{10} = 1-(T_2/T_1)$, $c_{11} = \Phi_1(2n+1, 2n+2m+2, 2n+2m+2, c_9, -v)$ and $c_{12} = \Phi_1(2m+1, 2n+2m+2, 2n+2m+2, c_{10}, v)$.

3.5. Bayesian credible intervals for R. We know that $\beta_1 | \alpha_0, \underline{r} \sim Gamma(\delta_1, \lambda_1)$ and $\beta_2 | \alpha_0, \underline{s} \sim Gamma(\delta_2, \lambda_2)$. Then, $2\lambda_1\beta_1 | \alpha_0, \underline{r} \sim \chi^2(2(n+a_1))$ and $2\lambda_2\beta_2 | \alpha_0, \underline{s} \sim \chi^2(2(m+a_2))$. Therefore,

$$W = \frac{2\lambda_2\beta_2 | \alpha_0, \underline{s} \; /2(m+a_2)}{2\lambda_1\beta_1 | \alpha_0, \underline{r} \; /2(n+a_1)}$$

is an F distributed random variable with $(2(m+a_2), 2(n+a_1))$ degrees of freedom and the $100(1-\gamma)\%$ Bayesian credible interval for R can be obtained as

$$\left(\frac{1}{1 + CF_{2(m+a_2),2(n+a_1);\frac{\gamma}{2}}}, \frac{1}{1 + CF_{2(m+a_2),2(n+a_1);1-\frac{\gamma}{2}}}\right) \tag{3.18}$$

where $C = \frac{(m+a_2)(b_1+T_1(r_n;\alpha_0))}{(n+a_1)(b_2+T_2(s_m;\alpha_0))}$, $F_{2(m+a_2),2(n+a_1);\frac{\gamma}{2}}$ and $F_{2(m+a_2),2(n+a_1);1-\frac{\gamma}{2}}$ are the lower and upper $\frac{\gamma}{2}$th percentile points of a F distribution with $(2(m+a_2), 2(n+a_1))$ degrees of freedom.

Moreover, this interval can be obtained independently of these parameters by using the empirical method given in Subsection 3.4. In this case, the posterior distributions of β_1 and β_2 have gamma distributions with parameters $(2n+1, 2T_1(r_n;\alpha_0))$ and $(2m+1, 2T_2(s_m;\alpha_0))$, respectively and the $100(1-\gamma)\%$ Bayesian credible interval for R can be obtained as

$$\left(\frac{1}{1+C_1F_{(4m+2),(4n+2);\frac{\gamma}{2}}}, \frac{1}{1+C_1F_{(4m+2),(4n+2);1-\frac{\gamma}{2}}}\right) \tag{3.19}$$

where $C_1 = \frac{(4m+2)T_1(r_n;\alpha_0)}{(4n+2)T_2(s_m;\alpha_0)}$, $F_{(4m+2),(4n+2);\frac{\gamma}{2}}$ and $F_{(4m+2),(4n+2);1-\frac{\gamma}{2}}$ are the lower and upper $\frac{\gamma}{2}$th percentile points of a F distribution with $(4m+2, 4n+2)$ degrees of freedom.

4. Numerical experiments

In this section, firstly the Monte Carlo simulations for the comparison of the derived estimates are presented, then two real life data sets are analysed.

4.1. Simulation study. In this subsection, we present some numerical results to compare the performance of the different estimates for different sample sizes and different priors. The performances of the point estimates are compared by using estimated risks (ERs). The performances of the confidence and credible intervals are compared by using average interval lengths and coverage probabilities (cps). The ER of θ, when θ is estimated by $\widehat{\theta}$, is given by

$$ER(\theta) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta}_i - \theta_i\right)^2,$$

under the SE loss function. Moreover, the ER of θ under the LINEX loss function is given by

$$ER(\theta) = \frac{1}{N}\sum_{i=1}^{N}\left(e^{v\left(\widehat{\theta}_i-\theta_i\right)} - v\left(\widehat{\theta}_i - \theta_i\right) - 1\right),$$

where N is the number of replications. All of the computations are performed by using MATLAB R2010a. All the results are based on 3000 replications.

We consider two cases separately to draw inference on R, namely when the common first shape parameter α is unknown and known. In both cases we generate the upper record values with the sample sizes; $(n,m) = (5,5)$, $(8,8)$, $(10,10)$, $(12,12)$, $(15,15)$ from the Burr Type XII distribution and different values of n and m, given in Table 1, are considered.

In Table 1, the ML and Bayes estimates of R and their corresponding ERs are listed when α is unknown. The Bayes estimates are computed by using Lindley's approximation and MCMC method under the SE and the LINEX ($v = -1$ and 1) loss functions for different prior parameters. In the Bayesian case, Prior 1: $(a_1,b_1) = (4,2)$, $(a_2,b_2) = (4,2)$, $(a_3,b_3) = (3,3)$, Prior 2: $(a_1,b_1) = (5,1)$, $(a_2,b_2) = (3,3/2)$, $(a_3,b_3) = (3,3/2)$ and Prior 3: $(a_1,b_1) = (5,1/2)$, $(a_2,b_2) = (3,3)$, $(a_3,b_3) = (3,3/2)$, are used for $R = 0.5006$, 0.7145 and 0.9095, respectively. Moreover, the 95% asymptotic confidence intervals, which are computed based on Fisher information and observation matrices, and HPD credible intervals with their cps are listed. From Table 1, the ERs of all estimates decrease as the sample sizes increase in all cases, as expected. The Bayes estimates under the SE and LINEX loss functions generally have smaller ER than that of ML estimates. Moreover, the ERs of the Bayes estimates based on Lindley's approximation are generally smaller than that of MCMC method. These estimates are close to each other as the sample sizes increase. The average lengths of the intervals decrease as the sample sizes increase. The asymptotic confidence intervals based on Fisher information and observation matrices are very similar, as expected. The average lengths of the HPD Bayesian credible intervals are smaller than that of the asymptotic confidence intervals.

In the MCMC case, we run three MCMC chains with fairly different initial values and generated 10000 iterations for each chain. To diminish the effect of the starting distribution, we generally discard the first half of each sequence and focus on the second half. To provide relatively independent samples for improvement of prediction accuracy, we calculate the Bayesian MCMC estimates by the means of every 5^{th} sampled values after discarding the first half of the chains (see Gelman et al. [20]). The scale reduction factor estimate $\sqrt{\widehat{R}} = \sqrt{Var(\psi)/W}$ is used to monitor convergence of MCMC simulations where ψ is the estimand of interest, $Var(\psi) = (n-1)W/n + B/n$ with the iteration number n for each chain, the between- and within- sequence variances B and W (see Gelman et al. [20]). In our case, the scale factor value of the MCMC estimates are found below 1.1 which is an acceptable value for their convergency.

