Optimal capital allocation with copulas

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Abstract

In this paper, we investigate optimal capital allocation problems for a portfolio consisting of different lines of risks linked by a Farlie-Gumbel-Morgenstern copula, modelling the dependence between them. Based on the Tail Mean-Variance principle, we examine the bivariate case and then the multivariate case. Explicit formulae for optimal capital allocations are obtained for exponential loss distributions. Finally, the results are illustrated by various numerical examples.

1. Introduction

A fundamental question in the fields such as quantitative risk management, insurance and finance is how much money to ask from a policyholder in a heterogeneous portfolio? A closely related question is how to allocate a given amount of capital between the different classes in a portfolio. The first question is known as the premium calculation problem and the second is the capital allocation problem. The two problems are well-known and have been studied, see e.g. Laeven and Goovaerts [13], Zaks et al. [20], Frostig et al. [10], Belles-Sampera [4], Bauer and Zanjani [2,3] and references therein.

In recent years, there has been growing interest in studying the optimal allocation problems because they play a central role in Solvency II. For example, Dhaene et al. [9] assumed a portfolio consisting of n risks X_1, X_2, $\cdots$, X_n and a company wished to allocate the total capital $d = d_1 + d_2 + \cdots + d_n$ to the corresponding risks. They formulated capital allocation as an optimization problem and provided a reasonable criterion, which is to set the amount d_i to X_i so that the potential loss measured by some appropriate

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distance measure is as small as possible. And further, they proposed to model the capital allocation problem by the following optimization problem:

$$\min_{d_1,\cdots,d_n} \sum_{i=1}^{n} v_i \mathbf{E}\left[\zeta_i D\left(\frac{X_i - d_i}{v_i}\right)\right], \quad \text{such that} \quad \sum_{i=1}^{n} d_i = d, \tag{1.1}$$

where the v_i, $i = 1, 2, \cdots, n$, are non-negative real numbers such that $\sum_{i=1}^{n} v_i = 1$, D is a non-negative function and the ζ_i, $i = 1, 2, \cdots, n$, are non-negative random variables such that $\mathbf{E}[\zeta_i] = 1$. The non-negative real number v_i is a measure of exposure or business volume of the ith unit, such as revenue, insurance premium, etc. D is a distance measurement function that measures the loss of allocation. The non-negative random variables ζ_i, $i = 1, 2, \cdots, n$, are used as weight factors to the different possible outcomes of $D(X_i - d_i)$. The framework is general and includes many optimisation criterion as special cases, such as Quantile, Haircut and Covariance. A special case of (1.1) is the following optimization problem:

$$\min_{d_1,\cdots,d_n} \sum_{i=1}^{n} v_i \mathbf{E}\left[(X_i - d_i)^2\right], \quad \text{such that} \quad \sum_{i=1}^{n} d_i = d,$$

which is showed in Dhaene et al. [9]. The solution to the equation above is

$$d_i = \frac{d}{n} + \mathbf{E}(X_i) - \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}(X_j), \quad i = 1, 2, \cdots, n.$$

Xu and Hu [17] generalized this idea and defined the following loss function:

$$L(\vec{p}) = \sum_{i=1}^{n} D(X_i - p_i).$$

Then, they proposed to model the capital allocation problem by the following optimization problem:

$$\min_{\vec{p}\in A} Pr(L(\vec{p}) \geq t), \quad \forall t \geq 0. \tag{1.2}$$

You and Li [19] further studied the capital allocation concerning mutually exchangeable random risks. Cai and Wei [5] proposed some new notions of dependence with applications in optimal allocation problems. Zaks and Tsanakas [21] studied the optimal capital allocation in a hierarchical corporate structure where stake holders at two organizational levels. Manesh and Khaledi [14] considered the allocations of policy limits and ordering relations for aggregate remaining claims.

