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The Weibull-Power Cauchy distribution: model, properties and applications

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Abstract

We propose a new three-parameter distribution with increasing, decreasing, reversed-J and upside-down bathtub shaped hazard rate, called the Weibull-power Cauchy distribution. We obtain explicit expressions for the mode, ordinary, negative and incomplete moments, mean deviations, mean residual life, quantile and generating functions, order statistics, Shannon entropy and reliability. We derive a power series for the quantile function using exponential partial Bell polynomials. A useful characterization of the new distribution is also presented. The method of maximum likelihood is used to estimate the model parameters. The importance of the new distribution is proved empirically by means of three real-life data sets.

Keywords: Cauchy distribution, half-Cauchy distribution, moments, power Cauchy distribution, T-X family, Weibull-X family

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1. Introduction

The Cauchy distribution is unimodal, symmetric and bell-shaped with much heavier tails than the normal distribution. It is also used for the analysis when outliers are presented in the data. It is well-known that the Cauchy distribution can arise as the ratio of two independent normal variates. The Cauchy distribution has received applications in many areas including physics, mathematics, econometrics, engineering, spectroscopy, biological analysis, clinical trials, stochastic modeling of decreasing failure rate life components, queueing theory, and reliability. For more details and discussion, the reader is referred to Johnson et al. [25](Ch:16), Krishnamoorthy [28](Ch:26) and Forbes et al. [18](Ch:10).

The cumulative distribution function (cdf) and probability density function (pdf) of the Cauchy (C) distribution with scale parameter $\sigma > 0$ (representing semi-interquartile range) are, respectively, given by

$$F_C(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x}{\sigma}\right) \tag{1.1}$$

and

$$f_C(x) = \frac{1}{\sigma\,\pi}\left[1 + \left(\frac{x}{\sigma}\right)^2\right]^{-1}, \quad x \in \Re, \quad \sigma > 0. \tag{1.2}$$

For the Cauchy distribution, the finite moments of order greater than or equal to one do not exist, hence the central limit theorem does not hold. Further, the maximum likelihood estimation of its parameters is not ideal because of no closed-form solution. Vribk [42] suggested the use of Edgeworth expansion to construct an accurate approximation to the sampling distribution of the maximum likelihood estimator (MLE) of this parameter. The method of moments is also not possible for this distribution.

Because of these facts, the Cauchy distribution serves as counter example for some well-accepted results and concepts in Statistics. This fact reduces the applicability of this distribution in modeling real life scenario. This also makes the choice of the Cauchy distribution as an unrealistic model. That is why extensions of the Cauchy distribution have been suggested in literature to overcome the problem of the moments and other useful properties, which are as follows:

1. Rider [39] pioneered the *generalized Cauchy (GC)* density given by

$$f_{GC1}(x) = \frac{\Gamma(m)}{\sigma\,\Gamma(\frac{1}{2})\,\Gamma(m-\frac{1}{2})}\left[1 + \left(\frac{x-\mu}{\sigma}\right)^2\right]^{-m}, \qquad m \geq 1, \quad x \in \Re, \quad \sigma > 0.$$

For $m = 1$, the GC distribution reduces to the Cauchy model and for $\mu = 0$, it reduces to the Student t-distribution with $(2m-1)$ degrees of freedom multiplied by $\sigma\,(2m-1)$.

By using the transformation $\mu = -\cos(\rho\pi)$ and $\sigma = -\sin(\rho\pi)$ in the last equation, it can be expressed as

$$f_{GC2}(x) = \frac{\sin^{2m-1}(\rho\pi)\,\Gamma(m)}{\Gamma(\frac{1}{2})\,\Gamma(m-\frac{1}{2})}\left\{\sin^2(\rho\pi) + \left[x + \cos(\rho\pi)\right]^2\right\}^{-m}, \qquad m \geq 1,\ 0 < \rho < 1.$$

This distribution is symmetric about $-\cos(\rho\pi)$, its odd moments, when they exist, vanish. The mean, median and mode are also equal to $\mu = -\cos(\rho\pi)$. The even moments about the mean are given by

$$\mu_{2r} = \frac{\Gamma(\frac{r+1}{2})\Gamma(m - \frac{r+1}{2})}{\Gamma(\frac{1}{r})\,\Gamma(m-\frac{1}{r})}, \quad r < 2m-1.$$

2. To overcome the problem of the non-existence of the moments and the MLE, Dahiya et al. [15] and Nadarajah and Kotz [32] introduced a *truncated Cauchy (TC)* density given by

$$f_{TC}(x) \quad = \quad \frac{1}{\sigma\,\pi}\left[1+\left(\frac{x}{\sigma}\right)^2\right]^{-1}\left[\tan^{-1}\left(\frac{B-\mu}{\delta}\right)-\tan^{-1}\left(\frac{A-\mu}{\sigma}\right)\right]^{-1},$$
$$-\infty < A \le x \le B < \infty, \quad \mu \in \Re, \quad \text{and}\, \sigma > 0.$$

Nadarajah and Kotz [33, 34] wrote an R-script for computing six quantities of interest, namely pdf, cdf, mean, variance, quantile function (qf) and random generation, for the TC distribution.

3. Manoukian and Nadeau [30], and Kravchuk [26] used the transformation $Z = \log|X|$ in (1.2), and obtained an interesting relation between the Cauchy and hyperbolic Secant (HS) $(\log\sigma, 1)$ distributions. The pdf of Z is given by

$$f_{HS}(z) = \frac{1}{\pi}\,\sec^{-1}\big(z-\log\sigma\big).$$

Kravchuk and Pollett [27] studied the properties of the Hedges-Lehmann scale estimator of the Cauchy distribution.

4. Ohakwe and Osu [35] suggested the use of the transformation

$$Y = \frac{X}{|X_m|} = \frac{\big(x_{-r},\ldots,x_{-1},x_0,x_1,\ldots,x_q\big)}{Max|x_{-r},\ldots,x_{-1},x_0,x_1,\ldots,x_q|}$$

to obtain a new version of the Cauchy density and computed raw moments, skewness, kurtosis, and cumulants. The pdf of Y is given by

$$f(y) = \frac{K}{\sigma\,\pi}\left[1+\left(\frac{y}{\sigma}\right)^2\right]^{-1}, \quad -1<y<1, \quad \sigma>0,$$

where $K^{-1} = \frac{2}{\pi}\tan^{-1}\left(\frac{y}{\sigma}\right)$ is the pdf stabilizing term, which can be obtained by integrating $f(y)$ and equating to unity. The last equation can be rewritten as

$$f(y) = \frac{1}{2\,\sigma\,\tan^{-1}\left(\frac{y}{\sigma}\right)}\left[1+\left(\frac{y}{\sigma}\right)^2\right]^{-1}, \quad -1<y<1.$$

5. Based on the beta-G generator pioneered by Eugene et al. [17], Alshawarbeh et al. [1, 2] introduced and studied another generalization of the Cauchy distribution, known as *beta-Cauchy (BC)* distribution. Its density function is given by

$$f_{BC}(x) \quad = \quad \frac{1}{\sigma\,\pi^a\,B(a,b)}\left[1+\left(\frac{x}{\sigma}\right)^2\right]^{-1}\left[\tan^{-1}\left(\frac{x}{\sigma}\right)\right]^{a-1}$$
$$\times\left[1-\pi^{-1}\,\tan^{-1}\left(\frac{x}{\sigma}\right)\right]^{b-1} \; x\in\Re,\ a,b,\sigma>0.$$

6. The half-Cauchy (HC) distribution is the folded standard Cauchy distribution around the origin so that positive values are observed. A random variable has the HC distribution with scale parameter $\sigma > 0$, if its cdf is given by

$$F_{HC}(x) = \frac{2}{\pi}\,\tan^{-1}\left(\frac{x}{\sigma}\right), \quad x>0. \tag{1.3}$$

The pdf corresponding to (1.3) is

$$f_{HC}(x) = \frac{2\pi^{-1}}{\sigma}\left[1+\left(\frac{x}{\sigma}\right)^2\right]^{-1}. \tag{1.4}$$

For $r<1$, the rth moment comes from (1.4) as $\mu'_r = \sigma^r\,\sec(r\pi/2)$. Further, three generalizations of the HC distribution are reported using the Marshall-Olkin [31], beta-G [17] and Kumaraswamy-G [13] generators. They are: the Marshall-Olkin HC (MOHC) by Jacob and Jayakumar [24], beta-HC (BHC) by Cordeiro and Lemonte [14] and Kumaraswamy-HC (KHC) by Ghosh [19]. The pdfs of the MOHC, BHC and KHC distributions are given by

$$
\begin{aligned}
f_{MOHC}(x) &= \frac{2\pi\, p}{\sigma}\left[1+\left(\frac{x}{\sigma}\right)^2\right]^{-1} \\
&\quad \times \left[\pi\, p + 2(1-p)\left(\frac{x}{\sigma}\right)^2\right]^{-1}, \quad \sigma>0, \quad 0<p<1, \\
f_{BHC}(x) &= \frac{2^a}{\sigma\,\pi^a\, B(a,b)}\left[1+\left(\frac{x}{\sigma}\right)^2\right]^{-1}\left[\tan^{-1}\left(\frac{x}{\sigma}\right)\right]^{a-1} \\
&\quad \times \left[1-2\pi^{-1}\,\tan^{-1}\left(\frac{x}{\sigma}\right)\right]^{b-1}, \quad a,b,\sigma>0, \\
f_{KHC}(x) &= \frac{a\, b\, 2^a}{\sigma\,\pi^a}\left[1+\left(\frac{x}{\sigma}\right)^2\right]^{-1}\left[\tan^{-1}\left(\frac{x}{\sigma}\right)\right]^{a-1} \\
&\quad \times \left[1-\left\{2\pi^{-1}\,\tan^{-1}\left(\frac{x}{\sigma}\right)\right\}^a\right]^{b-1}, \quad a,b,\sigma>0,
\end{aligned}
$$

respectively. Hamedani and Ghosh (2015) gave some characterizations of the KHC distribution.

