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On a conditioned Limit Structure of the Markov Branching Process

Azam Abdurakhimovich Imomov*

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Abstract: The principal aims are to investigate asymptotic properties of the stochastic population process as a continuous-time Markov chain called Markov Q-Process. We investigate asymptotic properties of the transition probabilities of the Markov Q-Process and their convergence to stationary measures.

Keywords: *Markov Branching processes; Markov Q-processes; transition function; q-matrix; limit theorems.*

1. Introduction and Preliminaries

Introducing the population of monotype individuals that are capable to perish and transforms into individuals of random number of the same type, we are interested in its evolution.

We consider an evolution process of this population in which each individual existing at epoch $t\in\mathcal{T}=[0;+\infty)$, independently of his history and of each other for a small time interval $(t;t+\varepsilon)$ transforms into $j\in\mathbb{N}_0\setminus\{1\}$ individuals with probability $a_j\varepsilon+o(\varepsilon)$ and, with probability $1+a_1\varepsilon+o(\varepsilon)$ each individual survives or makes evenly one descendant (as $\varepsilon\downarrow 0$), where $\mathbb{N}_0=\{0\}\cup\mathbb{N}$ and $\mathbb{N}=\{1,2,\ldots\}$. Here a_j are intensities of transformation that $a_j\geq 0$ for $j\in\mathbb{N}_0\setminus\{1\}$ and $0<a_0<-a_1=\sum_{j\in\mathbb{N}_0\setminus\{1\}}a_j<\infty$. Appeared new individuals undergo transformations under same way as above. Letting $Z(t), t\in\mathcal{T}$ be the population size at the moment $t\in\mathcal{T}$, we have the homogeneous continuous-time Markov Branching Process (MBP) which was first considered by Kolmogorov and Dmitriev [9]. The process $Z(t), t\in\mathcal{T}$ is a Markov chain with the state space on $\mathbb{N}_0$. The Markovian nature of this process yields that its transition functions

$$P_{ij}(t):=\mathsf{P}_i\{Z(t)=j\}=\mathsf{P}\{Z(t+\tau)=j\,|\,Z(\tau)=i\},$$

$\tau\in\mathcal{T}$, satisfy the branching property

$$P_{ij}(t)=\sum_{j_1+j_2+\cdots+j_i=j}P_{1j_1}(t)\cdot P_{1j_2}(t)\cdots P_{1j_i}(t). \tag{1.1}$$

for all $i,j\in\mathbb{N}_0$. Thus, for studying of evolution of the process $Z(t), t\in\mathcal{T}$ is suffice to set the transition functions $P_{1j}(t)$. These in turn, as it has been noted, are calculated using the local densities a_j by relation

$$P_{1j}(\varepsilon)=\delta_{1j}+a_j\varepsilon+o(\varepsilon),\quad \varepsilon\downarrow 0, \tag{1.2}$$

where δ_{ij} represents Kronecker's delta function. Probability generating functions (GFs) are the main analytical tool in our discussions on MBP. A GF version of relation (1.2) is

$$F(\rho;s)=s+f(s)\cdot\rho+o(\rho),\quad \rho\downarrow 0,$$

for all $0\leq s<1$, where

$$F(t;s)=\sum_{j\in\mathbb{N}_0}P_{1j}(t)s^j \quad\text{and}\quad f(s)=\sum_{j\in\mathbb{N}_0}a_js^j.$$

Using property (1.1) it is easy to see

$$F_i(t;s):=\sum_{j\in\mathbb{N}_0}P_{ij}(t)s^j=\left[F(t;s)\right]^i,\quad i\in\mathbb{N}. \tag{1.3}$$

By the formula (1.3) can be computed that

$$\mathsf{E}_iZ(t)=\sum_{j\in\mathbb{N}_0}jP_{ij}(t)=ie^{at}.$$

The last formula shows that long-term properties of MBP seem variously depending on the parameter a. Hence the MBP is classified as critical if $a=0$ and sub-critical or supercritical if $a<0$ or $a>0$ respectively.