Note that allocation criterion based on minimizing the loss function does not take into account the the factor of variability of loss function. We also note that both allocation rule (1.1) and (1.2) do not take into account tail risk. Xu and Mao [18] proposed a Tail Mean-Variance (TMV) framework to overcome this limitation. In more detail, they provided the following allocation rule:

$$\min_{d_1,\cdots,d_n} \left\{ \mathbf{E}\left[\sum_{i=1}^{n}(X_i - d_i)^2 | S > \text{VaR}_\kappa(S)\right] + \beta \text{Var}\left(\sum_{i=1}^{n}(X_i - d_i)^2 | S > \text{VaR}_\kappa(S)\right) \right\},$$

$$\text{such that} \quad \sum_{i=1}^{n} d_i = d, \tag{1.3}$$

where $\beta > 0$, $S = \sum_{i=1}^{n} X_i$ is the aggregate risk, $\text{VaR}_\kappa(S) = \inf(x \in \mathbf{R}, F(S) \geq \kappa)$ is the value at risk at level κ, $0 < \kappa < 1$ of S and $F(S) = \Pr(S \leq s)$ is the distribution of S. Xu and Mao [18] studied the optimal capital allocations when the risks have multivariate elliptical distributions.

In most papers described above, the dependence between different lines of business of a portfolio is due to construction of a multivariate distribution. Copulas are currently seen as flexible and effective tools to describe dependence between the random variables. In this paper, we propose introducing dependence with a copula. We use the Farlie-Gumbel-Morgenstern (FGM) copula, like Bargès et al. [1] and Cossette et al. [7], to describe the dependence. Based on the simplicity and tractability of FGM copula, we derive the closed-form expressions for the optimal capital allocations based on the Tail Mean-Variance model when the FGM copula represent dependence between the risk marginals.

The work of this paper can be seen as a complement to the research on the the optimal capital allocations based on the Tail Mean-Variance principle and extend the former results by introducing dependence with a copula. When the optimal capital allocation problems for a portfolio consisting of different lines of risks linked by a Farlie-Gumbel-Morgenstern copula are considered, we show that, the explicit capital allocation formulae can be obtained under the Tail Mean-Variance principle.

The rest of the paper is organized as follows. In Section 2, we describe basic definitions and properties of the FGM copula and the exponential distributions. In Section 3, we derive explicit capital allocation formulae for the bivariate case, where both losses are exponentially distributed with the dependence linked by the FGM copula. In Section 4, we extend the results to the multivariate case. Some numerical examples to calculate the optimal capital allocation are presented to illustrate the solution procedure in Section 5.

2. Definitions

Here, we briefly recall the characteristics and definition of the FGM copula. The FGM copula is a perturbation of the independence copula and it is not Archimedian. The FGM copula is a first order approximation of the Plackett copula (Nelsen [15], p.100) and of the Frank copula (p.133). Among the recent applications of the FGM copula, we mention Bargès et al. [1] and Cossette et al. [7] who deal with the application of the TVaR-based allocation rule using the FGM copula. The FGM copula is applied in the context of risk measurement by Gebizlioglu and Yagci [11] and in the context of sums of dependent r.v. by Geluk and Tang [12]. The FGM copula is also used in a ruin context by Cossette et al. [8], Xie and Zou [16]. The FGM copula is attractive due to its tractability and its simplicity.

We assume the lines of business of an portfolio are linked with a FGM copula. Since the exponential distribution is a classic distribution for the risk random variables and its practical mathematic properties allow explicit results, we assume marginal risks are distributed as exponentials.

2.1 The bivariate case

Firstly, we consider two business lines whose losses X_1 and X_2 follow the exponential distribution: $X_i \sim \text{Exp}(\lambda_i)$, $i = 1, 2$. Their probability density functions (pdf) f_{X_i} and cumulative distribution functions (cdf) F_{X_i} are given by

$$f_{X_i} = \lambda_i e^{-\lambda_i x_i}, \quad F_{X_i} = 1 - e^{-\lambda_i x_i}, \quad \text{for } i = 1, 2.$$

