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Research Article

KKM type maps and collectively coincidence theory

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ABSTRACT. In this paper, we present some properties of KKM maps and then use them to obtain a variety of collectively coincidence results for multivalued maps.

Keywords: Coincidence points, fixed points, set–valued maps.

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1. INTRODUCTION

In this paper using a fixed point theorem in the literature for KKM maps [3, 4, 5] with a result on reduction to finite dimensions, we establish several collectively coincidence results between two classes of set–valued maps defined on Hausdorff topological vector spaces. These classes include almost all the general maps in the literature and in particular we will consider KKM type maps (which include PK type maps), DKT type maps and $HLPY$ type maps. The KKM type maps are either of compact or coercive type. Usually the classes of maps considered are different but first we will present a coincidence result when the classes are the same, namely between maps of KKM type. To obtain our coincidence results we need to present some properties of KKM type maps. Our results extend and complement many papers in the literature; see [2, 5, 6, 7, 10, 11, 12, 13, 14] and the references therein.

We begin by recalling a result of Deguire and Lassonde [6, 16].

Theorem 1.1. *Let X be a compact Hausdorff topological space and let $\{Y_i\}_{i\in J}$ (J an index set) be a family of convex sets each in a Hausdorff topological vector space. For each $i \in J$ suppose $T_i : X \to Y_i$ and assume the following hold:*

$$T_i(x) \text{ has convex values for each } x \in X \tag{1.1}$$

and

$$T_i^{-1}(y_i) \text{ is open (in } X\text{) for each } y_i \in Y_i. \tag{1.2}$$

Also assume for each $x \in X$ that there exists a $j \in J$ with $T_j(x) \neq \emptyset$. Then there exists a subset $D \equiv \prod_{i\in J} D_i$ of $Y \equiv \prod_{i\in J} Y_i$ such that for each $x \in X$ there exists a $i \in J$ with $T_i(x) \cap D_i \neq \emptyset$. Moreover, for each $j \in J$, $D_j \subseteq Y_j$ is a polytope and all of these polytopes, except a finite number, consist of a single point.

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces

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and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$, where $f_{\star\, q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) For each $x \in X$, the set $p^{-1}(x)$ is acyclic,

(ii) p is a perfect map, i.e., p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

(i) p is a Vietoris map,

and

(ii) $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz ([9]). An upper semicontinuous map $\phi : X \to Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. An upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y.

We also discuss the following classes of maps in this paper. Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z, W)$ ([10]) if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup \{\, int\, S^{-1}(w) :\ w \in W\}$; here $S^{-1}(w) = \{z \in Z :\ w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\, S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$, i.e., $S(z) \neq \emptyset$. These maps are related to the DKT maps in the literature and $F \in DKT(Z, W)$ ([7]) if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note these maps were motivated from the $\Phi^\star$ maps. We say $G \in \Phi^\star(Z, W)$ ([2]) if W is convex and there exists a map $S : Z \to W$ with $S(x) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ and has convex values for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$.

Now we consider a general class of maps, namely the PK maps of Park. Let X and Y be Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z :\ Fix\, F \neq \emptyset \ \text{ for all } \ F \in \mathcal{X}(Z, Z)\},$$

where $Fix\, F$ denotes the set of fixed points of F.

The class $\mathcal{U}$ of maps is defined by the following properties:

(i) $\mathcal{U}$ contains the class C of single valued continuous functions;

(ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued;

(iii) $B^n \in \mathcal{F}(\mathcal{U}_c)$ for all $n \in \{1, 2, , \ldots, \}$; here $B^n = \{x \in \mathbf{R}^n :\ \|x\| \leq 1\}$.

We say $F \in PK(X, Y)$ if for any compact subset K of X there is a $G \in \mathcal{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$. Recall PK is closed under compositions ([12]).

Next we describe a class of maps more general than the PK maps in our setting. Let X be a convex subset of a Hausdorff topological vector space and Y a Hausdorff topological space. If $S, T : X \to 2^Y$ are two set valued maps such that $T(co\,(A)) \subseteq S(A)$ for each finite subset A of X then we call S a generalized KKM mapping w.r.t. T. Now the set valued map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM map $S : X \to 2^Y$ w.r.t. T the family $\{\,\overline{S(x)} : x \in X\}$ has the finite intersection property (the intersection of each finite subfamily is nonempty). We let

$$KKM(X,Y) = \{T : X \to 2^Y \,|\, T \text{ has the } KKM \text{ property}\}.$$

Note $PK(X,Y) \subset KKM(X,Y)$ (see [5]). Next we recall the following result [5].

Theorem 1.2. *Let X be a convex subset of a Hausdorff topological vector space and Y, Z be Hausdorff topological spaces.*

(i) $T \in KKM(X,Y)$ iff $T|_{\triangle} \in KKM(\triangle, Y)$ for each polytope $\triangle$ in X;
(ii) if $T \in KKM(X,Y)$ and $f \in C(Y,Z)$ then $f\,T \in KKM(X,Z)$;
(iii) if Y is a normal space, $\triangle$ a polytope of X and if $T : \triangle \to 2^Y$ is a set valued map such that for each $f \in C(Y,\triangle)$ we have that $f\,T$ has a fixed point in $\triangle$, then $T \in KKM(\triangle, Y)$.