In Table 2 and 3, the ML, UMVU and Bayesian estimates of R and their corresponding ERs are listed when α is known ($\alpha = 3$). In this case, the Bayes estimates are evaluated analytically under the SE and the LINEX ($v = -1$ and 1) loss functions for different prior parameters. Moreover, it is also computed by using Lindley's approximation and MCMC method. In the Bayesian case, Prior 4: $(a_1, b_1) = (6, 5/2)$, $(a_2, b_2) = (4, 2)$, Prior 5: $(a_1, b_1) = (12, 2)$, $(a_2, b_2) = (3, 3/2)$ and Prior 6: $(a_1, b_1) = (15, 5/4)$, $(a_2, b_2) = (2, 2)$ are used for $R = 0.5484$, 0.7506 and 0.9165, respectively. In addition, the empirical Bayes estimates are obtained. All point estimates of R are listed in Table 2. The exact and asymptotic confidence intervals are computed from Equations (3.2) and (3.4). The Bayesian and empirical Bayesian credible intervals are computed from Equations (3.18) and (3.19). The HPD credible interval is constructed by using the MCMC samples. All interval estimates of R are listed in Table 3.

From Table 2, the ERs of all estimates decrease as the sample sizes increase in all cases, as expected. The Bayes estimates with their corresponding ERs based on Lindley's approximation and MCMC method are very close to the exact values. The ERs of the ML, UMVU, Bayes and empiric Bayes (under the SE loss function) estimates are ordered as $ER(\widehat{R}_{BS}) < ER(\widehat{R}_{EBS}) < ER(\widehat{R}_{MLE}) < ER(\widehat{R}_U)$ when $R = 0.5484$, 0.7506 and $ER(\widehat{R}_{BS}) < ER(\widehat{R}_U) < ER(\widehat{R}_{MLE}) < ER(\widehat{R}_{EBS})$ when $R = 0.9165$. Moreover, the ERs of the Bayes estimates under the LINEX loss function have smaller than that of ML estimates. From Table 3, the average lengths of the intervals decrease as the sample sizes increase. The average lengths of the empirical Bayesian credible intervals are smallest, but their cps are not preferable. The HPD Bayesian credible intervals are more suitable than others in terms of the average lengths and cps.

In Table 4, the ML, UMVU and Bayesian estimates of R and their corresponding ERs are listed when α is known ($\alpha = 3$). In this case, the Bayes estimates are evaluated analytically and by using Lindley's approximation under the SE and the LINEX ($v = -1$ and 1) loss functions for the non informative prior. Moreover, the exact and asymptotic confidence intervals are computed from Equations (3.2) and (3.4). The point and interval estimates are computed for $R = 0.25$, 0.33, 0.50, $0, 70$, 0.90 and 0.92 when $(\beta_1, \beta_2) = (2, 6)$, $(2, 4)$, $(2, 2)$, $(7, 3)$, $(18, 2)$ and $(23, 2)$, respectively. From Table 4, the ERs of all estimates decrease as the sample sizes increase in all cases, as expected. The Bayes estimates under the SE loss function with their corresponding ERs are close to their response in the ML case. Moreover, the Bayes estimates with their corresponding ERs based on Lindley's approximation are very close the exact values. The ERs of the ML, UMVU and Bayes (under the SE loss function) estimates are ordered as $ER(\widehat{R}^*_{BS}) < ER(\widehat{R}_{MLE}) < ER(\widehat{R}_U)$ when $R = 0.25$, 0.33, 0.50, 0.70 and $ER(\widehat{R}_U) < ER(\widehat{R}_{MLE}) <$

$ER(\widehat{R}^*_{BS})$ when $R = 0.90$, 0.92. The ERs of ML and Bayes estimates have larger values when the true value of R is around 0.5 and it decreases as the true value of R approaches the extremes. Furthermore, the average lengths of the intervals decrease as the sample sizes increase. When $R = 0.25$, 0.90 and 0.92 the lengths of the asymptotic confidence intervals are smaller than that of exact confidence intervals, but for $R = 0.33$, 0.50 and 0,70 it is other way around.

On the other hand, to compare the performance of the different estimates of R, the graphs of MSEs and Biases are obtained for different n and m when α is unknown and known cases. When α is unknown, the graphs are plotted based on the MLE and Lindley methods in Figure 1. When α is known, the graphs are plotted based on the MLE, UMVUE and Lindley methods in Figure 2. For each choices of $(\beta_1, \beta_2, \alpha)$ or (β_1, β_2), we use the following procedure for the comparison of the estimates.

Step 1: For given $(\beta_1, \beta_2, \alpha)$ or (β_1, β_2), we compute R.

Step 2: For given different n and m, we generate a sample from the Burr Type XII distributions for the strength and the stress variables.

Step 3: The different estimates of R are computed.

Step 4: Steps 2-3 are repeated 3000 times, the MSEs and Biases are calculated and are given by $MSE(R_{s,k}) = \sum_{i=1}^{N}(\widehat{R}^{(i)} - R)^2/N$ and $Bias(R_{s,k}) = \sum_{i=1}^{N}(\widehat{R}^{(i)} - R)/N$.

From the Figures 1 and 2, it is observed that the MSEs and Biases of the estimates decrease when the sample size increases, as expected. The MSEs of the Bayes estimates under the SE and LINEX ($v = -1, 1$) loss functions are smaller than that of other estimates. Moreover, the MSE is small for the extreme values of R, but it is large when R is around 0.5 for all types of estimates. When R is around 0.5, the MSEs of UMVUE are greater than that of MLE and when R is around extreme values, the MSEs of UMVUE are smaller than that of MLE in Figure 2. Notice that the similar outcomes are observed in all Tables.

4.2. Real life examples. In this subsection, we consider the two real life data sets to illustrate the use of the methods proposed in this paper.

4.2.1. *Lifetime data for insulation specimens.* Nelson described the results of a life test experiment in which specimens of a type of electrical insulating fluid were subjected to a constant voltage stress in [37]. The length of time until each specimen failed, or "broke down," was observed. The results for seven groups of specimens, tested at voltages ranging from 26 to 38 kilovolts (kV) were presented. The data sets for 36kV and 38 kV, reported in Lawless [27], are considered and corresponding upper record values are given in Table 5. We fit the Burr Type XII distribution to the two data sets. The Kolmogorov-Smirnov (K-S) distances between the fitted and the empirical distribution functions and corresponding p-values, the parameters and the reliability (R) estimates are computed. All these results are presented in Table 6. From Table 6, we observe the Burr Type XII distribution provides an adequate fit for both of the data sets.

4.2.2. *Lifetime data for steel specimens.* Crowder gave the lifetimes of steel specimens tested at 14 different stress levels in [19]. The data sets for 38.5 and 36 stress levels are considered and corresponding upper record values are given in Table 7. Since all record values are greater than unity, we encounter the problem for the uniqueness of the ML estimates of the parameters. To overcome this situation, these data sets are divided by the corresponding maximum values. Then, we compute the K-S distances between the fitted and the empirical distribution functions based on the Burr Type XII distribution. Moreover, for this example it is recommended to compare the Burr Type XII distribution with common distributions such as Weibull and two-parameter bathtub-shaped based on

Table 1. Estimates of R when α is unknown and the true values of $R = 0.5006, 0.7145$ and 0.9095 by using the Priors 1-3.