We assume that the couple (X_1, X_2) has a bivariate distribution defined with FGM copula. The FGM copula is defined as follows:

$$C_\theta^{FGM}(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2),$$

for $(u_1, u_2) \in [0, 1] \times [0, 1]$. The dependence parameter θ takes value in $[-1, 1]$, where $\theta < 0 (> 0)$ corresponds to a negative (positive) dependence relation. The density of the bivariate FGM copula is

$$c_\theta^{FGM}(u_1, u_2) = \frac{\partial^2 C_\theta^{FGM}(u_1, u_2)}{\partial u_1 \partial u_2} = 1 + \theta(2\bar{u}_1 - 1)(2\bar{u}_2 - 1),$$

where $\bar{u}_i = 1 - u_i$, $i = 1, 2$. Then, the joint c.d.f. $F_{X_1, X_2}(x_1, x_2)$ of (X_1, X_2) with marginals F_{X_1} and F_{X_2} and defined with the FGM copula is given by

$$F_{X_1, X_2}(x_1, x_2) = C_\theta^{FGM}(F_{X_1}(x_1), F_{X_2}(x_2))$$
$$= F_{X_1}(x_1)F_{X_2}(x_2) + \theta F_{X_1}(x_1)F_{X_2}(x_2)(1 - F_{X_1}(x_1))(1 - F_{X_2}(x_2)).$$

The joint pdf of (X_1, X_2) is

$$f_{X_1, X_2}(x_1, x_2) = c_\theta^{FGM}(F_{X_1}(x_1), F_{X_2}(x_2))f_{X_1}(x_1)f_{X_2}(x_2)$$
$$= (1 + \theta)\lambda_1 e^{-\lambda_1 x_1}\lambda_2 e^{-\lambda_2 x_2} - \theta 2\lambda_1 e^{-2\lambda_1 x_1}\lambda_2 e^{-\lambda_2 x_2}$$
$$-\theta\lambda_1 e^{-\lambda_1 x_1}2\lambda_2 e^{-2\lambda_2 x_2} + \theta 2\lambda_1 e^{-2\lambda_1 x_1}2\lambda_2 e^{-2\lambda_2 x_2}. \tag{2.1}$$

2.2 The multivariate case

Now, we consider n business lines whose losses X_1, X_2, $\cdots$, X_n follow the exponential distribution: $X_i \sim \text{Exp}(\lambda_i)$, $i = 1, 2, \cdots, n$. We also assume that the n different risks jointed by a multivariate FGM n-copula, which has $2^n - n - 1$ parameters, is defined as follows:

$$C_\theta^{FGM}(u_1, \ldots, u_n) = u_1 \cdots u_n \left(1 + \sum_{q=2}^{n} \sum_{1 \le p_1 < \cdots < p_q \le n} \theta_{p_1 \ldots p_q} \bar{u}_{p_1} \cdots \bar{u}_{p_q}\right),$$

where $\theta \in [-1, 1]$, $\bar{u}_i = 1 - u_i$, $i = 1, 2, \cdots, n$. Its density can be written as

$$c_\theta^{FGM}(u_1, \ldots, u_n) = 1 + \sum_{q=2}^{n} \sum_{1 \le p_1 < \cdots < p_q \le n} \theta_{p_1 \ldots p_q}(2\bar{u}_{p_1} - 1) \cdots (2\bar{u}_{p_q} - 1).$$

Then, the joint pdf of $(X_1, \ldots, X_n)$ is

$$f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = c_\theta^{FGM}(F_{X_1}(x_1), \ldots, F_{X_n}(x_n))f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

$$= f_{X_1}(x_1) \cdots f_{X_n}(x_n)(1 + \sum_{q=2}^{n} \sum_{1 \le p_1 < \cdots < p_q \le n} \theta_{p_1 \ldots p_q}(1 - F_{X_{p_1}}(x_{p_1})) \cdots (1 - F_{X_{p_q}}(x_{p_q})))$$

$$= \omega(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n) + \sum_{q=2}^{n} \sum_{1 \le p_1 < \cdots < p_q \le n} \theta_{p_1 \ldots p_q} \times$$