Next we recall the following fixed point result for KKM maps. Recall a nonempty subset W of a Hausdorff topological vector space E is said to be admissible if for any nonempty compact subset K of W and every neighborhood V of 0 in E there exists a continuous map $h : K \to W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E (for example every nonempty convex subset of a locally convex space is admissible).

Theorem 1.3 ([4]). *Let X be an admissible convex set in a Hausdorff topological vector space E and $T \in KKM(X,X)$ be a closed compact map. Then T has a fixed point in X.*

Remark 1.1. *One could also consider $s - KKM$ maps* [4] *in this paper and we could obtain similar results to those in Section 2.*

Next we will present an analogue of Theorem 1.2 (ii) for $T\,f$ and this composition will be needed in a few results.

Theorem 1.4. *Let X be an admissible convex set in a Hausdorff topological vector space and Z a subset of a Hausdorff topological space. If $T \in KKM(X,Z)$ is an upper semicontinuous compact map with closed (in fact compact) values and $f \in C(Z,X)$ then $T\,f$ has a fixed point in Z.*

Proof. Now $T \in KKM(X,Z)$, $f \in C(Z,X)$ and Theorem 1.2 (ii) implies $f\,T \in KKM(X,X)$. Also $f\,T$ is a compact upper semicontinuous map with compact values (so $f\,F$ is a closed map [1]). Now Theorem 1.3 guarantees that $f\,T$ has a fixed point in X and consequently $T\,f$ has a fixed point in Z. □

Theorem 1.5. *Let X be an admissible convex set in a Hausdorff topological vector space, Y a convex set in a Hausdorff topological vector space and Y a normal space. If $T \in KKM(X,Y)$ is an upper semicontinuous map with compact values and $f \in C(Y,X)$, then $T\,f \in KKM(Y,Y)$.*

Proof. Note $T\,f : Y \to 2^Y$. From Theorem 1.2 (i), (iii) we need to show that for each polytope $\triangle$ in Y that $g(T\,f)$ has a fixed point in $\triangle$ for any $g \in C(Y,\triangle)$. Note from Theorem 1.2 (ii) since $T \in KKM(X,Y)$ and $g \in C(Y,\triangle)$ that $g\,T \in KKM(X,\triangle)$. Now from Theorem 1.4 (note $Z = \triangle$ is compact and $g\,T : X \to 2^{\triangle}$ is an upper semicontinuous compact map with compact values) guarantees that $(g\,T)\,f$ has a fixed point in $\triangle$. □

In Section 2, we will make use of the following two properties. Let C and X be convex subsets of a Hausdorff topological vector space E with $C \subseteq X$ and Y a Hausdorff topological space.

(i) If $T \in KKM(X,Y)$ then $G \equiv T|_C \in KKM(C,Y)$.

This can be seen from Theorem 1.2 (i). Note $T \in KKM(X,Y)$ so $T|_\triangle \in KKM(\triangle, Y)$ for each polytope $\triangle$ in X from Theorem 1.2 (i). Thus in particular for any polytope $\triangle$ in C we have $T|_\triangle \in KKM(\triangle, Y)$ so from Theorem 1.2 (i) we have $T|_C \in KKM(C,Y)$.

Alternatively we can prove it directly as follows. Let $S : C \to 2^Y$ be a generalized KKM map w.r.t. G, i.e., $G(co\,(A)) \subseteq S(A)$ for each finite subset A of C. We must show $\{\,\overline{S(x)} : x \in C\}$ has the finite intersection property. To see this let $S^\star : X \to 2^Y$ be given by

$$S^\star(x) = \begin{cases} S(x), & x \in C \\ Y, & x \in X \setminus C. \end{cases}$$

We claim $T(co\,(D)) \subseteq S^\star(D)$ for each finite subset D of X. Now either (a) $x \in C$ for all $x \in D$ or (b) there exists a $y \in D$ with $y \notin C$. Suppose first case (b) occurs. Then since $S^\star(y) = Y$ we have

$$T(co\,(D)) \subseteq Y = S^\star(y) = S^\star(D).$$

It remains to consider case (a). Then since C is convex we have $co\,(D) \subseteq C$ and since $S^\star(z) = S(z)$ for $z \in C$ we have

$$T(co\,(D)) = G(co\,(D)) \subseteq S(D) = S^\star(D).$$

Thus $S^\star : X \to 2^Y$ is a generalized KKM map w.r.t. T. Since $T \in KKM(X,Y)$ then $\{\,\overline{S^\star(x)} : x \in X\}$ has the finite intersection property. Now for any finite subset Ω of C (note $S^\star(z) = S(z)$ for $z \in C$) we have

$$\cap_{x\in\Omega}\, \overline{S(x)} = \cap_{x\in\Omega}\, \overline{S^\star(x)} \neq \emptyset,$$

so $G = T|_C \in KKM(C,Y)$.

(ii) If $T \in KKM(X,Y)$, $T(X) \subseteq Z \subseteq Y$ and Z is closed in Y then $T \in KKM(X,Z)$.