Recently, Rooks et al. [40] introduced a two-parameter *Power-Cauchy (PC)* distribution, a sub-model of the transformed beta distribution. The cdf of the PC distribution with shape parameter α and scale parameter σ is given by

$$(1.5) \qquad F(x) = 2\,\pi^{-1}\,\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}, \quad x>0 \quad \alpha,\sigma>0.$$

The pdf corresponding to (1.5) becomes

$$(1.6) \qquad f(x) = 2\,\pi^{-1}\left(\frac{\alpha}{\sigma}\right)\left(\frac{x}{\sigma}\right)^{\alpha-1}\left[1+\left(\frac{x}{\sigma}\right)^{2\alpha}\right]^{-1}.$$

Henceforth, we denote by $Y\ \sim$PC(α,σ) a random variable having the pdf (1.6) with parameters α and σ.

In the last two decades, there has been an increased interest in defining new generators for univariate continuous distributions by introducing additional shape parameter(s) to the baseline model, and thus obtaining flexible distributions from them. The addition of parameters has been proved useful in exploring skewness and tail properties, and also for improving the goodness-of-fit of the generated family.

Let $r(t)$ be the pdf of a random variable $T\in[a,b]$ for $-\infty\le a<b<\infty$ and let $F(x)$ be the cdf of a random variable X such that the link function $W(\cdot):[0,1]\longrightarrow[a,b]$ satisfies the following conditions: (i) $W(\cdot)$ is differentiable and monotonically non-decreasing, and (ii) $W(0)\to a$ and $W(1)\to b$.

Alzaatreh et al. [8] defined the cdf of the *T-X family* of distributions by

$$(1.7) \qquad G(x) = \int_a^{W[F(x)]} r(t)\,dt.$$

If $T\in(0,\infty)$, X is a continuous random variable and $W[F(x)] = -\log[1-F(x)]$. Then, the pdf corresponding to (1.7) is given by

$$(1.8) \qquad g(x) = \frac{f(x)}{1-F(x)}\,r\Big(-\log\big[1-F(x)\big]\Big) = h_f(x)\,r\Big(H_f(x)\Big),$$

where $h_f(x) = \frac{f(x)}{1-F(x)}$ and $H_f(x) = -\log[1-F(x)]$ are the hazard and cumulative hazard rate functions corresponding to any baseline pdf $f(x)$, respectively. For more details about the T-X family, see Alzaatreh et al. [3, 4, 8, 5, 7, 9], Alzaatreh and Knight [10], Alzaatreh and Ghosh [6] and Lee et al. [29].

Let T be a random variable having the Weibull distribution with one shape parameter $c > 0$, $r(t) = c\,t^{c-1}\,\mathrm{e}^{-t^{c}}$, $t \geq 0$. Then, the pdf of the *Weibull-X* family of distributions follows from equation (1.8) as

$$(1.9)\qquad g(x) = \frac{c\,f(x)}{[1-F(x)]}\Big\{-\log[1-F(x)]\Big\}^{c-1}\,\exp\Big\{-\big[-\log\big(1-F(x)\big)\big]^{c}\Big\}.$$

The cdf corresponding to (1.9) is given by

$$(1.10)\qquad G(x) = 1-\exp\Big\{-\big\{-\log[1-F(x)]\big\}^{c}\Big\}.$$

The pdf and cdf in (1.9) and (1.10) can also be rewritten as

$$(1.11)\qquad g(x) = c\;h_f(x)\,\big[H_f(x)\big]^{c-1}\,\exp\Big\{-[H_f(x)]^{c}\Big\}$$

and

$$(1.12)\qquad G(x) = 1-\exp\Big\{-[H_f(x)]^{c}\Big\}.$$

Some properties of the Weibull-X family have been studied in literature. See, for example, Alzaatreh et al. [4, 8].

The paper is unfolded as follows. In Section 2, we define a new generalization of the PC distribution, namely the *Weibull-Power-Cauchy (WPC)* model. In Section 3, we obtain some mathematical properties of the new distribution. Further, we derive a power series for its qf using exponential partial Bell polynomials. Section 4 refers to a useful characterization of the WPC distribution. In Section 5, the model parameters are estimated by the method of maximum likelihood. In Section 6, we explore the usefulness of the proposed distribution by means of three real data sets. Finally, Section 7 offers some concluding remarks.

2. The Weibull-Power Cauchy distribution

Inserting (1.5) and (1.6) in equations (1.9) and (1.10), the cdf and pdf of the WPC distribution are, respectively, given by

$$(2.1)\qquad G(x) = 1-\exp\left\{-\left[-\log\left(1-2\,\pi^{-1}\,\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right)\right]^{c}\right\}$$

and

$$\begin{aligned} g(x) \;&=\; 2\,c\,\pi^{-1}\left(\frac{\alpha}{\sigma}\right)\left(\frac{x}{\sigma}\right)^{\alpha-1}\left[1-2\pi^{-1}\,\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right]^{-1} \\ &\quad\times\left\{-\log\left[1-2\,\pi^{-1}\,\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right]\right\}^{c-1}\left[1+\left(\frac{x}{\sigma}\right)^{2\alpha}\right]^{-1} \\ (2.2)\qquad &\quad\times\exp\left\{-\left[-\log\left(1-2\,\pi^{-1}\,\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right)\right]^{c}\right\},\; x>0,\; c,\alpha,\sigma>0, \end{aligned}$$

respectively. Henceforth, a random variable having pdf (2.2) is denoted by $X \sim \mathrm{WPC}(c,\alpha,\sigma)$.

The survival function (sf), $S(x)$, hazard rate function (hrf), $h(x)$, and cumulative hazard rate function (chrf), $H(x)$, of X are given by

$$S(x) = \exp\left\{-\left[-\log\left(1-2\,\pi^{-1}\,\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right)\right]^{c}\right\},$$

$$\begin{aligned} h(x) \;&=\; 2\,c\,\pi^{-1}\left(\frac{\alpha}{\sigma}\right)\left(\frac{x}{\sigma}\right)^{\alpha-1}\left[1+\left(\frac{x}{\sigma}\right)^{2\alpha}\right]^{-1}\left[1-2\,\pi^{-1}\,\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right]^{-1} \\ &\quad\times\left\{-\log\left[1-2\,\pi^{-1}\,\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right]\right\}^{c-1} \end{aligned}$$

and

$$H(x) = \left\{ -\log \left[1 - 2\,\pi^{-1}\, \tan^{-1} \left(\frac{x}{\sigma} \right)^{\alpha} \right] \right\}^{c},$$

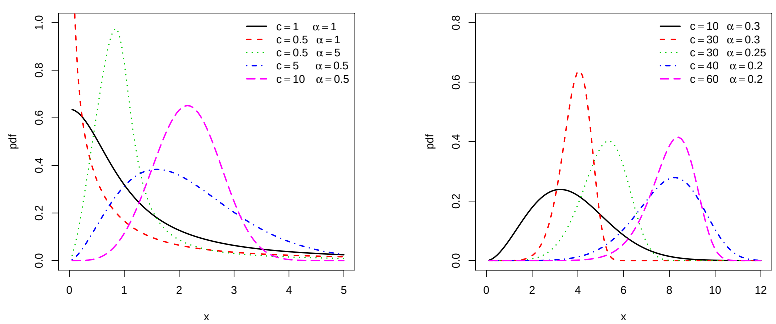
respectively.

Special cases of the WPC distribution:

(i) If $\alpha = 1$ in (2.2), the WPC distribution reduces to the WHC distribution with parameters c and σ.

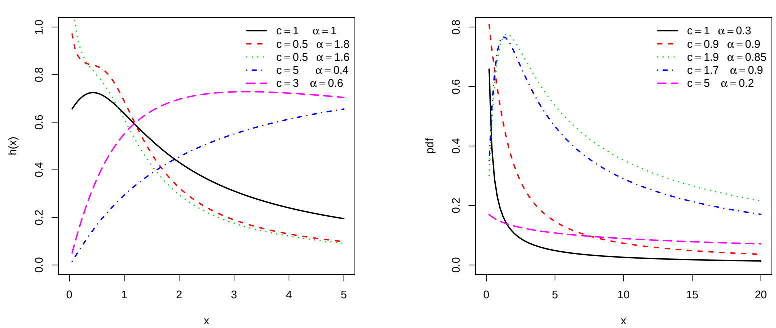
(ii) If $c = 1$ in (2.2), the WPC distribution is identical to the PC distribution with parameters α and σ.