			Bayes under the SE		Bayes estimates under the LINEX $v=-1$		Bayes estimates under the LINEX $v=1$		Asymptotic C.I. based on Fisher	Asymptotic C.I. based on Observ.	HPD Credible Interval
(n,m)	R	$\widehat{R}_{MLE}$	$\widehat{R}_{BS,Lindley}$	$\widehat{R}_{BS,MCMC}$	$\widehat{R}_{BL,Lindley}$	$\widehat{R}_{BL,MCMC}$	$\widehat{R}_{BL,Lindley}$	$\widehat{R}_{BL,MCMC}$			
(5,5)	0.5006	0.5016	0.4976	0.5005	0.4947	0.5069	0.5018	0.4941	(0.2224,0.7808)	(0.2221,0.7819)	(0.2852,0.7157)
		0.0263	0.0053	0.0070	0.0136	0.0035	0.0046	0.0035	0.5584/0.8640	0.5598/0.8653	0.4305/0.9917
(5,10)		0.5157	0.4817	0.4999	0.4825	0.5053	0.4794	0.4945	(0.2650,0.7663)	(0.2647,0.7665)	(0.3000,0.6965)
		0.0190	0.0039	0.0063	0.0026	0.0032	0.0019	0.0032	0.5013/0.8879	0.5018/0.8882	0.3965/0.9840
(8,5)		0.4879	0.5173	0.5004	0.5197	0.5060	0.5171	0.4947	(0.2292,0.7468)	(0.2290,0.7471)	(0.2985,0.7047)
		0.0205	0.0036	0.0065	0.0017	0.0032	0.0028	0.0033	0.5176/0.8925	0.5181/0.8929	0.4062/0.9843
(8,8)		0.5002	0.5007	0.5002	0.5053	0.5050	0.4961	0.4954	(0.2700,0.7304)	(0.2700,0.7304)	(0.3127,0.6877)
		0.0155	0.0039	0.0065	0.0020	0.0033	0.0020	0.0033	0.4604/0.8980	0.4604/0.8977	0.3750/0.9743
(10,10)		0.5035	0.5018	0.5024	0.5064	0.5065	0.4972	0.4982	(0.2940,0.7129)	(0.2940,0.7129)	(0.3276,0.6773)
		0.0112	0.0037	0.0054	0.0019	0.0027	0.0019	0.0027	0.4189/0.9233	0.4189/0.9230	0.3498/0.9780
(10,15)		0.4913	0.4875	0.4897	0.4915	0.4934	0.4834	0.4861	(0.2982,0.6844)	(0.2982,0.6844)	(0.3250,0.6532)
		0.0088	0.0037	0.0047	0.0018	0.0023	0.0019	0.0023	0.3862/0.9383	0.3862/0.9383	0.3281/0.9800
(12,8)		0.4982	0.5070	0.5031	0.5115	0.5074	0.5025	0.4988	(0.2844,0.7120)	(0.2844,0.7120)	(0.3266,0.6808)
		0.0115	0.0036	0.0053	0.0019	0.0026	0.0018	0.0026	0.4276/0.9253	0.4276/0.9253	0.3543/0.9753
(12,12)		0.4990	0.4996	0.4993	0.5038	0.5029	0.4954	0.4956	(0.3060,0.6920)	(0.3060,0.6920)	(0.3347,0.6638)
		0.0089	0.0036	0.0047	0.0018	0.0023	0.0018	0.0023	0.3860/0.9390	0.3860/0.9390	0.3291/0.9820
(15,5)		0.4986	0.5344	0.5137	0.5358	0.5187	0.5360	0.5088	(0.2584,0.7389)	(0.2582,0.7392)	(0.3252,0.7061)
		0.0160	0.0050	0.0056	0.0023	0.0028	0.0089	0.0028	0.4803/0.9100	0.4808/0.9103	0.3809/0.9387
(15,15)		0.5011	0.5007	0.5008	0.5044	0.5039	0.4971	0.4977	(0.3261,0.6761)	(0.3261,0.6761)	(0.3484,0.6529)
		0.0055	0.0025	0.0032	0.0013	0.0016	0.0013	0.0016	0.3500/0.9647	0.3500/0.9647	0.3045/0.9890
(5,5)	0.7145	0.7116	0.6738	0.7051	0.6598	0.7099	0.6813	0.7001	(0.4753,0.9415)	(0.4728,0.9527)	(0.5125,0.8805)
		0.0203	0.0085	0.0053	0.0193	0.0026	0.0033	0.0027	0.4662/0.8367	0.4799/0.8410	0.3680/0.9920
(5,12)		0.7327	0.6611	0.7077	0.6520	0.7117	0.6645	0.7036	(0.5253,0.9362)	(0.5219,0.9379)	(0.5298,0.8674)
		0.0135	0.0074	0.0041	0.0129	0.0020	0.0026	0.0021	0.4109/0.8572	0.4160/0.8626	0.3375/0.9929
(8,8)		0.7085	0.7013	0.7054	0.7042	0.7090	0.6983	0.7018	(0.5143,0.9027)	(0.5139,0.9032)	(0.5401,0.8584)
		0.0120	0.0029	0.0050	0.0014	0.0025	0.0015	0.0025	0.3884/0.8963	0.3893/0.8990	0.3182/0.9750
(8,15)		0.7182	0.6990	0.7073	0.7015	0.7102	0.6963	0.7042	(0.5465,0.8898)	(0.5462,0.8901)	(0.5554,0.8468)
		0.0085	0.0031	0.0040	0.0015	0.0020	0.0016	0.0020	0.3433/0.9010	0.3439/0.9017	0.2914/0.9793
(10,8)		0.7056	0.7065	0.7059	0.7098	0.7091	0.7032	0.7025	(0.5204,0.8909)	(0.5202,0.8910)	(0.5476,0.8535)
		0.0102	0.0028	0.0046	0.0014	0.0023	0.0015	0.0023	0.3705/0.9093	0.3707/0.9097	0.3059/0.9690
(10,10)		0.7096	0.7050	0.7067	0.7083	0.7097	0.7018	0.7036	(0.5346,0.8845)	(0.5346,0.8846)	(0.5546,0.8483)
		0.0087	0.0031	0.0043	0.0015	0.0021	0.0016	0.0022	0.3499/0.9073	0.3500/0.9077	0.2937/0.9717
(12,12)		0.7087	0.7059	0.7064	0.7089	0.7090	0.7030	0.7037	(0.5481,0.8692)	(0.5481,0.8692)	(0.5643,0.8395)
		0.0071	0.0031	0.0039	0.0015	0.0019	0.0016	0.0020	0.3212/0.9267	0.3211/0.9267	0.2752/0.9713
(12,15)		0.7102	0.7050	0.7063	0.7076	0.7087	0.7023	0.7038	(0.5579,0.8626)	(0.5579,0.8626)	(0.5695,0.8342)
		0.0067	0.0033	0.0039	0.0016	0.0019	0.0017	0.0020	0.3047/0.9130	0.3046/0.9130	0.2647/0.9627
(15,10)		0.7071	0.7096	0.7081	0.7125	0.7107	0.7066	0.7055	(0.5469,0.8673)	(0.5469,0.8673)	(0.5674,0.8410)
		0.0074	0.0032	0.0040	0.0016	0.0020	0.0016	0.0020	0.3204/0.9130	0.3204/0.9130	0.2736/0.9583
(15,15)		0.7092	0.7066	0.7069	0.7091	0.7091	0.7041	0.7047	(0.5651,0.8532)	(0.5652,0.8532)	(0.5766,0.8294)
		0.0056	0.0030	0.0035	0.0015	0.0017	0.0015	0.0018	0.2881/0.9247	0.2881/0.9247	0.2527/0.9673