Let $S : X \to 2^Z$ be a generalized KKM map w.r.t. T, i.e., $T(co\,(A)) \subseteq S(A)$ for each finite subset A of X. We must show $\{\,\overline{S(x)}^Z : x \in X\}$ has the finite intersection property. Note since $S : X \to 2^Y$ is a generalized KKM map w.r.t. T then since $T \in KKM(X,Y)$ we have that $\{\,\overline{S(x)}\,(= \overline{S(x)}^Y) : x \in X\}$ has the finite intersection property. However note for $x \in X$ that

$$\overline{S(x)}^Z = \overline{S(x)}^Y \cap Z = \overline{S(x)}^Y\ (= \overline{S(x)})$$

since Z is closed in Y (note $S(X) \subseteq Z$ so $\overline{S(x)}^Y \subseteq Z$). Thus $\{\,\overline{S(x)}^Z : x \in X\} = \{\,\overline{S(x)}\,(= \overline{S(x)}^Y) : x \in X\}$ has the finite intersection property.

We also note the following product result. Let I be an index set.

Theorem 1.6. *Let X be a convex set in a Hausdorff topological vector space and $\{Y_i\}_{i\in I}$ be a family of Hausdorff topological spaces. Suppose $T_i \in KKM(X, Y_i)$ for each $i \in I$ and let $T : X \to 2^Y$ (here $Y = \prod_{i\in I} Y_i$) be defined by $T(x) = \prod_{i\in I} T_i(x)$ for $x \in X$. Also assume for each $i \in I$ that there exists a compact set $K_i \subseteq Y_i$ with $T_i(X) \subseteq K_i$. Then $T \in KKM(X,K)$ where $K = \prod_{i\in I} K_i$.*

Proof. Let $S : X \to 2^K$ be a generalized KKM map w.r.t. $T : X \to 2^K$. We must show $\{\,\overline{S(x)} : x \in X\}$ has the finite intersection property. Without loss of generality assume $S(x)$ is closed for each $x \in X$. Let $\{x_j\}_{j=1}^n$ (for some $n \in \mathbf{N}$) be an arbitrary finite family of X. We need to show $S(x_1) \cap, \ldots, \cap S(x_n) \neq \emptyset$.

Fix $i \in I$ and let $P_i : K \to K_i$ be the projection of K onto K_i. Note $T_i = P_i T$ and also note $P_i T(co\,(A)) \subseteq P_i S(A)$, i.e., $T_i(co\,(A)) \subseteq S_i(A)$ for each finite subset A of X, where $S_i = P_i S$, so S_i is a generalized KKM map w.r.t. T_i. Since $T_i \in KKM(X, K_i)$ (see Section 1) then $\overline{S_i(x_1)} \cap, \ldots, \cap \overline{S_i(x_n)} \neq \emptyset$. Note as well, for example, that $\overline{S_i(x_1)} = \overline{P_i S(x_1)} = P_i S(x_1) = S_i(x_1)$ (recall [8, pp. 126] that $P_i : K = \prod_{j \neq i} K_j \times K_i \to K_i$ is a closed map or alternatively note $S(x_1)$ is a compact set). Thus $S_i(x_1) \cap, \ldots, \cap S_i(x_n) \neq \emptyset$. Let $y_i \in S_i(x_1) \cap, \ldots, \cap S_i(x_n)$. We can do this argument for each $i \in I$ and choose y_i (for each $i \in I$) as above. Let $y = (y_i)_{i \in I}$. First note $y_i \in S_i(x_1) = P_i S(x_1)$ for each $i \in I$. Thus $y = (y_i)_{i \in I} \in S(x_1)$. Similarly note $y_i \in S_i(x_2) = P_i S(x_2)$ for each $i \in I$ so $y \in S(x_2)$ and continue to obtain $y \in S(x_3), \ldots, y \in S(x_n)$. Thus $y \in S(x_1) \cap, \ldots, \cap S(x_n)$. □

Remark 1.2. *(i) Suppose we do not assume "$S(x)$ is closed for each $x \in X$" in the proof of Theorem 1.6. Then we follow the proof to obtain $\overline{S_i(x_1)} \cap, \ldots, \cap \overline{S_i(x_n)} \neq \emptyset$. Then note, for example, that $P_i \overline{S(x_1)}$ is a closed set (since $\overline{S(x_1)}$ is a compact set) and so since $S_i(x_1) = P_i S(x_1) \subseteq P_i \overline{S(x_1)}$ we have $\overline{S_i(x)} \subseteq P_i \overline{S(x_1)}$ (in fact we also have $\overline{P_i S(x_1)} \subseteq \overline{P_i S(x_1)}$ since P_i is continuous). As a result $\emptyset \neq \overline{S_i(x_1)} \cap, \ldots, \cap \overline{S_i(x_n)} \subseteq P_i \overline{S(x_1)} \cap, \ldots, \cap P_i \overline{S(x_n)}$. Now let $y_i \in \overline{S_i(x_1)} \cap, \ldots, \cap \overline{S_i(x_n)}$ and $y = (y_i)_{i \in I}$.*

(ii) One could rephrase the statement of Theorem 1.6 as follows. Let X be a convex set in a Hausdorff topological vector space and $\{Y_i\}_{i \in I}$ be a family of Hausdorff topological spaces. For each $i \in I$ suppose $T_i : X \to 2^{Y_i}$ and there exists a compact set $K_i \subseteq Y_i$ with $T_i(X) \subseteq K_i$. In addition assume $T_i \in KKM(X, K_i)$ for each $i \in I$ and let $T : X \to 2^K$ (here $K = \prod_{i \in I} K_i$) be defined by $T(x) = \prod_{i \in I} T_i(x)$ for $x \in X$. Then $T \in KKM(X, K)$