Figures 1 and 2 display some plots of the density of X when $\sigma = 1$ for different values of c and α. The plots in Figure 1 indicate that the WPC distribution can produce various shapes such as symmetrical, left-skewed, right-skewed and reversed-J. For $c \leq 1$, the density is always reversed-J shaped. For $c > 1$, the peakedness increases as α decreases. For $c > 30$, the density is left-skewed, otherwise right-skewed. Figure 2 shows that the hrf of the WPC distribution has IFR (increasing failure rate), DFR (decreasing failure rate), reversed-J and UBT (upside-down-bathtub) shapes.



(a) (b)

Figure 1. Plots of the WPC density for different values of c and α.



(a) (b)

Figure 2. Plots of the hrf of the WPC distribution for some values of c and α.

3. Properties of the WPC distribution

In this section, we provide some mathematical properties of the new distribution. Lemma 1 below gives the relations between the WPC and Weibull and exponential distributions.

3.1. Transformation.

3.1. Lemma. *(i) If a random variable Y follows the Weibull distribution with shape parameter c and unit scale parameter, then* $X = \sigma \left[\cot\left(\frac{\pi}{2}\exp(-Y)\right)\right]^{\frac{1}{\alpha}} \sim \mathrm{WPC}(c, \alpha, \sigma)$.
(ii) If a random variable Y follows the standard exponential distribution, then $X = \sigma \left[\cot\left(\frac{\pi}{2}\exp(-Y)\right)\right]^{\frac{1}{\alpha}} \sim \mathrm{PC}(\alpha, \sigma)$.
(iii) If a random variable Y follows the Weibull distribution with shape parameter c and unit scale parameter, then $X = \sigma \cot\left(\frac{\pi}{2}\exp(-Y)\right) \sim \mathrm{WHC}(c, \sigma)$.

3.2. Mode.

3.2. Lemma. *The mode of the WPC distribution is the solution of the equation,* $w(x) = 0$*, where*

$$\begin{aligned} w(x) &= (\alpha-1)\left[1+\left(\frac{x}{\sigma}\right)^{2\alpha}\right] - 2\alpha\left(\frac{x}{\sigma}\right)^{2\alpha} + \frac{2\alpha\left(\frac{x}{\sigma}\right)^{\alpha}}{\pi\left[1-2\,\pi^{-1}\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right]} \\ &\times\left\{1-\frac{c\left[1-\left\{-\log\left[1-2\,\pi^{-1}\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right]\right\}\right]^{c}-1}{\left\{-\log\left[1-2\,\pi^{-1}\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right]\right\}}\right\}. \end{aligned}$$

Proof. Setting $g'(x) = 0$ is equivalent to,

$$\begin{aligned} g'(x) &= \frac{2\,c\,\pi^{-1}\,\alpha\left(\frac{x}{\sigma}\right)^{\alpha}}{x^2\left[1+\left(\frac{x}{\sigma}\right)^{2\alpha}\right]^2\left[1-2\,\pi^{-1}\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right]} \\ &\times\left\{-\log\left[1-2\,\pi^{-1}\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right]\right\}^{c-1} \\ &\times\exp\left\{-\left[-\log\left(1-2\,\pi^{-1}\tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right)\right]^{c}\right\}\times w(x) = 0. \end{aligned}$$

Hence, the critical values of $g(x)$ are the solution of $w(x) = 0$. □

3.3. Quantile function. The following Lemma 3 provides the qf for the WPC distribution.

3.3. Lemma. *The qf of the WPC distribution is given by (for $0 < u < 1$)*

$$Q(u) = \sigma\left\{\cot\left(\frac{\pi}{2}\exp\left\{-\left[-\log(1-u)\right]^{\frac{1}{c}}\right\}\right)\right\}^{\frac{1}{\alpha}}. \tag{3.1}$$

3.4. Moments. By using Lemma 3.1, the nth moment of X can be expressed as

$$\mathbb{E}(X^n) = c\,\sigma^n \int_0^{\infty} \cot\left(\frac{\pi}{2}\exp(-y)\right)^{\frac{n}{\alpha}} y^{c-1}\exp(-y^c)\,dy. \tag{3.2}$$

The calculations in this section involve several special functions, including the complete and incomplete gamma function defined by (for $\alpha > 0$) $\Gamma(\alpha) = \int_0^{\infty} w^{\alpha-1}\,\mathrm{e}^{-\mathrm{w}}\,\mathrm{dw}$ and $\gamma(\alpha, x) = \int_0^{x} w^{\alpha-1}\,\mathrm{e}^{-\mathrm{w}}\,\mathrm{dw}$, respectively, and the complementary incomplete gamma function defined by $\Gamma(\alpha, x) = \int_x^{\infty} w^{\alpha-1}\,\mathrm{e}^{-\mathrm{w}}\,\mathrm{dw}$.

The following power series can be obtained in MATHEMATICA

$$\cot(x)^s = \sum_{i=0}^{\infty} a_i(s)\, x^{2i-s}, \tag{3.3}$$

where $a_0(s) = 1$, $a_1(s) = -s/3$, $a_2(s) = s(5s-7)/90$, etc. Using this expansion, we have

$$\cot\left(\frac{\pi}{2}\,\exp(-x)\right)^{\frac{n}{\alpha}} = \sum_{i=0}^{\infty} b_i\left(\frac{n}{\alpha}\right)\,\exp\left\{\left(\frac{n}{\alpha}-2i\right)\,x\right\},$$

where $b_i\left(\frac{n}{\alpha}\right) = \left(\frac{\pi}{2}\right)^{2i-n/\alpha}\,a_i\left(\frac{n}{\alpha}\right)$.

The nth ordinary moment of X can be expressed as

$$\mu_n' = \mathbb{E}(X^n) = c\,\sigma^n \sum_{i=0}^{\infty} b_i\,\left(\frac{n}{\alpha}\right)\int_0^{\infty} x^{c-1}\,\exp\left\{\left(\frac{n}{\alpha}-2i\right)\,x - x^c\right\}\,dx.$$

Let $I(r,s,c) = \int_0^{\infty} x^{r-1}\,\exp\left(s\,x - x^c\right)dx$. We can write

$$\mu_n' = c\,\sigma^n \sum_{i=0}^{\infty} b_i\,\left(\frac{n}{\alpha}\right)\,I\Big(c, \frac{n}{\alpha}-2i, c\Big). \tag{3.4}$$

First, we obtain a representation for the integral $I(r,s,c)$ using the gamma function. By expanding $\mathrm{e}^{s\,x}$ in power series, we obtain

$$I(r,s,c) = \sum_{j=0}^{\infty} \frac{s^j}{j!}\int_0^{\infty} x^{r+j-1}\,\exp\left(-x^c\right)\,dx \tag{3.5}$$

For $r>0$ and $c>0$, we have

$$\int_0^{\infty} x^{r-1}\,\exp\left(-x^c\right)\,dx = c^{-1}\,\Gamma\Big(\frac{r}{c}\Big).$$

and then

$$I(r,s,c) = c^{-1}\sum_{j=0}^{\infty} \frac{s^j}{j!}\,\Gamma\left(\frac{r+j}{c}\right). \tag{3.6}$$

Combining (3.4) and (3.6) gives

$$\mu_n' = \sigma^n \sum_{i,j=0}^{\infty} \frac{\left(\frac{n}{\alpha}-2i\right)^j\,b_i\left(\frac{n}{\alpha}\right)}{j!}\,\Gamma\left(\frac{j}{c}+1\right). \tag{3.7}$$

A second representation for the integral (3.5) follows from the Wright generalized hypergeometric function defined by

$${}_p\Psi_q\left[\begin{array}{c}(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q)\end{array};x\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(\alpha_j + A_j\,n\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_j + B_j\,n\right)}\,\frac{x^n}{n!}.$$

We can write

$$\begin{aligned} I(r,s,c) &= \sum_{j=0}^{\infty}\frac{s^j}{j!}\int_0^{\infty} x^{j+c-1}\,\exp\left(-x^c\right)\,dx = \frac{1}{c}\sum_{j=0}^{\infty}\frac{s^j}{j!}\,\Gamma\left(\frac{j}{c}+1\right) \\ &= \frac{1}{c}\,{}_1\Psi_0\left[\begin{array}{c}(1,1/c)\\-\end{array};s\right] \end{aligned} \tag{3.8}$$

provided that $c>1$. Combining (3.4) and (3.8) gives a second way to obtain the moments of X.