The probability $P_{i0}(t)=\mathsf{P}_i\{Z(t)=0\}$ denotes the MBP is dying out in time $t\in\mathcal{T}$. The extinction probability $q=\lim_{t\to\infty}P_{10}(t)$ can be considered as the probability of all descendants of one initial particle eventually will be lost. Sevastyanov [15] proved that is the least non-negative root of $f(s)=0$ and that $q=1$ if the process is non-supercritical. It directly follows from last results that $q=F(t;q)$ for any $t\in\mathcal{T}$. Moreover $F(t;s)\to q$ as $t\to\infty$ uniformly for all $0\leq s\leq r<1$. Owing to the last assertion the function $R(t;s)=q-F(t;s)$ plays an important role in observing limit behaviors of MBP.

Let the random variable $\mathcal{T}=\inf\{t\in\mathcal{T}:Z(t)=0\}$ be the extinction time of MBP. An asymptote of probability of this variable has first been observed by Sevastyanov [15]. Exertions of this variable treated also by Heatcote, Seneta, Vere-Jones [4], Nagaev, Badalbaev [11] and Zolotarev [17]. Put the conditioned distribution

$$\mathsf{P}_i^{\mathcal{T}(t)}\{*\}:=\mathsf{P}_i\{*\,|\,t<\mathcal{T}<\infty\}.$$

In the discrete-time situation probabilities $\mathsf{P}_i^{\mathcal{T}(t+\tau)}\{*\}$ converges as $\tau\to\infty$ to a distribution measure, which defines the Markov chain called the Q-process; see Athreya and Ney [2, pp.56–60]. The Q-process was considered first by Lamperti and Ney [10]. Some properties of it were discussed by Pakes [12–14], Formanov and Imomov [3] and by author [8].

A closer look shows that in MBP case the limit $\lim_{\tau\to\infty}\mathsf{P}_i^{\mathcal{T}(t+\tau)}\{Z(t)=j\}$ has an honest probability measures $\mathbf{Q}(t)\equiv\{\mathcal{Q}_{ij}(t)\}$ which defines the homogeneous continuous-time stochastic process as Markov chain called the Markov Q-Process; see Imomov [5–7]. Let $W(t)$ be the state size in the Markov Q-Process at the moment $t\in\mathcal{T}$. Then $W(0)\overset{d}{=}Z(0)$ and

* *State Testing Centre under the Cabinet of Ministers of Uzbekistan;*
Institute of Mathematics at the National University of Uzbekistan.
* *Email: imomov_azam@mail.ru*

$$\mathsf{P}_i\{W(t)=j\}=\mathcal{Q}_{ij}(t)=\lim_{\tau\to\infty}\mathsf{P}_i^{T(t+\tau)}\{Z(t)=j\}.$$

The paper is devoted to investigate structural and asymptotic properties of the Markov Q-Process. In particular we compute the q-matrix and the GF version of Kolmogorov backward equation implied by $\mathbf{Q}(t)$.

2. The Markov Q-process

As it was said in first Section the limit of conditional transition function $\mathsf{P}_i^{T(t+\tau)}\{Z(t)=j\}$ as $\tau\to\infty$ and for all $t\in\mathcal{T}$ is an honest probability measure $\mathbf{Q}$. This measure defines so-called Markov Q-Process (MQP) to be the continuous-time Markov chain $\{W(t), t\in\mathcal{T}\}$ with the state space $\mathcal{E}\subset\mathbb{N}$. The transition function $\mathbf{Q}(t)=\{\mathcal{Q}_{ij}(t)\}$ is form of (see[6])

$$\mathcal{Q}_{ij}(t)=\lim_{\tau\to\infty}\mathsf{P}_i^{T(t+\tau)}\{Z(t)=j\}=\frac{jq^{j-i}}{i\beta^t}P_{ij}(t), \qquad (2.1)$$

for $i,j\in\mathcal{E}$, where $\beta=\exp\{f'(q)\}$ and $f(q)=0$. It is easy to be convinced that $0<\beta\le 1$ decidedly. To wit $\beta=1$ if $a=0$ and $\beta<1$ otherwise. The random function $W(t)$ is the state size at the moment $t\in\mathcal{T}$ in MQP.