Theorem 1.7. *Let X be an admissible convex compact set in a Hausdorff topological vector space and Y be an admissible convex compact set in a Hausdorff topological vector space. Also assume $F \in KKM(X, Y)$ and $G \in KKM(Y, X)$ are upper semicontinuous maps with compact values. Then $F\,G$ has a fixed point in Y.*

Proof. Let $P_1 : X \times Y \to X$ (respectively, $P_2 : X \times Y \to Y$) be the projections of $X \times Y$ onto X (respectively $X \times Y$ onto Y). Now Theorem 1.5 guarantees that $F\,P_1 \in KKM(X \times Y, Y)$ and $G\,P_2 \in KKM(X \times Y, X)$. Let $H : X \times Y \to X \times Y$ be given by $H(x, y) = G\,P_2(x, y) \times F\,P_1(x, y) = G(y) \times F(x)$ for $(x, y) \in X \times Y$. Now Theorem 1.6 guarantees that $H \in KKM(X \times Y, X \times Y)$ (also note H is an upper semicontinuous compact map with compact values). Now Theorem 1.3 guarantees that there exists a $(x, y) \in X \times Y$ with $(x, y) \in G(y) \times F(x)$. □

Remark 1.3. *(i) One could also obtain an analogue of Theorem 1.7 if one replaces the compactness of the spaces X and Y with compactness of the maps F and G.*

(ii) If X was compact (or more generally paracompact) then X is normal since Hausdorff paracompact spaces are normal ([8]).

Theorem 1.8. *Let X_0 be a convex set in a Hausdorff topological vector space and Y_0 be an admissible convex compact set in a Hausdorff topological vector space. Let Z_0 be a subset of a Hausdorff topological space with Z_0 a normal space. Also assume $T \in KKM(X_0, Y_0)$ is an upper semicontinuous map with compact values and $H \in KKM(Y_0, Z_0)$ is an upper semicontinuous map with compact values. Then $H\,T \in KKM(X_0, Z_0)$.*

Proof. Note $H\,T : X_0 \to 2^{Z_0}$. From Theorem 1.2 (i), (iii) (note Z_0 is normal) we need to show that for each polytope $\triangle$ in X that $f(H\,T)$ has a fixed point in $\triangle$ for any $f \in C(Z_0, \triangle)$. Note from Theorem 1.2 (ii) since $H \in KKM(Y_0, Z_0)$ and $f \in C(Z_0, \triangle)$ that $f\,H \in KKM(Y_0, \triangle)$ (also note $f\,H$ is an upper semicontinuous compact map with compact values). Now from above (see (i)) we have $T \in KKM(\triangle, Y_0)$ is an upper semicontinuous compact map with compact values. Theorem 1.7 (with $F = T$, $G = f\,H$, $X = \triangle$, $Y = Y_0$) guarantees that $(f\,H)\,T$ has a fixed point in $\triangle$. □

2. Coincidence results

In this section, we present coincidence results between two classes of set valued maps. Usually the classes are different but we begin by presenting a coincidence result when the classes are the same, namely between maps of KKM type.

Theorem 2.9. *Let X be an admissible convex compact set in a Hausdorff topological vector space and Y be an admissible convex set in a Hausdorff topological vector space and suppose Y is a normal space. Also assume $F \in KKM(X, Y)$ is an upper semicontinuous map with compact values and $G \in KKM(Y, X)$ is an upper semicontinuous map with compact values. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Proof. Now Theorem 1.8 (with $H = F$, $T = G$, $Y_0 = X$, $X_0 = Y$ and $Z_0 = Y$) guarantees that $F\,G \in KKM(Y, Y)$ and $F\,G$ is an upper semicontinuous ([1, pp. 472]), compact map with compact values, so a closed map ([1]). Theorem 1.3 guarantees a $y \in Y$ with $y \in F\,G(y)$. □

Throughout this section I and J will denote index sets.

Theorem 2.10. *Let X be a convex compact admissible set in a Hausdorff topological vector space, let $\{Y_i\}_{i\in J}$ be a family of convex compact sets each in a Hausdorff topological vector space and let $Y = \prod_{j\in J} Y_j$ be admissible. For each $i \in J$ let $F_i \in KKM(X, Y_i)$ be an upper semicontinuous map with compact values and let $G \in KKM(Y, X)$ be an upper semicontinuous map with compact values. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$ (here $F(z) = \prod_{i\in J} F_i(z)$, $z \in X$).*

Proof. Now Theorem 1.6 guarantees that $F \in KKM(X, Y)$ is an upper semicontinuous ([1, pp. 472]), compact map with compact values and note $G \in KKM(Y, X)$ is an upper semicontinuous map with compact values. Now apply Theorem 2.9. □

In the remainder of this section we present a number of collectively coincidence results between two different classes of set valued maps. We begin when one family consists of compact sets and then we extend when the compact family is replaced by either compact or coercive type maps.