We now derive a third formula for μ'_n by assuming that $s \geq 0$ and $c = p/q$, where $p \geq 1$ and $q \geq 1$ are co-prime integers. We require the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{\prod\limits_{j=1}^{m} \Gamma\left(b_j + t\right) \prod\limits_{j=1}^{n} \Gamma\left(1 - a_j - t\right)}{\prod\limits_{j=n+1}^{p} \Gamma\left(a_j + t\right) \prod\limits_{j=m+1}^{q} \Gamma\left(1 - b_j - t\right)} \, x^{-t} dt,$$

where $\mathrm{i} = \sqrt{-1}$ is the complex unit and L denotes an integration path (see Section 9.3 in Gradshteyn and Ryzhik [21] for a description of this path). The Meijer G-function contains many integrals with elementary and special functions. Some of these integrals are included in Prudnikov et al. [37]. The Wright generalized hypergeometric function is due to Wright [43]. This function exists if $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$.

Based on the result $\mathrm{e}^{-g(x)} = G_{0,1}^{1,0}\left(g(x) \,\middle|\, \begin{matrix} - \\ 0 \end{matrix}\right)$ for an arbitrary function $g(\cdot)$ and using the Meijer G-function defined above, we have

$$I(c, s, c) = \int_0^{\infty} x^{c-1} \exp\left(s\, x\right) \, G_{0,1}^{1,0}\left(x^c \,\middle|\, \begin{matrix} - \\ 0 \end{matrix}\right) dx.$$

Using equation (2.24.1.1) in Prudnikov et al. [37], we obtain (for $s \geq 0$)

$$I(c, s, c) = \frac{p^{p/q-1/2}(-s)^{-p/q}}{(2\pi)^{(p+q)/2-1}} G_{q,p}^{p,q}\left(\frac{(p/q)^q p^p}{(-s)^p q^q} \,\middle|\, \begin{matrix} \frac{1-p/q}{p}, \frac{2-p/q}{p}, \ldots, \frac{p-p/q}{p} \\ 0, \frac{1}{q}, \ldots, \frac{q-1}{q} \end{matrix}\right). \tag{3.9}$$

Note that the condition $c = p/q$ in the last equation is not restrictive since every real number can be approximated by a rational number. Inserting the last equation in (3.4), we can obtain the nth negative moment of X as

$$\mathbb{E}(X^{-n}) = c\, \sigma^{-n} \sum_{i=0}^{\infty} b_i \left(-\frac{n}{\alpha}\right) I\left(c, -\frac{n}{\alpha} - 2i, c\right).$$

where $I\left(c, -\frac{n}{\alpha} - 2i, c\right)$ is given by (3.9) with $s = -\frac{n}{\alpha} - 2i$.

Further, the central moments (μ_n) and cumulants (κ_n) of X are obtained from the ordinary moments by

$$\mu_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \, \mu_1'^k \, \mu'_{n-k} \qquad \text{and} \qquad \kappa_n = \mu'_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \kappa_k \, \mu'_{n-k},$$

respectively, where $\kappa_1 = \mu'_1$. The skewness and kurtosis of X can be calculated from the ordinary moments above using well-known relationships. The nth descending factorial moment of X (for $n = 1, 2, \ldots$) is

$$E[X^{(n)}] = E[X(X-1) \times \cdots \times (X-n+1)] = \sum_{j=0}^{n} s(n,j)\, \mu'_{j,X},$$

where $s(n,j) = (j!)^{-1} \left[d^j j^{(n)}/dx^j\right]_{x=0}$ is the Stirling number of the first kind.

3.5. Incomplete moments and mean deviations. First, we determine the nth incomplete moment of X. We can write from equations (3.4) and (3.6)

$$\begin{aligned} m_n(y) &= \int_0^y x^n\, f(x)\, dx = c\,\sigma^n \sum_{i,j=0}^{\infty} \frac{\left(\frac{n}{\alpha}-2i\right)^j b_i \left(\frac{n}{\alpha}\right)}{j!} \\ &\times \int_0^y x^{c+j-1}\, \exp\left(-x^c\right)\, dx. \end{aligned}$$

Further, we have (for $r>0$ and $c>0$)

$$\int_0^y x^{r-1}\, \exp\left(-x^c\right)\, dx = c^{-1}\, \Gamma\Big(\frac{r}{c}, y^c\Big),$$

and then

$$m_n(y) = c\,\sigma^n \sum_{i,j=0}^{\infty} \frac{\left(\frac{n}{\alpha}-2i\right)^j b_i \left(\frac{n}{\alpha}\right)}{(c+j)\, j!}\, \gamma\Big(1+\frac{j}{c}, y^c\Big). \tag{3.10}$$

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves, which are useful in economics, reliability, demography, insurance and medicine. For a given probability π, they are defined by $B(\pi) = m_1(q)/(\pi\,\mu_1')$ and $L(\pi) = m_1(q)/\mu_1'$, respectively, where $m_1(q)$ can be determined from (3.10) with $n=1$ and $q=Q(\pi)$ is calculated from (3.1).

The amount of scatter in a population is measured to some extent by the totality of deviations from the mean and median defined by $\delta_1 = \int_0^\infty |x-\mu_1'|\, g(x)dx$ and $\delta_2(x) = \int_0^\infty |x-M|\, g(x)dx$, respectively, where μ_1' denotes the mean and $M=Q(0.5)$ denotes the median of X. These measures can be determined from $\delta_1 = 2\mu_1'\, G(\mu_1') - 2m_1(\mu_1')$ and $\delta_2 = \mu_1' - 2m_1(M)$, where $G(\mu_1')$ is evaluated from (2.1) and $m_1(\cdot)$ is given by (3.10) with $n=1$.

A further application of the first incomplete moment is related to the mean residual life given by $v(t) = \left[\mu_1' - m_1(t)\right]/[1-G(t)]$.

Table 1 provides the mode, median, mean, variance, skewness and kurtosis of the WPC for various values of c and α with fixed $\sigma=1$ From Table 1 we note that, for fixed $c\geq 2$, the mean and variance are decreasing functions of α. Also, the results in Table 1 indicate that the WPC distribution can be left-skewed or right-skewed.

Table 1. Mode, median, mean, variance, skewness and kurtosis of X for some values of c and α with $\sigma = 1$.

c	α	Mode	Median	Mean	Variance	Skewness	Kurtosis
2	0.3	0	1.9860	18.9243	38090.4964	1875.3993	7.923E+08
	0.4	0	1.6730	5.9347	424.6559	64.4304	77248.8555
	0.5	0.0967	1.5094	3.3841	47.3411	15.644	1477.8141
	1.0	0.8544	1.2286	1.4862	1.1753	2.3402	14.9209
	5.0	1.0405	1.0420	1.0424	0.0218	0.0117	3.4171
5	0.3	1.1705	3.0711	4.0730	12.9334	2.0027	9.6076
	0.4	1.5434	2.3199	2.6877	3.1271	1.2934	5.5384
	0.5	1.5910	1.9605	2.1334	1.2741	0.9070	4.1311
	1.0	1.3861	1.4002	1.4086	0.1492	0.1459	2.8982
	5.0	1.0806	1.0696	1.0640	0.0039	-0.5669	3.6058
10	0.3	3.2307	3.5687	3.7241	2.7380	0.5312	3.1587
	0.4	2.5366	2.5965	2.6294	0.8026	0.2289	2.8320
	0.5	2.1529	2.1454	2.1460	0.3539	0.0379	2.7833
	1.0	1.4969	1.4647	1.4498	0.0441	-0.3854	3.1258
	5.0	1.0871	1.0793	1.0752	0.0011	-0.7966	4.1051
15	0.3	3.7492	3.7541	3.7653	1.3377	0.0868	2.7570
	0.4	2.7479	2.6970	2.6777	0.3982	-0.1342	2.8191
	0.5	2.2649	2.2116	2.1886	0.1757	-0.2764	2.9487
	1.0	1.5174	1.4871	1.4722	0.0214	-0.5927	3.5031
	5.0	1.0883	1.0826	1.0795	0.0005	-0.892	4.3953
20	0.3	4.1237	3.8509	3.8197	0.8169	-0.1542	2.8209
	0.4	2.8980	2.7490	2.7170	0.2424	-0.3350	3.0109
	0.5	2.3441	2.2456	2.2186	0.1063	-0.4513	3.1960
	1.0	1.5321	1.4985	1.4852	0.0127	-0.7081	3.7889
	5.0	1.0892	1.0843	1.0818	0.0003	-0.9455	4.5825

3.6. Quantile expansion. The kernel of the qf of X in (3.1) using (3.3) is given by

$$\begin{aligned}&\left\{\cot\left(\frac{\pi}{2}\exp\left\{-\left[-\log(1-u)\right]^{\frac{1}{c}}\right\}\right)\right\}^{\frac{1}{\alpha}}\\ &=\sum_{i=0}^{\infty} b_i\left(\frac{1}{\alpha}\right)\exp\left\{p_i\left[-\log(1-u)\right]^{\frac{1}{c}}\right\},\end{aligned}\tag{3.11}$$

where $p_i = (1/\alpha - 2i)$ and $b_i(1/\alpha) = (\pi/2)^{2i-1/\alpha}\, a_i(1/\alpha)$ (for $i \geq 0$).