Table 1 continued

					Bayes estimates under the LINEX						
			Bayes under the SE		$v=-1$		$v=1$		Asymptotic C.I.	Asymptotic C.I.	HPD Credible
(n,m)	R	$\widehat{R}_{MLE}$	$\widehat{R}_{BS,Lindley}$	$\widehat{R}_{BS,MCMC}$	$\widehat{R}_{BL,Lindley}$	$\widehat{R}_{BL,MCMC}$	$\widehat{R}_{BL,Lindley}$	$\widehat{R}_{BL,MCMC}$	based on Fisher	based on Observ.	Interval
(5,5)	0.9095	0.9112	0.8523	0.9001	0.8347	0.9011	0.8615	0.8990	(0.7964,0.9988)	(0.7918,0.9989)	(0.8131,0.9563)
		0.0043	0.0147	0.0010	0.0504	0.0005	0.0052	0.0005	0.2024/0.7897	0.2070/0.8033	0.1432/0.9705
(5,8)		0.9144	0.8483	0.9002	0.8370	0.9013	0.8551	0.8991	(0.8106,0.9989)	(0.8055,0.9991)	(0.8080,0.9723)
		0.0037	0.0112	0.0009	0.0118	0.0004	0.0037	0.0005	0.1868/0.7838	0.1919/0.7990	0.1643/0.9993
(8,8)		0.9089	0.8965	0.9032	0.8964	0.9040	0.8965	0.9024	(0.8216,0.9935)	(0.8203,0.9939)	(0.8249,0.9673)
		0.0026	0.0008	0.0009	0.0004	0.0005	0.0004	0.0005	0.1719/0.8363	0.1737/0.8433	0.1424/0.9873
(8,12)		0.9096	0.8919	0.9017	0.8920	0.9024	0.8917	0.9009	(0.8283,0.9895)	(0.8269,0.9902)	(0.8259,0.9640)
		0.0021	0.0009	0.0009	0.0005	0.0004	0.0004	0.0004	0.1613/0.8570	0.1633/0.8637	0.1381/0.9947
(10,10)		0.9055	0.8990	0.9018	0.8995	0.9025	0.8985	0.9011	(0.8266,0.9843)	(0.8260,0.9848)	(0.8301,0.9621)
		0.0019	0.0006	0.0009	0.0003	0.0004	0.0003	0.0005	0.1577/0.8900	0.1588/0.8927	0.1320/0.9853
(10,15)		0.9033	0.8941	0.8984	0.8947	0.8990	0.8935	0.8977	(0.8283,0.9783)	(0.8277,0.9788)	(0.8284,0.9574)
		0.0015	0.0007	0.0008	0.0003	0.0004	0.0004	0.0004	0.1500/0.9263	0.1511/0.9293	0.1291/0.9897
(12,10)		0.9034	0.9012	0.9016	0.9017	0.9022	0.9007	0.9009	(0.8283,0.9785)	(0.8280,0.9787)	(0.8337,0.9597)
		0.0018	0.0006	0.0009	0.0003	0.0004	0.0003	0.0005	0.1502/0.9083	0.1507/0.9090	0.1260/0.9770
(12,12)		0.9049	0.9001	0.9017	0.9007	0.9023	0.8995	0.9012	(0.8335,0.9762)	(0.8333,0.9764)	(0.8357,0.9581)
		0.0014	0.0006	0.0008	0.0003	0.0004	0.0003	0.0004	0.1427/0.9153	0.1431/0.9167	0.1224/0.9863
(15,8)		0.9013	0.9047	0.9021	0.9052	0.9027	0.9044	0.9015	(0.8251,0.9776)	(0.8250,0.9776)	(0.8355,0.9601)
		0.0019	0.0006	0.0009	0.0003	0.0004	0.0003	0.0005	0.1525/0.9147	0.1526/0.9147	0.1246/0.9750
(15,15)		0.9000	0.8975	0.8983	0.8980	0.8988	0.8970	0.8978	(0.8344,0.9656)	(0.8343,0.9657)	(0.8371,0.9517)
		0.0011	0.0006	0.0007	0.0003	0.0004	0.0003	0.0004	0.1312/0.9563	0.1313/0.9563	0.1146/0.9847

Notes: The first row represents the average estimates and the second row represents corresponding ERs for each choice of (n,m). But, for the last three columns, the first row represents 95% confidence interval and the second row represents their lengths and cp's.

Table 2. Estimates of R when α is known ($\alpha = 3$) and the true values of $R = 0.5484, 0.7506$ and 0.9165 by using the Priors 4-6.