$$(\sum_{l=0}^{q} \sum_{(\epsilon_1, \ldots, \epsilon_q) \in \tau_{l,q}} (-1)^l \omega(x_{p_1}, \ldots, x_{p_q}; x_{i_{q+1}}, \ldots, x_{i_n}; 2^{\epsilon_1}\lambda_{p_1}, \ldots, 2^{\epsilon_q}\lambda_{p_q}, \lambda_{i_{q+1}}, \cdots, \lambda_{i_n})),$$

$$\tag{2.2}$$

where $\omega(x_1, x_2, \cdots, x_n; \alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha_1 e^{-\alpha_1 x_1}\alpha_2 e^{-\alpha_2 x_2} \cdots \alpha_n e^{-\alpha_n x_n}$, $i_{q+1}, \cdots, i_n$ are the missing indexes of $p_1, \cdots, p_q$ to complete $1, \cdots, n$ and $\tau_{l,q}$ are the sets of q-tuples composed of l zeros and $q - l$ ones, i.e., $\tau_{l,q}$, $l = 0, 1, \cdots, q$ and $q = 2, \cdots, n$ are defined as $\tau_{0,q} = \{(1, 1, \cdots, 1)_{1 \times q}\}$, $\tau_{1,q} = \{(1, 1, \cdots, 0)_{1 \times q}, \cdots, (0, 1, \cdots, 1)_{1 \times q}\}$, $\tau_{2,q} = \{(1, 1, \cdots, 0, 0)_{1 \times q}, \cdots, (0, 0, \cdots, 1)_{1 \times q}\}$, $\cdots$, $\tau_{q,q} = \{(0, 0, \cdots, 0)_{1 \times q}\}$.

3. Calculation of the optimal capital allocation for two risks

In this section, we obtain the close formed expression of the optimal capital allocation based on the Tail Mean-Variance (TMV) model for two exponentially distributed risks linked by a FGM copula. Let X_1 and X_2 are two exponentially distributed with λ_1 and λ_2. In order to simplify the calculation, we assume that $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq 2\lambda_2$, and $\lambda_2 \neq 2\lambda_1$. Applying the similar technique proposed below, one can obtains the adjusted results without these restrictions.

Theorem 3.1. Let X_1 and X_2 are two exponentially distributed with λ_1 and λ_2. The dependence structure for (X_1, X_2) is defined by the bivariate FGM copula with parameter $\theta \in [-1, 1]$. Then the optimal allocation solution $\vec{d}^* = (d_1^*, d_2^*)$ to the TMV model

$$\min_{d_1, d_2} \left\{ \mathbf{E}\left[\sum_{i=1}^{2}(X_i - d_i)^2 | S > \mathrm{VaR}_\kappa(S) \right] + \beta \mathrm{Var}\left(\sum_{i=1}^{2}(X_i - d_i)^2 | S > \mathrm{VaR}_\kappa(S) \right) \right\},$$

$$\text{such that} \quad d_1 + d_2 = d,$$

is presented as follows:

$$d_1 = \frac{1}{2(1 - 2\beta(2E^{X_1 X_2} + (E^{X_1} - E^{X_2})^2 - (E^{X_1^2} + E^{X_2^2})))} \times$$

$$\left(2\beta(E^{X_1 X_2^2} - E^{X_1^2 X_2} + E^{X_1^3} - E^{X_2^3}) + (E^{X_1} - E^{X_2})(1 - 2\beta(E^{X_1^2} + E^{X_2^2})) \right.$$

$$\left. + d(1 - 4\beta(E^{X_1 X_2} - E^{X_2^2} + E^{X_2}(E^{X_2} - E^{X_1}))) \right),$$ (3.1)

$$d_2 = \frac{1}{2(1 - 2\beta(2E^{X_1 X_2} + (E^{X_1} - E^{X_2})^2 - (E^{X_1^2} + E^{X_2^2})))} \times$$

$$\left(-2\beta(E^{X_1 X_2^2} - E^{X_1^2 X_2} + E^{X_1^3} - E^{X_2^3}) - (E^{X_1} - E^{X_2})(1 - 2\beta(E^{X_1^2} + E^{X_2^2})) \right.$$