Combining equalities (1.2) and (2.1) we obtain the following representation for probabilities $\mathcal{Q}_{1j}(\varepsilon)$:

$$\mathcal{Q}_{1j}(\varepsilon)=\delta_{1j}+p_j\varepsilon+o(\varepsilon),\quad j\in\mathcal{E}, \qquad (2.2)$$

as $\varepsilon\downarrow 0$, with probability densities $p_0=0$, $p_1=a_1-\ln\beta$, and $p_j=jq^{j-1}a_j\ge 0$ for $j\in\mathcal{E}\setminus\{1\}$, where a_j are evolution intensities of MBP $Z(t)$. It follows from (2.2) that GF of intensities p_j has the form of

$$g(s):=\sum_{j\in\mathcal{E}}p_js^j=s\left[f'(qs)-f'(q)\right]. \qquad (2.3)$$

Needles to see that this GF is infinitesimal because $g(1)=\sum_{j\in\mathcal{E}}p_j=0$. So the GF $g(s)$ completely defines the process $\{W(t), t\in\mathcal{T}\}$, where p_j are intensities of process evolution that $p_j>0$ for $j\in\mathcal{E}\setminus\{1\}$ and

$$0<-p_1=\sum_{j\in\mathcal{E}\setminus\{1\}}p_j<\infty.$$

2.1. Construction, existence and uniqueness

Let's now discuss basic properties of transition matrix $\mathbf{Q}(t)=\{\mathcal{Q}_{ij}(t)\}$. Herewith we will follow methods and facts from monograph of Anderson [1]. First we prove the following theorem.

Theorem 1. *Let $\{W(t), t\in\mathcal{T}\}$ be the MQP given by infinitesimal GF $g(s)$. Then the transition matrix $\mathbf{Q}(t)$ is standard. Its components $\mathcal{Q}_{ij}(t)$ are positive and uniformly continuous function of $t\in\mathcal{T}$ for all $i,j\in\mathcal{E}$.*

Proof. According to the branching property (1.1), we see that $P_{ij}(\varepsilon)=\delta_{ij}+ia_{j-i+1}\varepsilon+o(\varepsilon)$ as $\varepsilon\downarrow 0$. Hence seeing representation (2.1)

$$\begin{cases}\mathcal{Q}_{ii}(\varepsilon)=1+(ia_1-\ln\beta)\varepsilon+o(\varepsilon),\\ \mathcal{Q}_{ij}(\varepsilon)=jq^{j-i}a_{j-i+1}\varepsilon+o(\varepsilon),\end{cases} \qquad (2.4)$$

as $\varepsilon\downarrow 0$ and for all $i,j\in\mathcal{E}$. Considering the representations (2.4), we have the following relations:

$$\begin{aligned}\sum_{j\in\mathcal{E}}\left|\mathcal{Q}_{ij}(\varepsilon)-\delta_{ij}\right|&=\sum_{j\in\mathcal{E}\setminus\{i\}}\mathcal{Q}_{ij}(\varepsilon)+\left|\mathcal{Q}_{ii}(\varepsilon)-1\right|\\&=\sum_{j\in\mathcal{E}\setminus\{i\}}\mathcal{Q}_{ij}(\varepsilon)+1-\mathcal{Q}_{ii}(\varepsilon)\\&\le 2\left|1-\mathcal{Q}_{ii}(\varepsilon)\right|\to 0,\end{aligned}$$

as $\varepsilon\downarrow 0$. So that $\mathcal{Q}_{ij}(t)$ is standard.