Theorem 2.11. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i\in J}$ be a family of compact sets each in a Hausdorff topological space. For each $i \in J$ suppose $F_i : X \equiv \prod_{j\in I} X_j \to Y_i$ and $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values. For each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and in addition there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Finally suppose for each $y \in Y$ there exists a $j \in I$ with $S_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$ (here x_i (respectively, y_j) is the projection of x (respectively, y) on X_i (respectively, Y_j)).*

Proof. From Theorem 1.1 (note Y is compact) there exists a subset $C = \prod_{i\in I} C_i$ of X (as described in Theorem 1.1 so C is homeomorphic to a convex compact set of finite dimension) such that for each $y \in Y$ there exists a $j \in I$ with $S_j(y) \cap C_j \neq \emptyset$.

For $i \in J$ let $F_i^{\star} = F_i|_C$ (the restriction of F_i to C) and note (see (i) in Section 1) that $F_i \in KKM(C, Y_i)$. Now $i \in I$ let $L_i : Y \to C_i$ and $G_i^{\star} : Y \to C_i$ be given by

$$L_i(y) = S_i(y) \cap C_i \quad \text{and} \quad G_i^{\star}(y) = G_i(y) \cap C_i \quad \text{for} \quad y \in Y.$$

For $i \in I$ note $L_i(y) \subseteq G_i^\star(y)$ for $y \in Y$, $L_i(y)$ has convex values for each $y \in Y$ (note S_i is convex valued and C_i is convex) and for $x \in C_i$ note

$$L_i^{-1}(x) = \{z \in Y : x \in L_i(z) = S_i(z) \cap C_i\} = \{z \in Y : x \in S_i(z)\} = S_i^{-1}(x)$$

which is open in Y. Next note for each $y \in Y$ there exists (see the beginning of the proof) a $j \in I$ with $S_j(y) \cap C_j \neq \emptyset$, i.e., $L_j(y) \neq \emptyset$.

Note $E_i = \{y \in Y : L_i(y) \neq \emptyset\}, i \in I$ is an open covering of Y (recall the fibres of L_i are open in Y). Now from [8, Lemma 5.1.6, pp. 301] there exists a covering $\{Q_i\}_{i\in I}$ of Y where Q_i is closed in Y and $Q_i \subset E_i$ for all $i \in I$. For each $i \in I$ let $N_i : Y \to C_i$ and $P_i : Y \to C_i$ be given by

$$N_i(y) = \begin{cases} G_i^\star(y), & y \in Q_i \\ C_i, & y \in Y\backslash Q_i \end{cases} \quad \text{and} \quad P_i(y) = \begin{cases} L_i(y), & y \in Q_i \\ C_i, & y \in Y\backslash Q_i \end{cases}$$

and we claim $N_i \in \Phi^\star(Y, C_i)$. To see this first note for $i \in I$ that $P_i(y) \neq \emptyset$ for $y \in Y$ (if $y \in Q_i$ then $P_i(y) = L_i(y) \neq \emptyset$ since $Q_i \subset E_i$ whereas if $y \in Y\backslash Q_i$ then $P_i(y) = C_i$). Next we note for $i \in I$ that $P_i(y) \subseteq N_i(y)$ for $y \in Y$ (if $y \in Q_i$ then $P_i(y) = L_i(y) = S_i(y) \cap C_i \subseteq G_i(y) \cap C_i = G_i^\star(y) = N_i(y)$ whereas if $y \in Y\backslash Q_i$ then $P_i(y) = C_i = N_i(y)$). Also note for $i \in I$ that $P_i(y)$ is convex valued for $y \in Y$ and for $x \in C_i$ note

$$\begin{aligned} P_i^{-1}(x) = \{z \in Y : x \in P_i(z)\} &= \{z \in Y\backslash Q_i : x \in P_i(z) = C_i\} \cup \{z \in Q_i : x \in L_i(z)\} \\ &= (Y\backslash Q_i) \cup \{z \in Q_i : x \in L_i(z)\} \\ &= (Y\backslash Q_i) \cup [Q_i \cap \{z \in Y : x \in L_i(z)\}] \\ &= (Y\backslash Q_i) \cup \left[Q_i \cap L_i^{-1}(x)\right] \\ &= Y \cap \left[(Y\backslash Q_i) \cup L_i^{-1}(x)\right] = (Y\backslash Q_i) \cup L_i^{-1}(x) \end{aligned}$$

which is open in Y (note $L_i^{-1}(x)$ is open in Y and Q_i is closed in Y). Thus $N_i \in \Phi^\star(Y, C_i)$ for each $i \in I$. Now for each $i \in I$ from [2] there exists a continuous (single valued) selection $g_i : Y \to C_i$ of N_i with $g_i(y) \in P_i(y) \subseteq N_i(y)$ for $y \in Y$.

Let $F(x) = \prod_{i\in J} F_i^\star(x)$ for $x \in C$ and note Theorem 1.6 guarantees that $F \in KKM(C, Y)$ is an upper semicontinuous ([1]) compact map with compact values . Now let $g(y) = \prod_{i\in I} g_i(y)$ for $y \in Y$ and note $g : Y \to C$ is continuous. Now Theorem 1.2 (ii) guarantees that $g\,F \in KKM(C, C)$ and note C is a convex compact set of finite dimension so Theorem 1.3 guarantees a $x \in C$ with $x \in g\,F(x)$. Now let $y \in F(x)$ with $x = g(y)$. Note $y \in F(x)$ so $y_j \in F_j^\star(x)$ for all $j \in J$, i.e., $y_j \in F_j(x)$ for all $j \in J$ (note $x \in C$). Also we have $x_i = g_i(y) \in P_i(y) \subseteq N_i(y)$ for $i \in I$. Now since $\{Q_i\}_{i\in I}$ is a covering of Y then there exists a $i_0 \in I$ with $y \in Q_{i_0}$ so $x_{i_0} \in N_{i_0}(y) = G_{i_0}^\star(y) = G_{i_0}(y) \cap C_{i_0}$, i.e., $x_{i_0} \in G_{i_0}(y)$. □