				Bayes estimates under the SE				Bayes estimates under the LINEX $v=-1$				Bayes estimates under the LINEX $v=1$			
(n,m)	R	$\widehat{R}_{MLE}$	$\widehat{R}_U$	$\widehat{R}_{BS}$	$\widehat{R}_{BS,Lindley}$	$\widehat{R}_{BS,MCMC}$	$\widehat{R}_{EBS}$	$\widehat{R}_{BL}$	$\widehat{R}_{BL,Lindley}$	$\widehat{R}_{BL,MCMC}$	$\widehat{R}_{EBL}$	$\widehat{R}_{BL}$	$\widehat{R}_{BL,Lindley}$	$\widehat{R}_{BL,MCMC}$	$\widehat{R}_{EBL}$
(5,5)	0.5484	0.5475	0.5511	0.5508	0.5503	0.5508	0.5461	0.5562	0.5565	0.5561	0.5502	0.5453	0.5442	0.5453	0.5421
		0.0191	0.0222	0.0108	0.0108	0.0108	0.0181	0.0054	0.0054	0.0054	0.0091	0.0054	0.0054	0.0054	0.0091
(8,8)		0.5477	0.5499	0.5502	0.5500	0.5502	0.5468	0.5543	0.5545	0.5543	0.5495	0.5461	0.5455	0.5461	0.5440
		0.0127	0.0139	0.0085	0.0085	0.0085	0.0122	0.0043	0.0043	0.0042	0.0061	0.0043	0.0043	0.0043	0.0061
(10,10)		0.5427	0.5442	0.5462	0.5461	0.5462	0.5420	0.5497	0.5499	0.5497	0.5443	0.5427	0.5422	0.5427	0.5397
		0.0100	0.0107	0.0072	0.0072	0.0072	0.0097	0.0036	0.0036	0.0036	0.0048	0.0036	0.0036	0.0036	0.0048
(12,12)		0.5446	0.5460	0.5472	0.5471	0.5472	0.5439	0.5503	0.5504	0.5503	0.5459	0.5441	0.5438	0.5441	0.5420
		0.0085	0.0091	0.0064	0.0064	0.0064	0.0083	0.0032	0.0032	0.0032	0.0042	0.0032	0.0032	0.0032	0.0042
(15,15)		0.5433	0.5444	0.5456	0.5455	0.5456	0.5428	0.5483	0.5484	0.5483	0.5444	0.5430	0.5427	0.5430	0.5412
		0.0069	0.0072	0.0056	0.0056	0.0056	0.0067	0.0028	0.0028	0.0028	0.0034	0.0028	0.0028	0.0028	0.0034
(5,5)	0.7506	0.7347	0.7496	0.7513	0.7502	0.7513	0.7291	0.7543	0.7534	0.7543	0.7321	0.7482	0.7471	0.7482	0.7261
		0.0140	0.0153	0.0062	0.0062	0.0062	0.0137	0.0031	0.0031	0.0031	0.0070	0.0031	0.0031	0.0031	0.0068
(8,8)		0.7426	0.7520	0.7522	0.7516	0.7522	0.7387	0.7545	0.7540	0.7545	0.7406	0.7498	0.7492	0.7498	0.7368
		0.0086	0.0090	0.0047	0.0047	0.0047	0.0085	0.0023	0.0023	0.0023	0.0043	0.0024	0.0023	0.0024	0.0042
(10,10)		0.7417	0.7493	0.7499	0.7494	0.7498	0.7384	0.7519	0.7516	0.7519	0.7400	0.7478	0.7474	0.7478	0.7368
		0.0067	0.0069	0.0041	0.0041	0.0041	0.0066	0.0020	0.0020	0.0020	0.0033	0.0021	0.0021	0.0021	0.0033
(12,12)		0.7404	0.7467	0.7484	0.7481	0.7484	0.7377	0.7502	0.7499	0.7502	0.7390	0.7465	0.7462	0.7465	0.7363
		0.0062	0.0064	0.0038	0.0038	0.0038	0.0062	0.0019	0.0019	0.0019	0.0031	0.0019	0.0019	0.0019	0.0031
(15,15)		0.7463	0.7514	0.7511	0.7509	0.7511	0.7440	0.7527	0.7525	0.7527	0.7450	0.7496	0.7494	0.7496	0.7429
		0.0045	0.0046	0.0032	0.0032	0.0032	0.0045	0.0016	0.0016	0.0016	0.0022	0.0016	0.0016	0.0016	0.0022
(5,5)	0.9165	0.9029	0.9155	0.9165	0.9162	0.9166	0.8977	0.9172	0.9167	0.9172	0.8986	0.9159	0.9156	0.9159	0.8967
		0.0042	0.0037	0.0012	0.0012	0.0012	0.0044	0.0006	0.0006	0.0006	0.0023	0.0006	0.0006	0.0006	0.0022
(8,8)		0.9080	0.9157	0.9156	0.9154	0.9156	0.9042	0.9161	0.9158	0.9161	0.9051	0.9151	0.9150	0.9151	0.9040
		0.0023	0.0021	0.0010	0.0010	0.0010	0.0024	0.0005	0.0005	0.0005	0.0012	0.0005	0.0005	0.0005	0.0012
(10,10)		0.9085	0.9146	0.9147	0.9145	0.9147	0.9056	0.9151	0.9149	0.9151	0.9060	0.9142	0.9141	0.9142	0.9052
		0.0017	0.0016	0.0008	0.0008	0.0008	0.0018	0.0004	0.0004	0.0004	0.0009	0.0004	0.0004	0.0004	0.0009
(12,12)		0.9077	0.9129	0.9132	0.9131	0.9132	0.9052	0.9136	0.9135	0.9136	0.9056	0.9129	0.9128	0.9128	0.9049
		0.0015	0.0014	0.0008	0.0008	0.0008	0.0016	0.0004	0.0004	0.0004	0.0008	0.0004	0.0004	0.0004	0.0008
(15,15)		0.9079	0.9121	0.9119	0.9118	0.9119	0.9059	0.9122	0.9121	0.9122	0.9062	0.9115	0.9115	0.9115	0.9057
		0.0012	0.0011	0.0007	0.0007	0.0007	0.0012	0.0004	0.0004	0.0004	0.0006	0.0004	0.0004	0.0004	0.0006

Note: The first row represents the average estimates and the second row represents corresponding ERs for each choice of (n,m).

Table 3. Confidence intervals of R when α is known ($\alpha = 3$) and the true values of $R = 0.5484, 0.7506$ and 0.9165.

(n, m)	R	Exact C.I.	Asymptotic C.I.	Bayesian Credible I.	HPD Bayes C.I.	Empirical Bayes C.I.
(5,5)	0.5484	(0.2855,0.7877)	(0.2895,0.8055)	(0.3461,0.7476)	(0.3508,0.7483)	(0.3704,0.7145)
		0.5023/0.9500	0.5160/0.8940	0.4014/0.9473	0.3975/0.9450	0.3441/0.8173
(8,8)		(0.3351,0.7456)	(0.3386,0.7568)	(0.3722,0.7220)	(0.3761,0.7227)	(0.4019,0.6864)
		0.4104/0.9427	0.4182/0.9023	0.3498/0.9440	0.3466/0.9407	0.2845/0.8067
(10,10)		(0.3507,0.7240)	(0.3531,0.7323)	(0.3808,0.7066)	(0.3840,0.7068)	(0.4102,0.6700)
		0.3732/0.9527	0.3792/0.9240	0.3257/0.9497	0.3228/0.9463	0.2598/0.8203
(12,12)		(0.3683,0.7110)	(0.3709,0.7182)	(0.3922,0.6973)	(0.3953,0.6979)	(0.4226,0.6617)
		0.3427/0.9463	0.3473/0.9223	0.3051/0.9457	0.3026/0.9470	0.2391/0.8207
(15,15)		(0.3839,0.6948)	(0.3861,0.7005)	(0.4026,0.6845)	(0.4050,0.6847)	(0.4326,0.6502)
		0.3109/0.9460	0.3144/0.9303	0.2819/0.9440	0.2797/0.9437	0.2175/0.8207
(5,5)	0.7506	(0.4760,0.8997)	(0.5253,0.9441)	(0.5877,0.8828)	(0.6015,0.8910)	(0.5712,0.8554)
		0.4237/0.9497	0.4189/0.8860	0.2951/0.9460	0.2895/0.9497	0.2842/0.8043
(8,8)		(0.5429,0.8789)	(0.5762,0.9090)	(0.6105,0.8681)	(0.6215,0.8748)	(0.6129,0.8424)
		0.3360/0.9463	0.3328/0.9070	0.2576/0.9470	0.2533/0.9493	0.2295/0.8180
(10,10)		(0.5632,0.8678)	(0.5905,0.8929)	(0.6177,0.8593)	(0.6272,0.8652)	(0.6244,0.8339)
		0.3046/0.9533	0.3024/0.9190	0.2416/0.9483	0.2380/0.9417	0.2095/0.8160
(12,12)		(0.5791,0.8579)	(0.6020,0.8789)	(0.6243,0.8521)	(0.6329,0.8574)	(0.6336,0.8261)
		0.2787/0.9423	0.2769/0.9200	0.2278/0.9390	0.2246/0.9377	0.1925/0.8077
(15,15)		(0.6035,0.8524)	(0.6226,0.8700)	(0.6373,0.8470)	(0.6448,.08518)	(0.6511,0.8237)
		0.2489/0.9470	0.2474/0.9333	0.2098/0.9433	0.2070/0.9417	0.1726/0.8297
(5,5)	0.9165	(0.7433,0.9697)	(0.8033,0.9952)	(0.8394,0.9676)	(0.8508,0.9733)	(0.8115,0.9540)
		0.2264/0.9540	0.1918/0.8933	0.1282/0.9613	0.1225/0.9527	0.1425/0.8057
(8,8)		(0.7945,0.9631)	(0.8309,0.9851)	(0.8494,0.9614)	(0.8582,0.9662)	(0.8395,0.9498)
		0.1686/0.9447	0.1541/0.9227	0.1121/0.9473	0.1080/0.9500	0.1103/0.8150
(10,10)		(0.8105,0.9594)	(0.8394,0.9776)	(0.8531,0.9582)	(0.8610,0.9625)	(0.8481,0.9469)
		0.1489/0.9497	0.1382/0.9363	0.1050/0.9493	0.1016/0.9547	0.0988/0.8303
(12,12)		(0.8199,0.9558)	(0.8438,0.9715)	(0.8551,0.9551)	(0.8620,0.9591)	(0.8527,0.9439)
		0.1360/0.9410	0.1277/0.9457	0.1001/0.9350	0.0970/0.9473	0.0912/0.8127
(15,15)		(0.8315,0.9523)	(0.8506,0.9653)	(0.8578,0.9516)	(0.8640,0.9552)	(0.8594,0.9412)
		0.1208/0.9250	0.1147/0.9423	0.0937/0.8953	0.0912/0.9097	0.0818/0.7570

Note: The first row represents a 95% confidence interval and the second row represents their lengths and cp's.