We can obtain using Mathematica

$$p_i\left[-\log(1-u)\right]^{\frac{1}{c}} = u^{1/c}\sum_{j=0}^{\infty} q_j^{(i)}\,\frac{u^j}{j!},$$

where $q_j^{(i)} = q_j(p_i, c)$ and $q_0^{(i)} = p_i$, $q_1^{(i)} = p_i/2c$, $q_2^{(i)} = (3+5c)p_i/(12c^2)$, $q_3^{(i)} = (1+5c+6c^2)p_i/(8c^3)$, etc.

The exponential partial Bell polynomials in formal double series expansion are defined by Comtet [12](p.133) as

$$\exp\left(z\sum_{m\geq 1}^{\infty} a_m\,\frac{t^m}{m!}\right) = \sum_{n,k\geq 0}^{\infty}\frac{B_{n,k}}{n!}\,t^n\,z^k,\tag{3.12}$$

where

$$B_{n,k} = B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}) = \sum \frac{n!}{c_1!\, c_2! \ldots (1!)^{c_1} (2!)^{c_2} \ldots}\, a_1^{c_1}\, a_2^{c_2}, \ldots,$$

and the summation takes place over all integers $c_1, c_2, \ldots \geq 0$ such that $c_1 + 2c_2 + 3c_3 + \cdots = n$ and $c_1 + c_2 + c_3 + \cdots = k$.

These exponential partial Bell polynomials can be computed using Maple by IncompleteBellB(n, k, z[1], z[2],. . . ,z[n = k+1]) and using Mathematica by BellY[n,k,{x1,. . . ,n-k+1}].

Using (3.12), we obtain

$$\exp\left(u^{1/c} \sum_{j=0}^{\infty} q_j^{(i)} \frac{u^j}{j!}\right) = \exp\left(q_0^{(i)} u^{1/c}\right) \exp\left(u^{1/c} \sum_{j=1}^{\infty} q_j^{(i)} \frac{u^j}{j!}\right)$$

$$= \exp\left(q_0^{(i)} u^{1/c}\right) \sum_{n,k \geq 0}^{\infty} \frac{B_{n,k}^{(i)}}{n!}\, u^{n+k/c} = \sum_{n,k,j \geq 0}^{\infty} \frac{q_0^{(i)j}\, B_{n,k}^{(i)}}{n! j!}\, u^{n+(k+j)/c}, \tag{3.13}$$

where $B_{n,k}^{(i)} = B_{n,k}(q_1^{(i)}, q_2^{(i)}, \ldots, q_{n-k+1}^{(i)})$ for $n, k, i \geq 0$.

By combining (3.11) and (3.13), we can write

$$Q(u) = \sigma \cot\left(\frac{\pi}{2}\, \exp\left\{-\left[\,-\log(1-u)\right]^{\frac{1}{c}}\right\}\right)^{\frac{1}{\alpha}} = \sum_{n,k,j \geq 0}^{\infty} v_{n,k,j}\, u^{n+(k+j)/c}$$

where (for $n, k, j, i \geq 0$)

$$v_{n,k,j} = \frac{\sigma}{n! j!} \sum_{i=0}^{\infty} b_i(1/\alpha)\, q_0^{(i)j}\, B_{n,k}^{(i)}.$$

Since $0 < u < 1$, we can expand $u^{n+(k+j)/c}$ as follows

$$u^{n+(k+j)/c} = \sum_{r=0}^{\infty} s_r(n + [k+j]/c)\, u^r, \tag{3.14}$$

where

$$s_r(n + [k+j]/c) = \sum_{k=r}^{\infty} (-1)^{r+k} \binom{n + [k+j]/c}{k} \binom{k}{r}.$$

Hence,

$$Q(u) = \sum_{r=0}^{\infty} g_r\, u^r, \tag{3.15}$$

where $g_r = \sum_{n,k,j \geq 0}^{\infty} v_{n,k,j}\, s_r(n + [k+j]/c)$.

Let $W(\cdot)$ be any integrable function in the positive real line. We can write

$$\int_0^{\infty} W(x)\, g(x) dx = \int_0^1 W\left(\sum_{r=0}^{\infty} g_r\, u^r\right) du. \tag{3.16}$$

Equations (3.15) and (3.16) are the main results of this section. In fact, various WPC properties can follow by using the second integral for special $W(\cdot)$ functions, which are usually more simple than if they are based on the first integral. Established algebraic expansions to determine mathematical quantities of X based on these equations can be more efficient then using numerical integration of the pdf (2.2), which can be prone to rounding off errors among others. For the great majority of these quantities, we can adopt twenty terms in the power series (3.15).

3.7. Generating function. We now provide a simple representation for the moment generating function (mgf) of X, say $M(t) = \mathrm{E}(\mathrm{e}^{\mathrm{tX}})$ based on the power series (3.15). We can write

$$M(t) = \int_0^\infty \exp(t\,x)\, g(x)dx = \int_0^1 \exp\left[t\left(\sum_{r=0}^{\infty} g_r\, u^r\right)\right] du.$$

Using (3.12), we obtain

$$M(t) = \mathrm{e}^{h_0\, t} \int_0^1 \exp\left(t \sum_{r\geq 1}^{\infty} h_r\, \frac{u^r}{r!}\right) du = \mathrm{e}^{h_0\, t} \sum_{k\geq 0}^{\infty} \rho_k\, t^k,$$

where $h_r = r!\, g_r$, $\rho_k = \sum_{n\geq 0}^{\infty} B_{n,k}/(n+1)!$ and $B_{n,k} = B_{n,k}(h_1, h_2, \ldots, h_{n-k+1})$.

3.8. Shannon entropy.

3.4. Theorem. *The Shannon entropy of X is given by*

$$\begin{aligned} \eta_X &= \log\left(\frac{\sigma\, 2^{1/\alpha}}{\alpha}\right) + \sigma\left(1+\frac{1}{\alpha}\right)\sec\left(\frac{\pi}{2\alpha}\right) + \sum_{m=0}^{\infty} \tau_m\, \sigma^m \sec\left(\frac{m\,\pi}{2\alpha}\right) \\ &\quad -\Gamma\Big(1+\frac{1}{c}\Big) + \xi\,\Gamma\Big(1-\frac{1}{c}\Big) - \log\, c + 1, \end{aligned} \tag{3.17}$$

where ξ is the well-known Euler constant and

$$\tau_m = \sum_{k=1}^{\infty} \frac{(-2k)^m\, (\pi)^{2k}\, B_{2k}}{\Gamma(m+1)\Gamma(2k+1)} \left[\left(1-\frac{1}{\alpha}\right)\left(2^{2k}-1\right) + (-1)^{k+1}\left(1+\frac{1}{\alpha}\right)\right].$$

Proof. Based on Alzaatreh et al. [8], the Shannon entropy of the Weibull-X family is given by

$$\eta_X = -\mathbb{E}\left\{\log\left[g\Big(G^{-1}\Big[1-\exp(-y)\Big]\Big)\right]\right\} - \Gamma\Big(1+\frac{1}{c}\Big) + \xi\,\Gamma\Big(1-\frac{1}{c}\Big) - \log\, c + 1, \tag{3.18}$$

where $Y \sim PC(\alpha, \sigma)$.

We first obtain $-\mathbb{E}\left\{\log\left[g\Big(G^{-1}\Big[1-\exp(-y)\Big]\Big)\right]\right\}$, where $g(x)$ and $G(x)$ are the pdf and cdf of the WPC distribution. It follows that

$$\begin{aligned} g\Big(G^{-1}\Big[1-\exp(-y)\Big]\Big) &= \frac{2\alpha}{\pi\sigma}\cos\left(\tfrac{\pi}{2}\,\exp(-y)\right)\sin\left(\tfrac{\pi}{2}\,\exp(-y)\right) \\ &\quad \times\Big[\cot\left(\tfrac{\pi}{2}\,\exp(-y)\right)\Big]^{-\frac{1}{\alpha}}. \end{aligned}$$

Applying logarithm to the above equation, and then using power series for the logarithms of trigonometric functions [see Section 1.518 in Gradshteyn and Ryzhik [21],(p.55)], we have

$$\begin{aligned} \log\left[g\Big(G^{-1}\Big[1-\exp(-y)\Big]\Big)\right] &= \log\left(\frac{\alpha}{\sigma}\right) + \frac{1}{\alpha}\log\left(\frac{\pi}{2}\right) - \left(1+\frac{1}{\alpha}\right) y \\ &\quad - \sum_{m=0}^{\infty} \tau_m\, y^m. \end{aligned}$$

Next, solving for (3.18), we obtain

$$\eta_X = -\log\left(\frac{\alpha}{\sigma}\right) - \frac{1}{\alpha}\log\left(\frac{\pi}{2}\right) + \left(1+\frac{1}{\alpha}\right)\mathbb{E}(Y) + \sum_{m=0}^{\infty} \tau_m\, \mathbb{E}(Y^m). \tag{3.19}$$

Finally, equation (3.17) follows immediately from $\mathbb{E}(Y^m) = \sigma^m \sec\left(\frac{m\,\pi}{2\alpha}\right)$ and substituting (3.19) in (3.18). $\square$

3.9. Order Statistics. Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, \ldots, X_n$ is a random sample from the WPC distribution. Let $X_{i:n}$ denote the ith order statistic. Then, the pdf of $X_{i:n}$ can be expressed as

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i, n-i+1)}\, g(x)\, G(x)^{i-1}\, \{1 - G(x)\}^{n-i} \\ &= \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{B(i, n-i+1)}\, g(x)\, G(x)^{j+i-1} \\ &= \sum_{j=0}^{n-i} v_{i,j}\, \underbrace{(i+j)\, g(x)\, G(x)^{i+j-1}}_{h_{i+j}(x)}, \end{aligned}$$

where $B(,\cdot,\cdot)$ is the beta function,

$$v_{i,j} = \frac{(-1)^j}{B(i, n-i+1)\ (i+j)} \binom{n-i}{j}$$

and $h_{i+j}(x)$ is the exponentiated-WPC (for short, EWPC) with power parameter $(i+j)$. Note that the EWPC model does not exist in the literature and the same authors are working on it. It reveals that the pdf of the WPC order statistics is a linear combination of EWPC densities. So, several mathematical quantities of these order statistics can be derived from those of the EWPC distribution.