$$\left. + d(1 - 4\beta(E^{X_1 X_2} - E^{X_1^2} + E^{X_1}(E^{X_1} - E^{X_2}))) \right),$$ (3.2)

where

$$E^{X_i^k} = \mathbf{E}\left[X_i^k | S > \mathrm{VaR}_\kappa(S) \right] = \frac{1}{1 - \kappa}[(1+\theta)\xi_k(\mathrm{VaR}_\kappa(S); \lambda_i; \lambda_j) - \theta\xi_k(\mathrm{VaR}_\kappa(S); 2\lambda_i; \lambda_j)$$

$$-\theta\xi_k(\mathrm{VaR}_\kappa(S); \lambda_i; 2\lambda_j) + \theta\xi_k(\mathrm{VaR}_\kappa(S); 2\lambda_i; 2\lambda_j)], \quad i, j = 1, 2, \quad i \neq j, \quad k = 1, 2, 3,$$

$$E^{X_i^2 X_j} = \mathbf{E}\left[X_i^2 X_j | S > \mathrm{VaR}_\kappa(S) \right] = \frac{1}{1 - \kappa}[(1+\theta)\chi(\mathrm{VaR}_\kappa(S); \lambda_i; \lambda_j)$$

$$-\theta\chi(\mathrm{VaR}_\kappa(S); 2\lambda_i; \lambda_j) - \theta\chi(\mathrm{VaR}_\kappa(S); \lambda_i; 2\lambda_j) + \theta\chi(\mathrm{VaR}_\kappa(S); 2\lambda_i; 2\lambda_j)], \quad i, j = 1, 2, \ i \neq j,$$

$$E^{X_i X_j} = \mathbf{E}\left[X_i X_j | S > \mathrm{VaR}_\kappa(S) \right] = \frac{1}{1 - \kappa}[(1+\theta)\varphi(\mathrm{VaR}_\kappa(S); \lambda_i; \lambda_j)$$

$$-\theta\varphi(\mathrm{VaR}_\kappa(S); 2\lambda_i; \lambda_j) - \theta\varphi(\mathrm{VaR}_\kappa(S); \lambda_i; 2\lambda_j) + \theta\varphi(\mathrm{VaR}_\kappa(S); 2\lambda_i; 2\lambda_j)],$$

$$\xi_1(x; \alpha_i; \alpha_j) = \frac{\alpha_j e^{-\alpha_i x}(x + \frac{1}{\alpha_i})}{\alpha_j - \alpha_i} - \frac{\alpha_j e^{-\alpha_i x} - \alpha_i e^{-\alpha_j x}}{(\alpha_j - \alpha_i)^2},$$

$$\xi_2(x; \alpha_i; \alpha_j) = \frac{e^{-\alpha_i x}\alpha_j(\frac{2}{\alpha_i^2} + \frac{2x}{\alpha_i} + x^2)}{\alpha_j - \alpha_i} - \frac{2\alpha_j e^{-\alpha_i x}(x + \frac{1}{\alpha_i})}{(\alpha_j - \alpha_i)^2} + \frac{2(\alpha_j e^{-\alpha_i x} - \alpha_i e^{-\alpha_j x})}{(\alpha_j - \alpha_i)^3},$$

$$\xi_3(x; \alpha_i; \alpha_j) = \frac{e^{-\alpha_i x}\alpha_j(\frac{6}{\alpha_i^3} + \frac{6x}{\alpha_i^2} + \frac{3x^2}{\alpha_i} + x^3)}{\alpha_j - \alpha_i} - \frac{3e^{-\alpha_i x}\alpha_j(\frac{2}{\alpha_i^2} + \frac{2x}{\alpha_i} + x^2)}{(\alpha_j - \alpha_i)^2}$$

$$+ \frac{6\alpha_j e^{-\alpha_i x}(x + \frac{1}{\alpha_i})}{(\alpha_j - \alpha_i)^3} - \frac{6(\alpha_j e^{-\alpha_i x} - \alpha_i e^{-\alpha_j x})}{(\alpha_j - \alpha_i)^4},$$