Remark 2.4. *Since $PK(X, Y_i) \subseteq KKM(X, Y_i)$ for each $i \in J$ then one could replace KKM with PK in the statement of Theorem 2.11 (i.e., "for each $i \in J$ suppose $F_i \in KKM(X, Y_i)$" is replaced with "for each $i \in J$ suppose $F_i \in PK(X, Y_i)$" in the statement of Theorem 2.11). Note Theorem 1.6 will again guarantees that $F \in KKM(C, Y)$ in the proof of Theorem 2.11.*

Now Theorem 2.11 will generate a result for compact type maps.

Theorem 2.12. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i\in J}$ be a family of sets each in a Hausdorff topological space. For each $i \in J$ suppose $F_i : X \equiv \prod_{j\in I} X_j \to Y_i$ and $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values and in addition there exists a compact set K_i with $F_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and in addition there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Finally suppose for each $y \in Y$ there* *exists a $j \in I$ with $S_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. Note $K \equiv \prod_{i\in J} K_i\,(\subseteq Y)$ is compact. For $i \in I$ let $G_i^{\star\star} = G_i|_K$ and $S_i^{\star\star} = S_i|_K$ and note $G_i^{\star\star} : K \to X_i$ and $S_i^{\star\star} : K \to X_i$. Note for $i \in I$ that $S_i^{\star\star}(y) \subseteq G_i^{\star\star}(y)$ for $y \in K$ and $S_i^{\star\star}(y)$ has convex values for $y \in K$. Also for $i \in I$ and $x \in X_i$ note

$$(S_i^{\star\star})^{-1}(x) = \{z \in K : x \in S_i^{\star\star}(z)\} = K \cap \{z \in Y : x \in S_i(z)\} = K \cap S_i^{-1}(x)$$

which is open in $K \cap Y = K$.

Next note if $y \in K$ then there exists a $i_0 \in I$ with $S_{i_0}^{\star\star}(y) \neq \emptyset$ (since for any $w \in Y$ there exists a $j \in I$ with $S_j(w) \neq \emptyset$). Finally note for $i \in J$ since $F_i \in KKM(X, Y_i)$ and K_i is closed then (see (ii) in section 1) $F_i \in KKM(X, K_i)$. Now apply Theorem 2.11 with $\{X_i\}_{i\in I}$, $\{K_i\}_{i\in J}$, F_i, $G_j^{\star\star}$ and $S_j^{\star\star}$ so there exists a $x \in X$, a $y \in K$ and a $i_0 \in I$ with $y_j \in F_j(y)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}^{\star\star}(y)\,(= G_{i_0}(y))$. □

Remark 2.5. *In the statement of Theorem 2.12 we could replace "for each $i \in J$ suppose $F_i \in KKM(X, Y_i)$" with "for each $i \in J$ suppose $F_i \in PK(X, Y_i)$".*

Now we present a result for coercive type maps.

Theorem 2.13. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i\in J}$ be a family of sets each in a Hausdorff topological space. For each $i \in J$ suppose $F_i : X \equiv \prod_{j\in I} X_j \to Y_i$ and $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values. For each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and in addition there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Also suppose for each $y \in Y$ there exists a $j \in I$ with $S_j(y) \neq \emptyset$. Finally assume there is a compact subset K of Y and for each $i \in I$ a convex compact subset Z_i of X_i such that for each $y \in Y\backslash K$ there exists a $j \in I$ with $S_j(y) \cap Z_j \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. From Theorem 1.1 (note K is compact) there exists a subset $C = \prod_{i\in I} C_i$ of X (as described in Theorem 1.1 so C is homeomorphic to a convex compact set of finite dimension) such that for each $y \in K$ there exists a $j \in I$ with $S_j(y) \cap C_j \neq \emptyset$. Also (see the statement of Theorem 2.13) for each $y \in Y\backslash K$ there exists a $i \in I$ with $S_i(x) \cap Z_i \neq \emptyset$. Let

$$\Omega_i = co\,(Z_i \cup C_i) \;\text{ for }\; i \in I$$

which is a convex compact (see [1, pp. 126]) subset of X_i. Let $\Omega = \prod_{i\in I} \Omega_i$ which is a convex compact subset of X. For each $i \in I$ let $S_i^\star : Y \to \Omega_i$ and $G_i^\star : Y \to \Omega_i$ be given by

$$S_i^\star(y) = S_i(y) \cap \Omega_i \;\text{ and }\; G_i^\star(y) = G_i(y) \cap \Omega_i \;\text{ for }\; y \in Y.$$