3.10. Reliability estimation. The reliability parameter R is defined as $R = P(X > Y)$, where X and Y are independent random variables. Numerous applications of the reliability parameter have appeared in the literature such as the area of classical stress-strength model and the breakdown of a system having two components. If X and Y are two continuous random variables with cdfs $F_1(x)$ and $F_2(y)$ and their pdfs $f_1(x)$ and $f_2(y)$, respectively, the reliability R is given by

$$R = P(X > Y) = \int_{-\infty}^{\infty} F_2(x)\, f_1(x)\, dx. \tag{3.20}$$

3.5. Theorem. *Suppose that X and Y are two independent WPC random variables with parameters c_1 and c_2, respectively, but fixed α and σ. Then,*

$$R = 1 - \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\, \Gamma\Big(j\, \frac{c_2}{c_1} + 1\Big). \tag{3.21}$$

Proof. From equations (2.1) and (2.2) and by setting
$u = -\log\left[1 - 2\pi^{-1} \tan^{-1}\left(\frac{x}{\sigma}\right)^{\alpha}\right]$, we have $P(X_1 > X_2) = 1 - I$, where

$$\begin{aligned} I &= c_1 \int_0^{\infty} u^{c_1 - 1} \exp\left(-u^{c_1} - u^{c_2}\right)\, du \\ &= c_1 \sum_{k=0}^{\infty} \frac{(-1)^j}{j!} \int_0^{\infty} u^{c_1 + j c_2 - 1} \exp\left(-u^{c_1}\right)\, du \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\, \Gamma\Big(j \frac{c_2}{c_1} + 1\Big). \end{aligned} \tag{3.22}$$

□

4. Characterization of the WPC distribution

Glänzel [20] provides characterizations for some distributions based on truncated moments. Recently, Tahir et al. [41] proposed a class of distributions named the odd generalized exponential (OGE) family and used the following results from [20] to obtain one characterization for it.

4.1. Theorem. *Let $(\Omega, \Sigma, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $-\infty \le a < b < \infty$. Let $X : \Omega \to H$ be a continuous random variable whose distribution function $F(x)$ is defined on Ω and let q_1 and q_2 be two real functions defined on H such that*

$$\mathbb{E}\left[q_1(X) \mid X \ge x\right] = \mathbb{E}\left[q_2(X) \mid X \ge x\right] \eta(x), \quad x \in H,$$

for some real function η. Consider that $q_1,\ q_2 \in C^1(H)$, $\eta \in C^2(H)$ and $G(x)$ is twice continuously differentiable and strictly monotone function on the set H. Further, assume that the equation $q_2\eta - q_1 = 0$ has no real solution in the interior of H. Then, G is uniquely determined by the functions q_1, q_2 and η. Further, the density function of G is given by

$$G(x) = C \int_a^x \left| \frac{\eta'}{\eta\, q_2 - q_1} \right| \exp\left[-s(t)\right] dt\,,$$

where s is a solution of the differential equation $s' = \frac{\eta'\, q_2}{\eta\, q_2 - q_1}$ and C is a constant.

Theorem 4 provides a characterization for the Weibull-X family (1.9).

4.2. Theorem. *Let $Y : \Omega \to H$ be a continuous random variable. Then, X follows the Weibull-X family of distributions in* (1.9) *if and only if the functions in Theorem 3 can be chosen as*

$$\begin{aligned} q_2(x) &= 2\,\exp\Big\{-[H_f(x)]^c\Big\}, \\ q_1(x) &= [H_f(x)]^c\, q_2(x) \\ &\textit{and} \\ \eta &= [H_f(x)]^c + 0.5, \quad x \in H. \end{aligned}$$

Proof. Using (1.9), we can prove that

$$[1 - F(x)]\ \mathbb{E}\left[q_1(X) \mid X \ge x\right] = \exp\Big\{-2[H_f(x)]^c\Big\}\Big\{[H_f(x)]^c + 0.5\Big\},$$

$$[1 - F(x)]\ \mathbb{E}\left[q_2(X) \mid X \ge x\right] = \exp\Big\{-2[H_f(x)]^c\Big\}$$

and $\eta = [H_f(x)]^c + 0.5$.

Also, $\eta\, q_2 - q_1 = \exp\left\{-[H_f(x)]^c\right\} > 0$, which implies that $\eta\, q_2 - q_1 = 0$ has no solution in the interior of H.

Conversely, if $\eta = [H_f(x)]^c + 0.5$, then

$$s'(x) = \frac{\eta'\, q_2}{\eta\, q_2 - q_1} = 2\, h_f(x)\, [H_f(x)]^{c-1},$$

which implies that $s(x) = 2\,[H_f(x)]^c$. Hence,

$$\int_a^x \left| \frac{\eta'}{\eta\, q_2 - q_1} \right| \exp\Big(-s(t)\Big)\, dt = 1 - \exp\left\{-[H_g(x)]^c\right\},$$

which is the cdf of the Weibull-X family in (1.10). □

4.3. Corollary. *Let $X : \Omega \to H^+$ be a continuous random variable. Then, X follows the WPC distribution if and only if the functions in Theorem 4 can be chosen as*
$q_2(x) = 2\,\exp\left\{-[H_{Ca}(x)]^c\right\}$, $q_1(x) = [H_{Ca}(x)]^c\, q_2(x)$ *and* $\eta = [H_{Ca}(x)]^c + 0.5$.

4.4. Theorem. *Let $Y : \Omega \to H^+$ be a continuous random variable with cdf $F(x)$. Let $\psi(x)$ be a differential function on H^+ such that $\lim_{x\to\infty} \psi(x) = 1$. Then, for $\xi \neq 1$, $\mathbb{E}\left[\psi(x)|X < x\right] = \xi\, \psi(x)$, $x \in H^+$, if and only if $\psi(x) = \left[1 - F(x)\right]^{\frac{1}{\xi}-1}$, $x \in H^+$.*

4.5. Corollary. *If $\psi(x) = \exp\left\{-\left[H_f(x)\right]^c\right\}$ and $\xi = 0.5$ in Theorem 5, then we have a characterization of the Weibull-X family based in a single truncated moment.*

4.6. Corollary. *The WPC distribution can be characterized based in a single truncated moment, if we use $\psi(x) = \exp\left\{-\left[H_{Ca}(x)\right]^c\right\}$ and $\xi = 0.5$ in Theorem 5.*

5. Estimation

Inference can be carried out in three different ways: point estimation, interval estimation and hypothesis tests. Several approaches for parameter point estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimates(MLEs) enjoy desirable properties that can be used when constructing confidence intervals for the model parameters. Large sample theory for these estimates delivers simple approximations that work well in finite samples. Statisticians often seek to approximate quantities such as the density of a test-statistic that depend on the sample size in order to obtain better approximate distributions. The resulting approximation for the MLEs in distribution theory is easily handled either analytically or numerically.