$$\chi(x;\alpha_i;\alpha_j) = \frac{\alpha_j e^{-\alpha_i x}(\frac{2}{\alpha_i^2} + \frac{2x}{\alpha_i} + x^2)}{(\alpha_j - \alpha_i)^2} - \frac{4\alpha_j e^{-\alpha_i x}(x + \frac{1}{\alpha_i}) + 2\alpha_i e^{-\alpha_j x}(x + \frac{1}{\alpha_j})}{(\alpha_j - \alpha_i)^3}$$

$$+ \frac{6(\alpha_j e^{-\alpha_i x} - \alpha_i e^{-\alpha_j x})}{(\alpha_j - \alpha_i)^4},$$

$$\varphi(x;\alpha_i;\alpha_j) = -\frac{\alpha_j e^{-\alpha_i x}(x + \frac{1}{\alpha_i}) + \alpha_i e^{-\alpha_j x}(x + \frac{1}{\alpha_j})}{(\alpha_j - \alpha_i)^2} + \frac{2(\alpha_j e^{-\alpha_i x} - \alpha_i e^{-\alpha_j x})}{(\alpha_j - \alpha_i)^3}.$$

Proof. In order to find the optimal allocation solution $\vec{d}^* = (d_1^*, d_2^*)$, we utilize the method of Lagrange multiplier. Assume

$$l(d_1, d_2) = \mathbf{E}\left[(X_1 - d_1)^2 + (X_2 - d_2)^2 | S > \text{VaR}_\kappa(S)\right]$$

$$+ \beta \text{Var}\left((X_1 - d_1)^2 + (X_2 - d_2)^2 | S > \text{VaR}_\kappa(S)\right),$$

and

$$L(d_1, d_2, \gamma) = l(d_1, d_2) + \gamma(d - d_1 - d_2).$$

Since

$$\text{Var}\left((X_1 - d_1)^2 + (X_2 - d_2)^2 | S > \text{VaR}_\kappa(S)\right)$$

$$= \mathbf{E}\left[((X_1)^2 + (X_2)^2 - 2d_2 X_2)^2 | S > \text{VaR}_\kappa(S)\right] + (E^{X_1^2} + E^{X_2^2} - 2d_2 E^{X_2})^2 + 4d_1^2(E^{X_1^2}$$

$$- (E^{X_1})^2) + 4d_1[2d_2(E^{X_1 X_2} - E^{X_1}E^{X_2}) - (E^{X_1^3} - E^{X_1}E^{X_1^2}) - (E^{X_1 X_2^2} - E^{X_1}E^{X_2^2})],$$

we get

$$\frac{\partial L(d_1, d_2, \gamma)}{\partial d_1} = -2E^{X_1} + 2d_1 + 8\beta d_1(E^{X_1^2} - (E^{X_1})^2)$$

$$+ 4\beta[2d_2(E^{X_1 X_2} - E^{X_1}E^{X_2}) - (E^{X_1^3} - E^{X_1}E^{X_1^2}) - (E^{X_1 X_2^2} - E^{X_1}E^{X_2^2})] - \gamma = 0. \quad (3.3)$$

Similarly,

$$\frac{\partial L(d_1, d_2, \gamma)}{\partial d_2} = -2E^{X_2} + 2d_2 + 8\beta d_2(E^{X_2^2} - (E^{X_2})^2)$$

$$+ 4\beta[2d_1(E^{X_1 X_2} - E^{X_1}E^{X_2}) - (E^{X_2^3} - E^{X_2}E^{X_2^2}) - (E^{X_2 X_1^2} - E^{X_2}E^{X_1^2})] - \gamma = 0, \quad (3.4)$$

and

$$\frac{\partial L(d_1, d_2, \gamma)}{\partial \gamma} = d_1 + d_2 - d = 0. \quad (3.5)$$

From Eqs.(3.3)-(3.5), the results of (3.1) and (3.2) follow immediately. Now, we need to find the formulae of $E^{X_i^k}$, $E^{X_i^2 X_j}$ and $E^{X_1 X_2}$. Firstly, the explicit expression of $E^{X_1^3}$ is calculated as follow.