A positiveness of functions $\mathcal{Q}_{ij}(t)$ is obvious owing to (2.4). The Markovian nature of the process $\{W(t), t\in\mathcal{T}\}$ implies that the following Kolmogorov-Chapman equation holds:

$$\mathcal{Q}_{ij}(t+\varepsilon)=\sum_{k\in\mathcal{E}}\mathcal{Q}_{ik}(t)\mathcal{Q}_{kj}(\varepsilon).$$

So supposing $\varepsilon>0$ it follows that

$$\begin{aligned}\mathcal{Q}_{ij}(t+\varepsilon)-\mathcal{Q}_{ij}(t)&=\sum_{k\in\mathcal{E}}\mathcal{Q}_{ik}(\varepsilon)\mathcal{Q}_{kj}(t)-\mathcal{Q}_{ij}(t)\\&=\sum_{k\in\mathcal{E}\setminus\{i\}}\mathcal{Q}_{ik}(\varepsilon)\mathcal{Q}_{kj}(t)-\mathcal{Q}_{ij}(t)\cdot\left(1-\mathcal{Q}_{ii}(\varepsilon)\right).\end{aligned}$$

The last relation gives

$$\begin{aligned}-\left(1-\mathcal{Q}_{ii}(\varepsilon)\right)&\le-\mathcal{Q}_{ij}(t)\cdot\left(1-\mathcal{Q}_{ii}(\varepsilon)\right)\\&\le\mathcal{Q}_{ij}(t+\varepsilon)-\mathcal{Q}_{ij}(t)\\&\le\sum_{k\in\mathcal{E}\setminus\{i\}}\mathcal{Q}_{ik}(t)\mathcal{Q}_{kj}(\varepsilon)\\&\le\sum_{k\in\mathcal{E}\setminus\{i\}}\mathcal{Q}_{kj}(\varepsilon)=1-\mathcal{Q}_{ii}(\varepsilon),\end{aligned}$$

so $\left|\mathcal{Q}_{ij}(t+\varepsilon)-\mathcal{Q}_{ij}(t)\right|\le 1-\mathcal{Q}_{ii}(\varepsilon)$. Similarly

$$\left|\mathcal{Q}_{ij}(t-\varepsilon)-\mathcal{Q}_{ij}(t)\right|\le 1-\mathcal{Q}_{ii}\left(t-(t-\varepsilon)\right)=1-\mathcal{Q}_{ii}(\varepsilon).$$

Therefore we obtain $\left|\mathcal{Q}_{ij}(t+\varepsilon)-\mathcal{Q}_{ij}(t)\right|\le 1-\mathcal{Q}_{ii}(|\varepsilon|)$ for any $\varepsilon\ne 0$ and for all $i,j\in\mathcal{E}$. The obtained relation implies that $\mathcal{Q}_{ij}(t)$ is uniformly continuous function of $t\in\mathcal{T}$ because $\lim_{\varepsilon\downarrow 0}\mathcal{Q}_{ii}(\varepsilon)=1$ for all $i\in\mathcal{E}$.

The theorem is proved.

It easily be convinced that a GF version of (2.4) is

$$G_i(t;s):=\mathsf{E}_is^{W(t)}=\sum_{j\in\mathcal{E}}\mathcal{Q}_{ij}(t)s^j=\frac{qs}{i\beta^t}\left[\frac{\partial}{\partial x}\left(\frac{F(t;x)}{q}\right)^i\right]_{x=qs},$$

or more obviously that

$$G_i(t;s)=\left[\frac{F(t;qs)}{q}\right]^{i-1}G(t;s), \qquad (2.5)$$

where

$$G(t;s)=G_1(t;s)=\frac{s}{\beta^t}\left.\frac{\partial F(t;x)}{\partial x}\right|_{x=qs}.$$

Theorem 2. *All states of the Markov chain $\{W(t), t\in\mathcal{T}\}$ are stable. The transition function $\mathbf{Q}(t)=\{\mathcal{Q}_{ij}(t)\}$ is the Feller functions. These are differentiable and has a finite and continuous derivative by $t\in\mathcal{T}$. Its q-matrix has components*

$$q_{ij}=\begin{cases}ip_1+(i-1)\ln\beta, & i=j,\\ \dfrac{jp_{j-i+1}}{j-i+1}, & i\ne j,\end{cases} \qquad (2.6)$$

where p_i are in (2.2) and $q_{ij}\ge 0$ when $i\ne j$ and, $q_i:=q_{ii}<0$ for all $i,j\in\mathcal{E}$. Moreover it satisfies the identity

$$\mathcal{Q}'_{ij}(t+\tau)=\sum_{k\in\mathcal{E}}\mathcal{Q}'_{ik}(\tau)\mathcal{Q}_{kj}(t),\quad t,\tau\in\mathcal{T}, \qquad (2.7)$$

the backward Kolmogorov system for all $i,j\in\mathcal{E}$.