For $i \in I$ note $S_i^\star(y) \subseteq G_i^\star(y)$ for $y \in Y$, $S_i^\star(y)$ has convex values for each $y \in Y$ and for $x \in \Omega_i$ note

$$(S_i^\star)^{-1}(x) = \{z \in Y : x \in S_i^\star(z)\} = \{z \in Y : x \in S_i(z) \cap \Omega_i\} = \{z \in Y : x \in S_i(z)\} = S_i^{-1}(x)$$

which is open in Y. Finally for each $y \in Y$ there exists a $i_0 \in I$ with $S_{i_0}^\star(y) \neq \emptyset$. To see this let $y \in Y$. Then either $y \in K$ or $y \in Y\backslash K$. If $y \in K$ then there exists a $i \in I$ with $S_i(y) \cap C_i \neq \emptyset$ so $S_i(y) \cap \Omega_i \neq \emptyset$ since $C_i \subseteq \Omega_i$. If $y \in Y\backslash K$ then there exists a $j \in I$ with $S_j(y) \cap Z_j \neq \emptyset$ so $S_j(y) \cap \Omega_j \neq \emptyset$ since $Z_j \subseteq \Omega_j$.

For $i \in J$ let $F_i^\star = F_i|_\Omega$ and note (see (i) in Section 1) that $F_i^\star \in KKM(\Omega, Y_i)$ is upper semicontinuous with compact values. Also for $i \in J$ let $K_i = F_i^\star(\Omega) = F_i(\Omega)$ which is a compact subset of Y_i (note $F_i^\star$ is upper semicontinuous with compact values and Ω is compact).

Now we apply Theorem 2.12 with $\{\Omega_i\}_{i\in I}$, $\{Y_i\}_{i\in J}$, $F_i^\star$, $S_i^\star$ and $G_i^\star$ so there exists a $x \in \Omega$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in F_j^\star(x) = F_j(y)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}^\star(y)\,(= G_{i_0}(y) \cap \Omega_{i_0})$. □

Remark 2.6. *In the statement of Theorem 2.13 we could replace "for each $i \in J$ suppose $F_i \in KKM(X, Y_i)$" with "for each $i \in J$ suppose $F_i \in PK(X, Y_i)$".*

Next we will extend the results (Theorem 2.11, Theorem 2.12 and Theorem 2.13) to DKT and $HLPY$ type maps.

Theorem 2.14. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i\in J}$ be a family of compact sets each in a Hausdorff topological space. For each $i \in J$ suppose $F_i : X \equiv \prod_{j\in I} X_j \to Y_i$ and $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values. For each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and in addition there exists a map $T_j : Y \to X_j$ with $co\,(T_j(y)) \subseteq G_j(y)$ for $y \in Y$ and $T_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Finally suppose for each $y \in Y$ there exists a $j \in I$ with $T_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$ and an $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For $j \in I$ let $S_j(y) = co\,(T_j(y))$ for $y \in Y$. For $j \in I$, first note $S_j(y)$ has convex values for each $y \in Y$ and note $S_j(y) \subseteq G_j(y)$ for $y \in Y$. In addition for $j \in I$ from [15, Lemma 5.1] we have that $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Finally note if $y \in Y$ then there exists a $i_0 \in I$ with $T_{i_0}(y) \neq \emptyset$ and so $\emptyset \neq T_{i_0}(y) \subseteq co\,(T_{i_0}(y)) = S_{i_0}(y)$. Now Theorem 2.11 guarantees that there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$. □

Theorem 2.15. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i\in J}$ be a family of compact sets each in a Hausdorff topological space. For each $i \in J$ suppose $F_i : X \equiv \prod_{j\in I} X_j \to Y_i$ and $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values. For each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and in addition there exists a map $T_j : Y \to X_j$ with $co\,(T_j(y)) \subseteq G_j(y)$ for $y \in Y$ and $Y = \bigcup_{j\in I} \bigcup \{\, int\, T_j^{-1}(w) :\ w \in X_j\}$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For $j \in I$ let $R_j : Y \to X_j$ be given by

$$R_j(y) = \{z_j :\ y \in int\, T_j^{-1}(z_j)\},\ y \in Y$$

and let $S_j : Y \to X_j$ be given by

$$S_j(y) = co\,(R_j(y)) \ \text{ for } \ y \in Y.$$

For $j \in I$ first note that $S_j(y)$ has convex values for each $y \in Y$ and also note that $R_j(y) \subseteq T_j(y)$ for $y \in Y$ since if $z_j \in R_j(y)$ then $y \in int\, T_j^{-1}(z_j) \subseteq T_j^{-1}(z_j) = \{w \in Y :\ z_j \in T_j(w)\}$ so $z_j \in T_j(y)$ and putting this together yields $R_j(y) \subseteq T_j(y)$ for $y \in Y$. Thus for $j \in I$ we have $S_j(y) = co\,(R_j(y)) \subseteq co\,(T_j(y)) \subseteq G_j(y)$ for $y \in Y$.

Now for $j \in I$ notice for $x_j \in X_j$ that $R_j^{-1}(x_j) = \{z :\ x_j \in R_j(z)\} = int\, T_j^{-1}(x_j)$ so $R_j^{-1}(x_j)$ is open (in Y) and so from [15, Lemma 5.1] we have that $S_j^{-1}(x_j)$ is open (in Y).