Table 4. Estimates of R for the non informative prior when α is known ($\alpha = 3$).

				Bayes estimates under the SE		Bayes estimates under the LINEX				Exact C.I.	Asymptotic C.I.
						$v=-1$		$v=1$			
(n,m)	R	$\widehat{R}_{MLE}$	$\widehat{R}_{U}$	$\widehat{R}^{*}_{BS}$	$\widehat{R}^{*}_{BS,Lindley}$	$\widehat{R}^{*}_{BL}$	$\widehat{R}^{*}_{BL,Lindley}$	$\widehat{R}^{*}_{BL}$	$\widehat{R}_{BL,Lindley}$	length/cp	length/cp
(5,5)	0.25(0.33)	0.2604(0.3376)	0.2421(0.3231)	0.2750(0.3487)	0.2789(0.3519)	0.2822(0.3575)	0.2870(0.3624)	0.2681(0.3403)	0.2705(0.3412)	0.4442(0.4956)	0.4388(0.5056)
		0.0157(0.0197)	0.0170(0.0228)	0.0149(0.0177)	0.0150(0.0173)	0.0075(0.0089)	0.0077(0.0078)	0.0075(0.0088)	0.0072(0.0084)	0.9457(0.9473)	0.8807(0.8813)
	0.50(0.70)	0.5004(0.6828)	0.5005(0.6984)	0.5003(0.6707)	0.5003(0.6673)	0.5104(0.6788)	0.5133(0.6776)	0.4903(0.6623)	0.4873(0.6573)	0.5396(0.4851)	0.5658(0.4919)
		0.0218(0.0185)	0.0264(0.0207)	0.0187(0.0171)	0.0178(0.0168)	0.0093(0.0084)	0.0090(0.0080)	0.0093(0.0087)	0.0089(0.0088)	0.9497(0.9507)	0.8943(0.8907)
	0.90(0.92)	0.8790(0.9059)	0.8950(0.9196)	0.8641(0.8926)	0.8614(0.8905)	0.8672(0.8948)	0.8644(0.8925)	0.8609(0.8902)	0.8585(0.8885)	0.2825(0.2353)	0.2515(0.2029)
		0.0054(0.0036)	0.0044(0.0029)	0.0066(0.0046)	0.0070(0.0048)	0.0032(0.0022)	0.0033(0.0023)	0.0034(0.0024)	0.0036(0.0025)	0.9460(0.9487)	0.8937(0.8940)
(8,8)	0.25(0.33)	0.2598(0.3401)	0.2482(0.3310)	0.2699(0.3477)	0.2715(0.3490)	0.2745(0.3535)	0.2764(0.3555)	0.2654(0.3420)	0.2664(0.3424)	0.3628(0.4109)	0.3600(0.4176)
		0.0088(0.0114)	0.0091(0.0125)	0.0086(0.0107)	0.0086(0.0106)	0.0044(0.0054)	0.0044(0.0054)	0.0043(0.0053)	0.0042(0.0052)	0.9583(0.9563)	0.9193(0.9167)
	0.50(0.70)	0.5006(0.6895)	0.5006(0.6998)	0.5006(0.6809)	0.5006(0.6794)	0.5073(0.6862)	0.5085(0.6855)	0.4939(0.6754)	0.4926(0.6734)	0.4472(0.3963)	0.4627(0.4000)
		0.0139(0.0101)	0.0157(0.0107)	0.0125(0.0097)	0.0123(0.0097)	0.0062(0.0047)	0.0061(0.0047)	0.0063(0.0050)	0.0062(0.0050)	0.9563(0.9573)	0.9137(0.9220)
	0.90(0.92)	0.8909(0.9121)	0.9005(0.9205)	0.8817(0.9039)	0.8809(0.9032)	0.8832(0.9050)	0.8823(0.9042)	0.8801(0.9028)	0.8795(0.9023)	0.2030(0.1713)	0.1857(0.1540)
		0.0025(0.0016)	0.0022(0.0014)	0.0029(0.0020)	0.0030(0.0020)	0.0014(0.0010)	0.0015(0.0010)	0.0015(0.0010)	0.0015(0.0010)	0.9517(0.9567)	0.9137(0.9150)
(10,10)	0.25(0.33)	0.2556(0.3393)	0.2463(0.3321)	0.2639(0.3455)	0.2649(0.3464)	0.2675(0.3502)	0.2687(0.3515)	0.2603(0.3409)	0.2610(0.3412)	0.3233(0.3717)	0.3210(0.3767)
		0.0072(0.0093)	0.0074(0.0100)	0.0071(0.0088)	0.0072(0.0088)	0.0036(0.0045)	0.0037(0.0045)	0.0035(0.0044)	0.0035(0.0043)	0.9557(0.9590)	0.9237(0.9220)
	0.50(0.70)	0.4997(0.6887)	0.4997(0.6968)	0.4997(0.6817)	0.4997(0.6808)	0.5053(0.6860)	0.5061(0.6856)	0.4942(0.6773)	0.4934(0.6760)	0.4066(0.3574)	0.4183(0.3599)
		0.0114(0.0092)	0.0126(0.0096)	0.0104(0.0089)	0.0103(0.0089)	0.0052(0.0044)	0.0052(0.0044)	0.0052(0.0045)	0.0052(0.0045)	0.9617(0.9580)	0.9220(0.9220)
	0.90(0.92)	0.8900(0.9135)	0.8976(0.9200)	0.8826(0.9070)	0.8820(0.9066)	0.8838(0.9078)	0.8832(0.9074)	0.8814(0.9062)	0.8809(0.9059)	0.1803(0.1484)	0.1678(0.1360)
		0.0023(0.0014)	0.0020(0.0013)	0.0026(0.0017)	0.0027(0.0017)	0.0013(0.0008)	0.0013(0.0008)	0.0013(0.0008)	0.0013(0.0009)	0.9557(0.9440)	0.9197(0.9177)
(12,12)	0.25(0.33)	0.2545(0.3353)	0.2468(0.3292)	0.2616(0.3407)	0.2623(0.3413)	0.2646(0.3446)	0.2654(0.3455)	0.2586(0.3369)	0.2591(0.3371)	0.2962(0.3401)	0.2942(0.3438)
		0.0060(0.0081)	0.0061(0.0085)	0.0059(0.0077)	0.0059(0.0076)	0.0030(0.0039)	0.0030(0.0039)	0.0029(0.0038)	0.0029(0.0038)	0.9530(0.9520)	0.9197(0.9257)
	0.50(0.70)	0.4963(0.6869)	0.4961(0.6935)	0.4964(0.6810)	0.4964(0.6804)	0.5011(0.6846)	0.5017(0.6843)	0.4917(0.6773)	0.4912(0.6764)	0.3753(0.3290)	0.3846(0.3311)
		0.0096(0.0084)	0.0105(0.0087)	0.0090(0.0081)	0.0089(0.0081)	0.0045(0.0040)	0.0044(0.0040)	0.0045(0.0041)	0.0044(0.0041)	0.9570(0.9440)	0.9283(0.9227)
	0.90(0.92)	0.8928(0.9135)	0.8990(0.9189)	0.8867(0.9081)	0.8863(0.9078)	0.8876(0.9087)	0.8872(0.9084)	0.8857(0.9074)	0.8855(0.9072)	0.1599(0.1343)	0.1503(0.1247)
		0.0019(0.0011)	0.0017(0.0010)	0.0021(0.0013)	0.0021(0.0013)	0.0010(0.0006)	0.0010(0.0007)	0.0010(0.0007)	0.0011(0.0007)	0.9447(0.9550)	0.9157(0.9300)
(15,15)	0.25(0.33)	0.2534(0.3350)	0.2472(0.3301)	0.2592(0.3394)	0.2597(0.3398)	0.2616(0.3426)	0.2622(0.3432)	0.2568(0.3363)	0.2572(0.3364)	0.2663(0.3075)	0.2647(0.3103)
		0.0043(0.0060)	0.0043(0.0063)	0.0042(0.0058)	0.0043(0.0058)	0.0022(0.0029)	0.0022(0.0029)	0.0021(0.0029)	0.0021(0.0029)	0.9647(0.9603)	0.9393(0.9347)
	0.50(0.70)	0.4971(0.6951)	0.4970(0.7006)	0.4972(0.6900)	0.4972(0.6896)	0.5010(0.6929)	0.5014(0.6927)	0.4934(0.6871)	0.4930(0.6866)	0.3401(0.2939)	0.3472(0.2951)
		0.0074(0.0058)	0.0079(0.0060)	0.0070(0.0057)	0.0070(0.0057)	0.0035(0.0028)	0.0035(0.0028)	0.0035(0.0029)	0.0035(0.0029)	0.9630(0.9483)	0.9350(0.9293)
	0.90(0.92)	0.9004(0.9152)	0.9052(0.9195)	0.8956(0.9109)	0.8954(0.9108)	0.8962(0.9114)	0.8960(0.9112)	0.8950(0.9104)	0.8948(0.9103)	0.1342(0.1168)	0.1270(0.1099)
		0.0010(0.0008)	0.0009(0.0008)	0.0010(0.0009)	0.0010(0.0010)	0.0005(0.0005)	0.0005(0.0005)	0.0005(0.0005)	0.0005(0.0005)	0.9490(0.9630)	0.9337(0.9423)