Proof. It follows from relation (2.4) that for all $i\in\mathcal{E}$

$$q_i := \lim_{\varepsilon\downarrow 0}\frac{1-\mathcal{Q}_{ii}(\varepsilon)}{\varepsilon} < +\infty ,$$

that is all states are stable and also, the right-sided derivative $\mathcal{Q}'_{ij}(\varepsilon\downarrow 0)$ is finite. It follows from relation (2.5) that

$$\mathcal{Q}_{ij}(t) < \frac{G(t;s)}{s^j}\cdot\left[\frac{F(t;qs)}{q}\right]^{i-1}$$

for $0<s<1$. It can be convinced that $F(t;qs)/q$ is the GF of sub-critical MBP which converges increasing to one, so that $F(t;qs)/q<1$. Hence $\mathcal{Q}_{ij}(t)\to 0$ as $i\to\infty$. The last condition implies that $\mathcal{Q}_{ij}(t)$ is the Feller function. Therefore $\mathbf{Q}(t)$ has a stable q-matrix which components are $q_{ij}=\mathcal{Q}'_{ij}(\varepsilon\downarrow 0)$; see Anderson [1, p.43].

Next, since all states are stable then $\mathbf{Q}(t)= \mathcal{Q}_{ij}(t)$ is differentiable and has a finite and a continuous derivative on $\mathcal{T}$; see Anderson [1, p.10]. Let's compute this derivative. It follows from (4.1) that

$$\Delta\mathcal{Q}_{ij}(t)=\mathcal{Q}_{ij}(t+\varepsilon)-\mathcal{Q}_{ij}(t)=\frac{jq^{j-i}}{i\beta^t}\left[\frac{P_{ij}(t+\varepsilon)}{\beta^{\varepsilon}}-P_{ij}(t)\right]$$

$$=\frac{jq^{j-i}}{i\beta^t}\left[\Delta P_{ij}(t)+P_{ij}(t+\varepsilon)\cdot\varepsilon\cdot\ln\beta+o(\varepsilon)\right].$$

Taking limit as $\varepsilon\downarrow 0$ we receive from here that

$$\mathcal{Q}'_{ij}(t)=\frac{jq^{j-i}}{i\beta^t}\left[P'_{ij}(t)-P_{ij}(t)\ln\beta\right],\quad i,j\in\mathcal{E}. \qquad (2.8)$$

Being that $q_{ij}=\lim_{\varepsilon\downarrow 0}\mathcal{Q}_{ij}(\varepsilon)$, we should calculate $P'_{ij}(\varepsilon\downarrow 0)$. It follows from the general theory of MBP that

$$P'_{ij}(\varepsilon\downarrow 0)=\lim_{\varepsilon\to 0}P_{ij}(\varepsilon)/\varepsilon=ia_{j-i+1}.$$

Therefore we get

$$q_{ij}=\frac{jq^{j-i}}{i}\left[ia_{j-i+1}-\delta_{ij}\ln\beta\right].$$

The obtained result in context of densities p_j is equivalent to that (2.6). We see that $q_{ij}\ge 0$ when $i\ne j$ for all $i,j\in\mathcal{E}$ and, due to both p_1 and $\ln\beta$ are negative then $q_i:=q_{ii}<0$.

Owing to the Markovian nature of $W(t)$ it follows from the general theory of continuous-time Markov chain that the equation (2.7) holds. In particular, from here

$$\mathcal{Q}'_{ij}(t)=\sum_{k\in\mathcal{E}}q_{ik}\mathcal{Q}_{kj}(t).$$

The proof is completed.