Here, we consider the estimation of the unknown parameters of the new distribution by the maximum likelihood method. Let $x_1, \ldots, x_n$ be observed values from the WPC distribution given by (2.2) with parameters c, α and σ. The log-likelihood function for the vector of parameters $\mathbf{\Theta} = (c, \alpha, \sigma)^\top$ can be expressed as

$$\begin{aligned}
\ell &= n\log(2c\alpha) - n\log(\pi) - n\alpha\log(\sigma) + (\alpha-1)\sum_{i=1}^{n}\log x_i \\
&\quad -\sum_{i=1}^{n}\log\left[1+\left(\frac{x_i}{\sigma}\right)^{2\alpha}\right] - \sum_{i=1}^{n}\log\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right] \\
&\quad +(c-1)\sum_{i=1}^{n}\log\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\right\} \\
&\quad -\sum_{i=1}^{n}\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\right\}^{c}.
\end{aligned}$$

The components of the score vector $U(\mathbf{\Theta})$ are given by

$$\begin{aligned}
U_c &= \frac{n}{c} + \sum_{i=1}^{n}\log\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\right\} \\
&\quad -\sum_{i=1}^{n}\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\right\}^{c} \\
&\quad \times\ \log\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\right\}, \\
U_\alpha &= \frac{n}{\alpha} - n\,\log(\sigma) + \sum_{i=1}^{n}\log\, x_i + 2\sum_{i=1}^{n}\left\{\frac{\left(\frac{x_i}{\sigma}\right)^{\alpha}\log\left(\frac{x_i}{\sigma}\right)}{\left[\pi-2\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\left[1+\left(\frac{x_i}{\sigma}\right)^{2\alpha}\right]}\right\} \\
&\quad -2\,c\sum_{i=1}^{n}\left\{\frac{\left(\frac{x_i}{\sigma}\right)^{\alpha}\log\left(\frac{x_i}{\sigma}\right)\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\right\}^{c-1}}{\left[\pi-2\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\left[1+\left(\frac{x_i}{\sigma}\right)^{2\alpha}\right]}\right\}
\end{aligned}$$

$$
\begin{aligned}
& -2(c-1)\sum_{i=1}^{n}\left\{\frac{\left(\frac{x_i}{\sigma}\right)^{\alpha}\,\log\left(\frac{x_i}{\sigma}\right)\,\left[1+\left(\frac{x_i}{\sigma}\right)^{2\alpha}\right]^{-1}}{\left[\pi-2\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\log\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]}\right\} \\
& -2\sum_{i=1}^{n}\left\{\frac{\log\left(\frac{x_i}{\sigma}\right)}{\left[1+\left(\frac{x_i}{\sigma}\right)^{-2\alpha}\right]}\right\}, \\
U_{\sigma} \quad = \quad & -\frac{n\alpha}{\sigma}-\frac{2\,\alpha}{\sigma}\sum_{i=1}^{n}\left\{\frac{\left(\frac{x_i}{\sigma}\right)^{\alpha}}{\left[\pi-2\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\left[1+\left(\frac{x_i}{\sigma}\right)^{2\alpha}\right]}\right\} \\
& +\frac{2\,c\,\alpha}{\sigma}\sum_{i=1}^{n}\left\{\frac{\left(\frac{x_i}{\sigma}\right)^{\alpha}\left\{-\log\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\right\}}{\left[\pi-2\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\left[1+\left(\frac{x_i}{\sigma}\right)^{2\alpha}\right]}\right\} \\
& +\frac{2\,\alpha\,(c-1)}{\sigma}\sum_{i=1}^{n}\left\{\frac{\left(\frac{x_i}{\sigma}\right)^{\alpha}\,\left[1+\left(\frac{x_i}{\sigma}\right)^{2\alpha}\right]^{-1}}{\left[\pi-2\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]\,\log\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x_i}{\sigma}\right)^{\alpha}\right]}\right\} \\
& +\frac{2\,\alpha}{\sigma}\sum_{i=1}^{n}\left\{\left(\frac{x_i}{\sigma}\right)^{2\alpha}\,\left[1+\left(\frac{x_i}{\sigma}\right)^{2\alpha}\right]^{-1}\right\}.
\end{aligned}
$$

Setting U_c, U_α and U_σ equal to zero and solving the equations simultaneously yields the MLEs $\widehat{\boldsymbol{\Theta}} = (\widehat{c}, \widehat{\alpha}, \widehat{\sigma})^{\intercal}$. They can be solved numerically by using the R-language or any iterative methods such as the NR (Newton-Raphson), BFGS (Broyden-Fletcher-Goldfarb-Shanno), BHHH (Berndt-Hall-Hall-Hausman), NM (Nelder-Mead), SANN (Simulated-Annealing) and L-BFGS-B (Limited-Memory Quasi-Newton code for Bound-Constrained Optimization).

For interval estimation and hypothesis tests, we can use standard likelihood techniques based the observed information matrix, which can be obtained from the authors upon request.

5.1. Simulation study. We evaluate the performance of the maximum likelihood method for estimating the WPC parameters using Monte Carlo simulation for a total of twenty four parameter combinations and the process is repeated 1,000 times. Two different sample sizes $n = 100$ and 300 are considered. The MLEs, biases and the mean square erros (MSEs) of the parameter estimates are listed in Table 2. The MLEs of c, α and σ are determined by solving the nonlinear equations $U(\Theta) = \mathbf{0}$. From Table 2, we note that the ML method performs well for estimating the model parameters. Also, as the sample size increases, the biases (estimate minus actual) and the MSEs of the average estimates of MLEs decrease as expected.

6. Applications

In this section, we provide three applications to real data sets to illustrate the importance of the proposed distribution. The model parameters are estimated by the method of maximum likelihood and five well-recognized goodness-of-fit statistics are evaluated to compare the WPC distribution with other competing models: BHC, KHC, EHC and PC models.

Table 2. Average estimates, biases and MSEs for various parameter values.

	Actual values			Average estimates			Biases			MSEs		
n	c	α	σ	$\tilde{c}$	$\tilde{\alpha}$	$\tilde{\sigma}$	$\tilde{c}$	$\tilde{\alpha}$	$\tilde{\sigma}$	$\tilde{c}$	$\tilde{\alpha}$	$\tilde{\sigma}$
100	0.8	0.5	1.0	0.905	0.490	0.999	0.105	-0.010	-0.001	0.114	0.022	0.199
	0.8	1.0	2.0	0.890	0.988	1.961	0.090	-0.012	-0.039	0.083	0.084	0.169
	0.8	1.5	1.0	0.920	1.464	0.976	0.120	-0.036	-0.024	0.302	0.181	0.023
	0.8	2.0	2.0	0.900	1.964	1.956	0.100	-0.036	-0.044	0.138	0.319	0.048
	1.5	0.5	1.0	2.069	0.477	0.883	0.569	-0.023	-0.117	2.499	0.042	0.151
	1.5	1.0	2.0	2.300	0.936	1.793	0.800	-0.064	-0.207	2.330	0.180	0.313
	1.5	1.5	1.0	2.070	1.482	0.934	0.570	-0.018	-0.066	4.551	0.384	0.038
	1.5	2.0	2.0	2.193	1.871	1.876	0.693	-0.129	-0.124	5.209	0.622	0.121
	2.0	0.5	1.0	3.073	0.478	0.853	1.073	-0.022	-0.147	6.815	0.061	0.200
	2.0	1.0	2.0	3.184	0.968	1.761	1.184	-0.032	-0.239	1.632	0.247	0.369
	2.0	1.5	1.0	3.224	1.485	0.907	1.224	-0.015	-0.093	1.723	0.584	0.063
	2.0	2.0	2.0	3.230	1.927	1.844	1.230	-0.073	-0.156	3.311	0.870	0.182
300	0.8	0.5	1.0	0.826	0.498	0.996	0.026	-0.002	-0.004	0.014	0.007	0.040
	0.8	1.0	2.0	0.818	1.001	1.999	0.018	0.001	-0.001	0.012	0.024	0.054
	0.8	1.5	1.0	0.820	1.503	0.994	0.020	0.003	-0.006	0.013	0.057	0.005
	0.8	2.0	2.0	0.828	1.972	1.992	0.028	-0.028	-0.008	0.014	0.099	0.011
	1.5	0.5	1.0	1.618	0.496	0.962	0.118	-0.004	-0.038	0.249	0.014	0.047
	1.5	1.0	2.0	1.621	0.985	1.948	0.121	-0.015	-0.052	0.209	0.051	0.053
	1.5	1.5	1.0	1.613	1.485	0.982	0.113	-0.015	-0.018	0.195	0.116	0.006
	1.5	2.0	2.0	1.629	1.964	1.969	0.129	-0.036	-0.031	0.231	0.216	0.016
	2.0	0.5	1.0	2.248	0.497	0.951	0.248	-0.003	-0.049	0.886	0.021	0.064
	2.0	1.0	2.0	2.335	0.966	1.903	0.335	-0.034	-0.097	0.496	0.079	0.100
	2.0	1.5	1.0	2.249	1.479	0.974	0.249	-0.021	-0.026	0.759	0.175	0.010
	2.0	2.0	2.0	2.332	1.922	1.943	0.332	-0.078	-0.057	1.820	0.312	0.033

The first real data set represents breaking strengths of 100 yarn given by Duncan (1974). The data are: 66, 117, 132, 111, 107, 85, 89, 79, 91, 97, 138, 103, 111, 86, 78, 96, 93, 101, 102, 110, 95, 96, 88, 122, 115, 92, 137, 91, 84, 96, 97, 100, 105, 104, 137, 80, 104, 104, 106, 84, 92, 86, 104, 132, 94, 99, 102, 101, 104, 107, 99, 85, 95, 89, 102, 100, 98, 97, 104, 114, 111, 98, 99, 102, 91, 95, 111, 104, 97, 98, 102, 109, 88, 91, 103, 94, 105, 103, 96, 100, 101, 98, 97, 97, 101, 102, 98, 94, 100, 98, 99, 92, 102, 87, 99, 62, 92, 100, 96, 98.

The second real data set was originally reported by Proschan [36], which consists of 213 observations on the number of successive failures of the air conditioning system of a fleet of 13 Boeing 720 jet airplanes. The data are: 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71.