Notes: The first row represents the average estimates and the second row represents corresponding ERs for each choice of (n, m). But, for the last two columns, the first row represents the average length and second row represents their cp's. The corresponding results are reported within bracket in each cell for $R = 0.33$, 0.70 and 0.92.

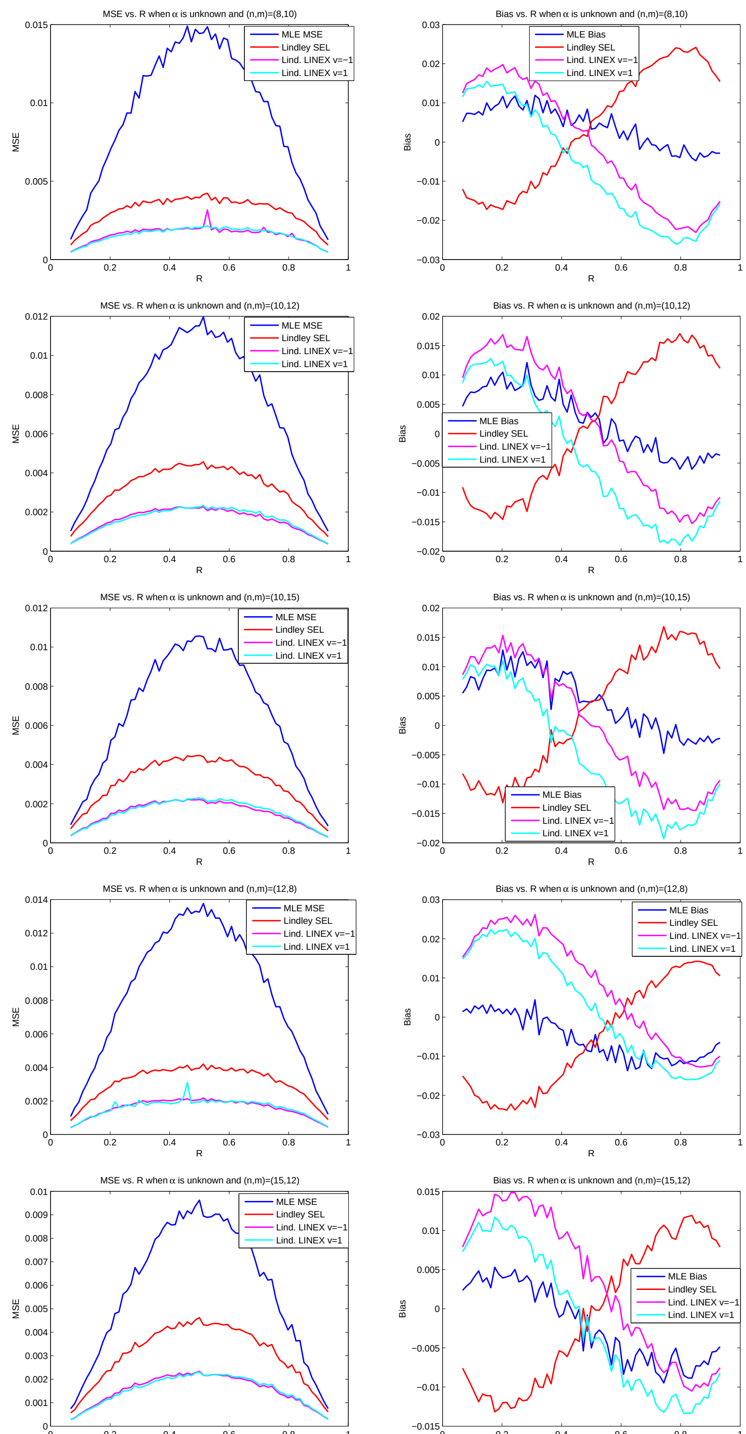

Figure 1. MSE and Bias for R when α is unknown and different pair of (n, m).