Let $\mathcal{G}_i(s)$ be the GF of q-matrix of $\mathbf{Q}(t)= \mathcal{Q}_{ij}(t)$ that is

$$\mathcal{G}_i(s):=\sum_{j\in\mathcal{E}}q_{ij}s^j=\sum_{j\in\mathcal{E}}\mathcal{Q}'_{ij}(\varepsilon\downarrow 0)s^j .$$

Using (4.6) it follows that

$$\mathcal{G}_i(s)= ip_1+(i-1)\ln\beta\ s^i+\sum_{j\in\mathcal{E}\setminus\{i\}}\frac{jp_{j-i+1}}{j-i+1}s^j$$

$$=(i-1)s^{i-1}\left[s\ln\beta+\sum_{j\in\mathcal{E}}\frac{p_j}{j}s^j\right]+s^{i-1}g(s).$$

On the other hand it is easy to see that

$$\sum_{j\in\mathcal{E}}\frac{p_j}{j}s^j=\int_0^s\frac{g(u)}{u}du .$$

Thence we obtain that

$$\mathcal{G}_i(s)=\frac{(i-1)m(s)+g(s)}{s}s^i , \qquad (2.9)$$

where $g(s)$ is defined in (2.3) and

$$m(s):=s\ln\beta+\int_0^s\frac{g(x)}{x}dx .$$

Now more general, consider

$$\mathcal{G}_i(t;s)=\sum_{j\in\mathcal{E}}\mathcal{Q}'_{ij}(t)s^j=\frac{\partial G_i(t;s)}{\partial t}.$$

After some calculations we make sure that the generalization of (2.9) for all $t\in\mathcal{T}$ is the following identity

$$\mathcal{G}_i(t;s)=\frac{(i-1)m\ F(t;s)\ +g\ F(t;s)}{F(t;s)}G_i(t;s) , \qquad (2.10)$$

that is the GF version of (2.8), where $F(t;s)=F(t;qs)/q$ is the GF of sub-critical MBP.

Theorem 3. *The transition function* $\mathbf{Q}(t)= \mathcal{Q}_{ij}(t)$ *is differentiable and is the unique solution of the backward Kolmogorov system, which is the unique GF solution of the differential equation*

$$\frac{\partial G(t;s)}{\partial t}=h\ F(t;s)\ G(t;s) , \qquad (2.11)$$

with $G(0;s)=s$, *where* $h(s)=g(s)/s$ *and* $F(t;s)=F(t;qs)/q$.

Proof. Following the proof of the Theorem 1 it follows that

$$0\le\mathcal{Q}_{ij}(t)-\mathcal{Q}_{ij}(t+\varepsilon)\le\mathcal{Q}_{ij}(t)\cdot\ 1-\mathcal{Q}_{ii}(\varepsilon)$$

and similarly

$$0\le\mathcal{Q}_{ij}(t-\varepsilon)-\mathcal{Q}_{ij}(t)\le\mathcal{Q}_{ij}(t-\varepsilon)\cdot\ 1-\mathcal{Q}_{ii}(\varepsilon)\ .$$

Hence for the difference $\Delta_\varepsilon G(t;s)=G(t-\varepsilon;s)-G(t+\varepsilon;s)$ we receive that

$$0\le\Delta_\varepsilon G(t;s)=\sum_{j\in\mathcal{E}}\left[\mathcal{Q}_{1j}(t-\varepsilon)-\mathcal{Q}_{1j}(t+\varepsilon)\right]s^j$$

$$\le\ 1-\mathcal{Q}_{11}(\varepsilon)\ \cdot\sum_{j\in\mathcal{E}}\left[\mathcal{Q}_{1j}(t-\varepsilon)+\mathcal{Q}_{1j}(t)\right]s^j$$

$$\le 2\ 1-\mathcal{Q}_{11}(\varepsilon)\ \cdot\sum_{j\in\mathcal{E}}\mathcal{Q}_{1j}(t-\varepsilon)s^j$$