The third real data set is taken from Gross and Clark [22](p.105), which gives the relief times of 20 patients receiving an analgesic. The data are: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The MLEs and the measures of goodness-of-fit including the log-likelihood function at the MLEs ($\hat{\ell}$) are evaluated using the L-BFGS-B optimization technique. The measures of goodness-of-fit including the Akaike information criterion (AIC), Bayesian information criterion (BIC), Anderson-Darling (A^*), Cramér–von Mises (W^*) and Kolmogorov-Smirnov (K-S) statistics are computed to compare the fitted models. The A* and W* statistics are described in details in Chen and Balakrishnan [11]. In general, the smaller the values of these statistics, the better the fit to the data. The required computations are carried out in the R-language introduced by R Development Core Team [38].

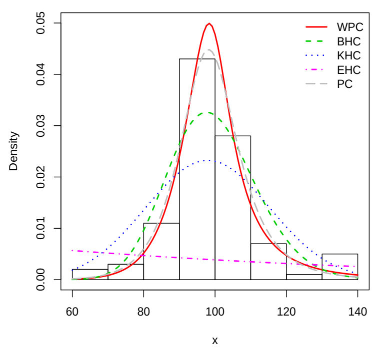
Table 3. MLEs and their standard errors (in parentheses) for data sets 1, 2 and 3.

Distribution	c	α	σ	a	b	θ
			Data set 1			
WPC	0.6964	21.2441	99.5837	-	-	-
	(0.1517)	(5.6077)	(1.0215)	-	-	-
BHC	-	-	163.1591	64.0357	120.3567	-
	-	-	(186.5076)	(23.0558)	(131.5742)	-
KHC	-	-	29.1153	30.8555	364.1721	-
	-	-	(15.7262)	(17.5677)	(107.5154)	-
EHC	-	-	328.1208	-	-	0.2075
	-	-	(115.9236)	-	-	(0.0796)
PC	-	13.8617	98.6978	-	-	-
	-	(1.2646)	(0.9599)	-	-	-
			Data set 2			
WPC	3.2915	0.3467	22.6104	-	-	-
	(1.5537)	(0.1648)	(12.5758)	-	-	-
BHC	-	-	109.1506	1.0056	2.0192	-
	-	-	(56.9868)	(0.1481)	(0.7031)	-
KHC	-	-	116.2468	0.9865	2.0874	-
	-	-	(72.0551)	(0.1413)	(0.8344)	-
EHC	-	-	110.8251	-	-	0.4912
	-	-	(37.4150)	-	-	(0.1391)
PC	-	1.1652	48.8228	-	-	-
	-	(0.0776)	(4.6173)	-	-	-
			Data set 3			
WPC	0.6231	9.4280	1.7542	-	-	-
	(0.2999)	(6.3112)	(0.0960)	-	-	-
BHC	-	-	0.6956	32.7461	10.4453	-
	-	-	(0.6002)	(31.8573)	(3.3862)	-
KHC	-	-	0.3810	23.2228	18.5007	-
	-	-	(0.6300)	(40.8843)	(9.6857)	-
EHC	-	-	6.5222	-	-	0.1994
	-	-	(5.7976)	-	-	(0.1899)
PC	-	5.0276	1.7343	-	-	-
	-	(1.0227)	(0.1076)	-	-	-

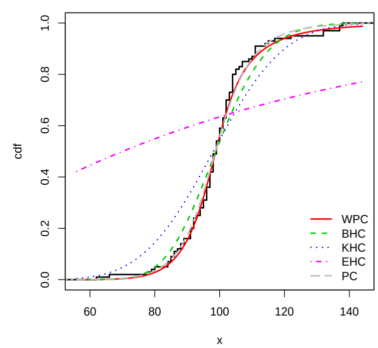
Table 3 lists the MLEs and their corresponding standard errors (in parentheses) of the model parameters for the data sets 1, 2 and 3. The numerical values of the model selection statistics AIC, BIC, A^*, W^* and K-S, and p-values are listed in Table 4. In general, the results from Table 4 indicate that the new distribution provides the best fit among the BHC, KHC, EHC and PC models. The histograms of the data sets 1, 2 and 3, and the estimated pdfs and cdfs of the WPC distribution and its competitive models are displayed in Figures 3, 4 and 5. These figures also support the results in favor of the WPC model as evident from the figures in Table 4.

Table 4. The statistics AIC, BIC, A^*, W^*, K-S and p-value (K-S) for data sets 1, 2 and 3.

Distribution	AIC	BIC	A^*	W^*	K-S	p-value (K-S)
			Data set 1			
WPC	769.4093	777.2248	0.4656	0.0792	0.0785	0.5690
BHC	790.4586	798.2742	2.3288	0.3931	0.1403	0.0391
KHC	810.9693	818.7848	2.7549	0.4561	0.1804	0.0030
EHC	1114.7440	1119.9550	2.3922	0.4017	0.5193	$2.2E^{-16}$
PC	770.0498	775.2601	0.7381	0.1205	0.0870	0.4348
			Data set 2			
WPC	1962.4300	1971.9920	0.4187	0.0624	0.0450	0.8607
BHC	1969.4880	1979.0500	0.7983	0.1208	0.0554	0.6413
KHC	1969.4800	1979.0420	0.8202	0.1251	0.0550	0.6518
EHC	1967.4890	1973.8640	0.8041	0.1219	0.0554	0.6425
PC	1973.1370	1979.5110	0.1377	0.9367	0.0585	0.5730
			Data set 3			
WPC	37.9428	40.9300	0.1497	0.0216	0.0871	0.9981
BHC	38.1601	41.1473	0.2863	0.0491	0.1311	0.8817
KHC	39.9508	42.9380	0.4510	0.0765	0.1519	0.7454
EHC	68.1659	70.1574	0.6283	0.1061	0.4310	0.0012
PC	36.7525	38.7439	0.2628	0.0429	0.0909	0.9965



(a) Estimated pdfs

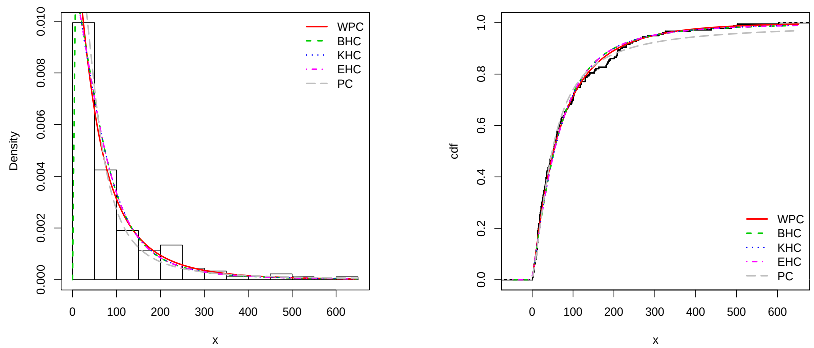


(b) Estimated cdfs

Figure 3. Plots of the estimated pdfs and cdfs of the WPC, BHC, KHC, EHC and PC models for data set 1.

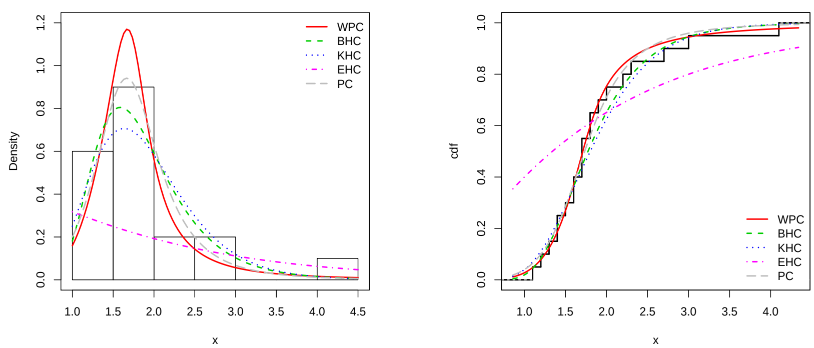
7. Concluding remarks

In this paper, we propose a generalization of the Power-Cauchy (PC) model called the *Weibull-Power-Cauchy (WPC)* distribution. We study some of its mathematical properties including mode, ordinary, incomplete and negative moments, generating and quantile function, mean deviations, mean residual life, Shannon entropy, order statistics and reliability. Two useful characterizations based on truncated moments are formulated for the *Weibull-X* family and the WPC distribution. The advantage of these characterizations is that the cdf is not required to have a closed-form and are given in terms of an integral whose integrand depends on the solution of a first order differential equation. The maximum likelihood method is used for estimating the model parameters. We fit



(a) Estimated pdfs (b) Estimated cdfs

Figure 4. Plots of the estimated pdfs and cdfs of the WPC, BHC, KHC, EHC and PC models for data set 2.



(a) Estimated pdfs (b) Estimated cdfs

Figure 5. Plots of the estimated pdfs and cdfs of the WPC, BHC, KHC, EHC and PC models for data set 3.

the proposed distribution to three real data sets to prove empirically its usefulness. The WPC model provides consistently better fits than other competing models. We hope that the new model will attract wider application in various fields.

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