$$E^{X_1^3} = \mathbf{E}\left[X_1^3 | S > \text{VaR}_\kappa(S)\right] = \int_{\text{VaR}_\kappa(S)}^\infty \mathbf{E}\left[X_1^3 | S = s\right] f_{S|S>\text{VaR}_\kappa(S)}(s)\mathrm{d}s$$

$$= \frac{\int_{\text{VaR}_\kappa(S)}^\infty \mathbf{E}\left[X_1^3 | S = s\right] f_S(s)\mathrm{d}s}{\Pr(S > \text{VaR}_\kappa(S))} = \frac{\int_{\text{VaR}_\kappa(S)}^\infty f_S(s) \int_0^s x^3 f_{X_1|S}(x|S=s)\mathrm{d}x\mathrm{d}s}{1 - F_S(\text{VaR}_\kappa(S))}$$

$$= \frac{\int_{\text{VaR}_\kappa(S)}^\infty \int_0^s x^3 f_{x_1,S}(x,s)\mathrm{d}x\mathrm{d}s}{1 - \kappa}. \quad (3.6)$$

Note that $S = X_1 + X_2$. From Eq.(2.1), we get

$$\int_0^s x^3 f_{x_1,S}(x,s)\mathrm{d}x = \int_0^s x^3 f_{x_1,x_2}(x,s-x)\mathrm{d}x$$

$$= (1 + \theta)\lambda_1\lambda_2\left(\frac{6(e^{-\lambda_2 s} - e^{-\lambda_1 s})}{(\lambda_2 - \lambda_1)^4} + \frac{6se^{-\lambda_1 s}}{(\lambda_2 - \lambda_1)^3} - \frac{3s^2 e^{-\lambda_1 s}}{(\lambda_2 - \lambda_1)^2} + \frac{s^3 e^{-\lambda_1 s}}{\lambda_2 - \lambda_1}\right)$$

$$-\theta 2\lambda_1\lambda_2\left(\frac{6(e^{-\lambda_2 s}-e^{-2\lambda_1 s})}{(\lambda_2-2\lambda_1)^4}+\frac{6se^{-2\lambda_1 s}}{(\lambda_2-2\lambda_1)^3}-\frac{3s^2e^{-2\lambda_1 s}}{(\lambda_2-2\lambda_1)^2}+\frac{s^3e^{-2\lambda_1 s}}{\lambda_2-2\lambda_1}\right)$$

$$-\theta\lambda_1 2\lambda_2\left(\frac{6(e^{-2\lambda_2 s}-e^{-\lambda_1 s})}{(2\lambda_2-\lambda_1)^4}+\frac{6se^{-\lambda_1 s}}{(2\lambda_2-\lambda_1)^3}-\frac{3s^2e^{-\lambda_1 s}}{(2\lambda_2-\lambda_1)^2}+\frac{s^3e^{-\lambda_1 s}}{2\lambda_2-\lambda_1}\right)$$

$$+\theta 2\lambda_1 2\lambda_2\left(\frac{6(e^{-2\lambda_2 s}-e^{-2\lambda_1 s})}{(2\lambda_2-2\lambda_1)^4}+\frac{6se^{-2\lambda_1 s}}{(2\lambda_2-2\lambda_1)^3}-\frac{3s^2e^{-2\lambda_1 s}}{(2\lambda_2-2\lambda_1)^2}+\frac{s^3e^{-2\lambda_1 s}}{2\lambda_2-2\lambda_1}\right). \quad (3.7)$$