$$=2G(t-\varepsilon;s)\cdot\ 1-\mathcal{Q}_{11}(\varepsilon)\ .$$

Since the function $\mathcal{Q}_{ij}(t)$ is standard, it follows from the last inequality that $\Delta_\varepsilon G(t;s)\to 0$ as $\varepsilon\downarrow 0$ implying that $G(t;s)$ is continuous function of $t\in\mathcal{T}$ uniformly for $0\le s<1$. Therefore it is differentiable. It can easily be seen that a GF version of the relation (2.2) is

$$G(\varepsilon;s)=s+g(s)\cdot\varepsilon+o(\varepsilon),\quad\varepsilon\downarrow 0, \qquad (2.12)$$

for $0\le s<1$. The equation (2.11) follows from (2.10) with $i=1$. The boundary condition $G(0;s)=s$ follows from (2.12). The uniqueness of the solution of (2.11) follows from the classical differential equations theory.

The theorem is proved.

From Theorem 3 we get the following assertion.

Corollary. *The differential equation (2.11) is equivalent to the following integral equation:*

$$\int_0^t h\ F(\tau;s)\ d\tau=\ln\frac{G(t;s)}{s}, \qquad (2.13)$$

with boundary condition $G(0;s)=s$.

2.2. Classification and a limit theorem

As it has been noticed above, that parameter $a=f'(1)$ plays a regulating role for MBP and is subdivided three types of process depending sign of a. Note that evolution of MQP is regulated in essence by the positive parameter β. Thus might be subdivide two types of process in depending on values of β. Putting together equalities (2.5) and (2.13), we write that

$$G_i(t;s) = s\Big[F(t;s)\Big]^{i-1}\exp\ \int_0^t h\ \ F(\tau;s)\ \ d\tau\ \ , \qquad (2.14)$$

where $F(t;s) = F(t;qs)/q$. Let $\alpha := g'(1)$ is finite. Direct differentiating in point $s=1$, it follows from (2.14) that

$$\mathsf{E}_i W(t) = \ i-1\ \ \beta^t + \mathsf{E}\,W(t)$$

and

$$\mathsf{E}\,W(t) = \begin{cases} 1+\gamma\ \ 1-\beta^t\ \ , & \beta<1. \\ \alpha t+1 \ \ , & \beta=1. \end{cases} \qquad (2.15)$$

Moreover we obtain the variance structure

$$\mathsf{Var}\Big[W(t)\Big|W(0)=i\Big] = \begin{cases} \Big[\gamma+\ \ i-1\ \ 1+\gamma\ \ \beta^t\Big]\ 1-\beta^t\ \ , & \beta<1, \\ \alpha i t \ \ , & \beta=1. \end{cases}$$

where $\gamma = \alpha\big/\big|\ln\beta\big|$. The formula (2.15) implies that when $\beta=1$

$$\mathsf{E}_i W(t) \sim \alpha t,\quad t\to\infty,$$

and if $0<\beta<1$

$$\mathsf{E}_i W(t) \to 1+\gamma,\quad t\to\infty.$$

Last findings give us evidence that in $\beta=1$ the MQP has transience property. Thereby MQP we classify as *restrictive* if $\beta<1$ and *explosive* if $\beta=1$.

Theorem 4. *The MQP is*

1) *positive if it is restrictive and* $\alpha := g'(1)$ *is finite*;

2) *null if it is explosive*.

Proof. To prove the assertion (i) from (2.13) we get

$$\ln Q_{11}(t) = \int_0^t h\ \ F(\tau;0)\ \ d\tau$$

$$= \int_0^{F(t;0)} \frac{h(x)}{\hat{f}(x)}dx \longrightarrow \int_0^1 \frac{h(x)}{\hat{f}(x)}dx,$$

since $F(t;0)\to 1$ as $t\to\infty$, where $\hat{f}(s) = f(qs)/q$. The condition $\alpha<\infty$ implies that integral in right-hand side converges. Hence $\lim_{t\to\infty} Q_{11}(t) > 0$. For part (ii) we recall that $q=1$ and $h(s)/s = f'(s)$ if $\beta=1$. Similarly

$$\ln Q_{11}(t) = \int_0^t h\ \ F(\tau;0)\ \ d\tau$$

$$= \int_0^{F(t;0)} \frac{h(x)}{f(x)}dx \longrightarrow \int_0^1 \frac{f'(x)}{f(x)}dx = -\infty.$$

So that $\lim_{t\to\infty} Q_{11}(t) = 0$.