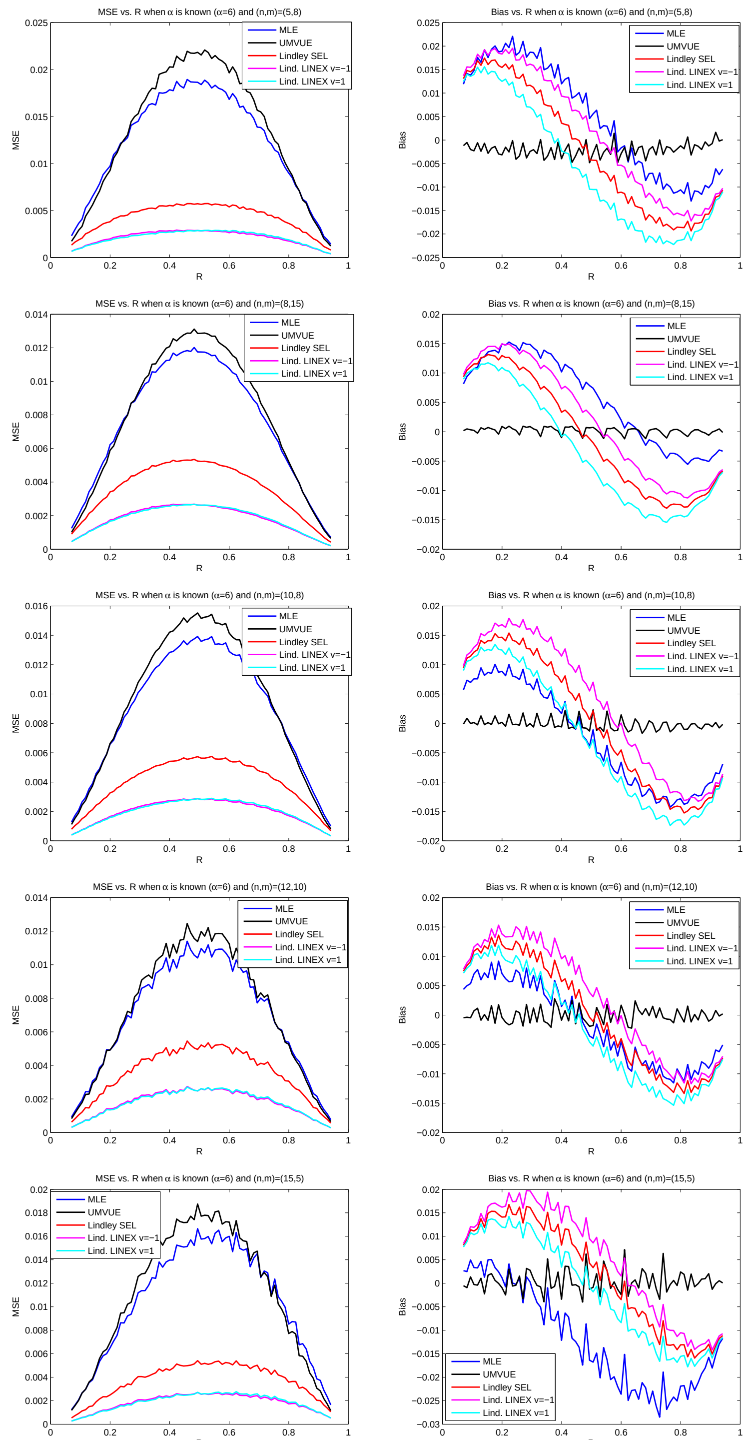

Figure 2. MSE and Bias for R when α is known ($\alpha = 6$) and different pair of (n, m).

Table 5. Upper record values from 36kV and 38kV data sets.

i	1	2	3	4
r	1.97	2.58	2.71	25.50
s	0.47	0.73	1.40	2.38

Table 6. K-S values and estimates when the first shape parameters (α) are common.

Kolmogorov-Smirnov and corresponding p values				
Data Set	K-S(MLE)	p-value	K-S(Lindley)	p-value
r	0.6111	>0.05	0.4796	>0.2
s	0.3879	>0.2	0.4180	>0.2
Parameter and reliability estimates				
Parameter	MLE		Lindley(SEL)	
β_1	0.5468		0.4227	
β_2	1.9134		0.4736	
α	2.2587		1.9249	
R	0.2222		0.3311	

the goodness-of-fit test. The Weibull distribution is considered by Baklizi [11] based on upper records for the reliability problem and the two-parameter bathtub-shaped distribution is considered by Selim [42] and Asgharzadeh et al. [8] based on upper records for the parameter and interval estimation problems. The Weibull distribution parameters have the unique ML estimates. However, the existence and uniqueness of the ML estimates of the two-parameter bathtub-shaped distribution parameters are not considered in literature, but these parameters are obtained for this example. The K-S values and the corresponding p-values, the parameters and the reliability (R) estimates are presented in Table 8. From Table 8, it is observed that the Burr Type XII distribution gives a better fit than the other distributions for both of the data sets.

Table 7. Upper record values from 38.5 and 36 stress levels.

i	1	2	3	4	5
r	60	83	140	–	–
s	173	218	288	394	585

5. Conclusion

In this paper, we have derived the estimates of the stress-strength reliability based on upper record values when the stress and strength variables follow the Burr Type XII distribution under the non-Bayesian and Bayesian frameworks. The first shape parameters of the distributions of the measurements are assumed to be the same. When the first shape parameters are unknown, the ML and Bayes estimates are obtained by using Lindley's approximation and MCMC method. It is observed that the performance of the Bayes estimates are better than ML estimates. When the first shape parameters are known, the Bayes estimates are obtained exactly and approximately by using Lindley and MCMC methods for the informative prior case. It is observed that the performance

Table 8. K-S values and estimates when the first shape parameters (α) are common.

Kolmogorov-Smirnov and corresponding p values				
	based on Burr Type XII		based on Weibul	based on bathtub
Data Set	K-S(MLE)	K-S(Lindley)	K-S(MLE)	K-S(MLE)
r	0.5104(p >0.2)	0.4464(p >0.2)	0.5058(p >0.2)	0.6626(0.05< p <0.1)
s	0.4431(p >0.2)	0.3098(p >0.2)	0.4636(0.1< p <0.2)	0.4601(0.1< p <0.2)
Parameter and reliability estimates				
	based on Burr Type XII		based on Weibull	based on bathtub
Parameter	MLE	Lindley(SEL)	MLE	MLE
β_1	4.3281	15.1596	0.006	0.0365
β_2	7.2135	14.3937	0.001	0.0056
α	2.0278	4.3117	1.7096	0.3008
R	0.3750	0.7283	0.8737	0.8672

of the Bayes estimates are better than ML and UMVU. Moreover, for the non informative prior case, it is observed that the performance of the Bayes estimates are better than others when the true values of the stress-strength reliability is not close to the extremes (0 or 1), while near the extremes the UMVU and ML estimates are better than the Bayes estimates. It is observed that the performance of the HPD Bayesian credible interval are better than others in all cases. When the first shape parameter is known, we observe that the stress-strength reliability estimates are very close for both exact and approximate methods. This is encouraging when the first shape parameter is unknown, because the stress-strength reliability estimates can be obtained from the approximate methods only. Furthermore, the Bayes estimates based on Lindley's approximation and the MCMC method are close to each other. Since the computation time for the MCMC method is much more than Lindley's approximation, the Bayes estimates based on Lindley's approximation are recommended.

To obtain the point and interval estimates of the stress-strength reliability are difficult due to lack of explicit form of the reliability when the measurements follow from the Burr Type XII distribution with no common parameters. More work is needed along that direction.

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