Now let $y \in Y$. Since $Y = \bigcup_{j\in I} \bigcup \{\, int\, T_j^{-1}(w) :\ w \in X_j\}$ there exists a $j \in I$ with $y \in int\, T_j^{-1}(w)$ for some $w \in X_j$ and so $w \in R_j(y)$, i.e., $R_j(y) \neq \emptyset$ and as a result $\emptyset \neq R_j(y) \subseteq co\,(R_j(y)) = S_j(y)$. Now Theorem 2.11 guarantees the result. □

Remark 2.7. *In the statement of Theorem 2.14 and Theorem 2.15 we could replace "for each $i \in J$ suppose $F_i \in KKM(X, Y_i)$" with "for each $i \in J$ suppose $F_i \in PK(X, Y_i)$".*

Now we obtain an analogue of Theorem 2.12 for DKT or $HLPY$ type maps.

Theorem 2.16. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i\in J}$ be a family of sets each in a Hausdorff topological space. For each $i \in J$ suppose $F_i : X \equiv \prod_{j\in I} X_j \to Y_i$ and $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values and in addition there exists a compact set K_i with $F_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and in addition there exists a map $T_j : Y \to X_j$ with $co\,(T_j(y)) \subseteq G_j(y)$ for $y \in Y$ and $T_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Also suppose for each $y \in Y$ there exists a $j \in I$ with $T_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For $j \in I$ let S_j be as in Theorem 2.14 and note S_j has the same properties as in Theorem 2.14 and now apply Theorem 2.12 to obtain the result. □

Theorem 2.17. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i\in J}$ be a family of sets each in a Hausdorff topological space. For each $i \in J$ suppose $F_i : X \equiv \prod_{j\in I} X_j \to Y_i$ and $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values and in addition there exists a compact set K_i with $F_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and in addition there exists a map $T_j : Y \to X_j$ with $co\,(T_j(y)) \subseteq G_j(y)$ for $y \in Y$ and $Y = \bigcup_{j\in I} \bigcup \{\, int\, T_j^{-1}(w) : \; w \in X_j\}$. Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For $j \in I$ let R_j and S_j be as in Theorem 2.15 and note S_j has the same properties as in Theorem 2.15 and now apply Theorem 2.12 to obtain the result. □

Remark 2.8. *In the statement of Theorem 2.16 and Theorem 2.17 we could replace "for each $i \in J$ suppose $F_i \in KKM(X, Y_i)$" with "for each $i \in J$ suppose $F_i \in PK(X, Y_i)$.*

Now we present our two results for coercive type maps.

Theorem 2.18. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i\in J}$ be a family of sets each in a Hausdorff topological space. For each $i \in J$ suppose $F_i : X \equiv \prod_{j\in I} X_j \to Y_i$ and $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values. For each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and in addition there exists a map $T_j : Y \to X_j$ with $co\,(T_j(y)) \subseteq G_j(y)$ for $y \in Y$ and $T_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Also suppose for each $y \in Y$ there exists a $j \in I$ with $T_j(y) \neq \emptyset$. Finally assume there is a compact subset K of Y and for each $i \in I$ a convex compact subset Z_i of X_i such that for each $y \in Y\backslash K$ there exists a $j \in I$ with $T_j(y) \cap Z_j \neq \emptyset$ (or, alternatively $co\,(T_j(y)) \cap Z_j \neq \emptyset$). Then there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For $i \in I$ let S_i be as in Theorem 2.14 and note S_i has the same properies as in Theorem 2.14. Let $y \in Y\backslash K$. We need only consider the case when there exists a $j \in I$ with $T_j(y) \cap Z_j \neq \emptyset$, but here $\emptyset \neq T_j(y) \cap Z_j \subseteq co\,(T_j(y)) \cap Z_j = S_j(y) \cap Z_j$. Now apply Theorem 2.13. □

Theorem 2.19. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i\in J}$ be a family of sets each in a Hausdorff topological space. For each $i \in J$ suppose $F_i : X \equiv \prod_{j\in I} X_j \to Y_i$ and $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values. For each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and in addition there exists a map $T_j : Y \to X_j$ with $co\,(T_j(y)) \subseteq G_j(y)$ for $y \in Y$ and $Y = \bigcup_{j\in I} \bigcup \{\, int\, T_j^{-1}(w) : \; w \in X_j\}$. Finally assume there is a compact subset K of Y and for each $i \in I$ a convex compact subset Z_i of X_i such that for each $y \in Y\backslash K$ there exists a $j \in I$ with $R_j(y) \cap Z_j \neq \emptyset$ (or, alternatively $co\,(R_j(y)) \cap Z_j \neq \emptyset$) where $R_j : Y \to X_j$ is given by $R_j(y) = \{z_j : \; y \in int\, T_j^{-1}(z_j)\}$ for $y \in Y$. Then there exist a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For $i \in I$ let S_i be as in Theorem 2.15 and note S_i has the same properies as in Theorem 2.15. Let $y \in Y \backslash K$. We need only consider the case when there exists a $j \in I$ with $R_j(y) \cap Z_j \neq \emptyset$, but here $\emptyset \neq R_j(y) \cap Z_j \subseteq co\,(R_j(y)) \cap Z_j = S_j(y) \cap Z_j$. Now apply Theorem 2.13. □

Remark 2.9. *In the statement of Theorem 2.18 and Theorem 2.19 we could replace "for each $i \in J$ suppose $F_i \in KKM(X, Y_i)$" with "for each $i \in J$ suppose $F_i \in PK(X, Y_i)$.*

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