The theorem is proved.

We complete the paper with stating of the following limit theorem without proof.

Theorem 5 [6]. L*et* $\alpha := g'(1)$ *is finite*.

1. *If MQP is restrictive, then the variable* $W(t)$ *tends in mean square and with probability one to a random variable* W *having finite mean and variance*:

$$\mathsf{E}\,W = 1+\gamma \quad \textit{and} \quad \mathsf{Var}\,W = \gamma.$$

2. *If MQP is explosive, then for any* $x>0$

$$\mathsf{P}_i\left\{\frac{W(t)}{\mathsf{E}\,W(t)} \le x\right\} = 1-e^{-x}-2e^{-2x},$$

as $t\to\infty$, *for all* $i\in\mathcal{E}$.

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References

[1] Anderson, W.(1991). Continuous-Time Markov Chains: An Applications-Oriented Approach. New York: Springer.

[2] Athreya, K.B. and Ney, P.E.(1972). Branching processes. New York: Springer.

[3] Formanov, Sh.K. and Imomov, A.A.(2011). On asymptotic properties of Q-processes. Uzbek Mathematical Journal, 3, 175-183. (in Russian)

[4] Heatcote, C.R., Seneta E. and Vere-Jones.(1967). A refinement of two theorems in the theory of branching process. Theory of Probab. and its Appl., 12(2), 341-346.

[5] Imomov, A.A.(2014). On long-term behavior of continuous-time Markov Branching Processes allowing Immigration. Journal of Siberian Federal University. Mathematics and Physics, 7(4), 429-440.

[6] Imomov, A.A.(2012). On Markov analogue of Q-processes with continuous time. Theory of Probability and Mathematical Statistics, 84, 57-64.

[7] Imomov, A.A.(2005). A differential analog of the main lemma of the theory of Markov branching processes and its applications. Ukrainian Math. Journal, 57(2), 307–315.

[8] Imomov, A.A.(2002). Some asymptotical behaviors of Galton-Watson branching processes under condition of non-extinctinity of it remote future. Abstracts of Comm. of 8th Vilnius Conference: Probab. Theory and Math. Statistics, Vilnius, Lithuania, p.118.

[9] Kolmogorov, A.N and Dmitriev, N.A.(1947). Branching stochastic process. Reports of Academy of Sciences of USSR, 56, 7-10. (Russian)

[10] Lamperti, J. and Ney, P.E.(1968). Conditioned branching processes and their limiting diffusions. Theory of Probability and its Applications, 13, 126-137.

[11] Nagaev, A.V. and Badalbaev, I.S.(1967). A refinement of certain theorems on branching random process. Litovskiy Matematicheskiy Sbornik, 7(1), 129-136.

[12] Pakes, A.G.(2010). Critical Markov branching process limit theorems allowing infinite variance. Advances in Applied Probability, 42, 460-488.

[13] Pakes, A.G.(1999). Revisiting conditional limit theorems for the mortal simple branching process. Bernoulli, 5(6), 969-998.

[14] Pakes, A.G.(1971). Some limit theorems for the total progeny of a branching process. Advances in Applied Probability, 3, 176-192.

[15] Sevastyanov, B.A.(1951). The theory of Branching stochastic process. Uspekhi Matematicheskikh Nauk, 6(46), 47-99. (in Russian)

[16] Sevastyanov, B.A.(1971). Branching processes, Moscow: Nauka. (Russian)

[17] Zolotarev, V.M.(1957). More exact statements of several theorems in the theory of branching processes. Theory of Probability and its Applications, 2, 245-253.