Define

$$\lambda_1\lambda_2\int_{\mathrm{VaR}_\kappa(S)}^\infty\left(\frac{6(e^{-\lambda_2 s}-e^{-\lambda_1 s})}{(\lambda_2-\lambda_1)^4}+\frac{6se^{-\lambda_1 s}}{(\lambda_2-\lambda_1)^3}-\frac{3s^2e^{-\lambda_1 s}}{(\lambda_2-\lambda_1)^2}+\frac{s^3e^{-\lambda_1 s}}{\lambda_2-\lambda_1}\right)\mathrm{d}s$$

$$=\frac{e^{-\lambda_1\mathrm{VaR}_\kappa(S)}\lambda_2\left(\frac{6}{\lambda_1^3}+\frac{6\mathrm{VaR}_\kappa(S)}{\lambda_1^2}+\frac{3(\mathrm{VaR}_\kappa(S))^2}{\lambda_1}+(\mathrm{VaR}_\kappa(S))^3\right)}{\lambda_2-\lambda_1}$$

$$-\frac{3e^{-\lambda_1\mathrm{VaR}_\kappa(S)}\lambda_2\left(\frac{2}{\lambda_1^2}+\frac{2\mathrm{VaR}_\kappa(S)}{\lambda_1}+(\mathrm{VaR}_\kappa(S))^2\right)}{(\lambda_2-\lambda_1)^2}$$

$$+\frac{6\lambda_2 e^{-\lambda_1\mathrm{VaR}_\kappa(S)}(\mathrm{VaR}_\kappa(S)+\frac{1}{\lambda_1})}{(\lambda_2-\lambda_1)^3}-\frac{6(\lambda_2 e^{-\lambda_1\mathrm{VaR}_\kappa(S)}-\lambda_1 e^{-\lambda_2\mathrm{VaR}_\kappa(S)})}{(\lambda_2-\lambda_1)^4}$$

$$=\xi_3(\mathrm{VaR}_\kappa(S);\lambda_1;\lambda_2). \quad (3.8)$$

Plugging Eqs.(3.7) and (3.8) into Eq.(3.6), we can derive the formula of

$$E^{X_1^3}=\frac{1}{1-\kappa}\left((1+\theta)\xi_3(\mathrm{VaR}_\kappa(S);\lambda_1;\lambda_2)-\theta\xi_3(\mathrm{VaR}_\kappa(S);2\lambda_1;\lambda_2)\right.$$
$$\left.-\theta\xi_3(\mathrm{VaR}_\kappa(S);\lambda_1;2\lambda_2)+\theta\xi_3(\mathrm{VaR}_\kappa(S);2\lambda_1;2\lambda_2)\right)$$

The formula of $E^{X_2^3}$ is symmetrically given by

$$E^{X_2^3}=\frac{1}{1-\kappa}\left((1+\theta)\xi_3(\mathrm{VaR}_\kappa(S);\lambda_2;\lambda_1)-\theta\xi_3(\mathrm{VaR}_\kappa(S);2\lambda_2;\lambda_1)\right.$$
$$\left.-\theta\xi_3(\mathrm{VaR}_\kappa(S);\lambda_2;2\lambda_1)+\theta\xi_3(\mathrm{VaR}_\kappa(S);2\lambda_1;2\lambda_2)\right).$$

Since the formulae of E^{X_i} and $E^{X_i^2}$ can be calculated similarly, we omit the proof. Now, we give the formula of $E^{X_1^2 X_2}$.

$$E^{X_1^2 X_2}=\mathbf{E}\left[X_1^2 X_2|S>\mathrm{VaR}_\kappa(S)\right]=\int_{\mathrm{VaR}_\kappa(S)}^\infty\mathbf{E}\left[X_1^2 X_2|S=s\right]f_{S|S>\mathrm{VaR}_\kappa(S)}(s)\mathrm{d}s$$

$$=\int_{\mathrm{VaR}_\kappa(S)}^\infty\mathbf{E}\left[X_1^2(s-X_1)|S=s\right]f_{S|S>\mathrm{VaR}_\kappa(S)}(s)\mathrm{d}s$$

$$=\frac{\int_{\mathrm{VaR}_\kappa(S)}^\infty\mathbf{E}\left[X_1^2(s-X_1)|S=s\right]f_S(s)\mathrm{d}s}{\Pr(S>\mathrm{VaR}_\kappa(S))}=\frac{\int_{\mathrm{VaR}_\kappa(S)}^\infty\int_0^s x^2(s-x)f_{x_1,S}(x,s)\mathrm{d}x\mathrm{d}s}{1-\kappa}.$$

Note that

$$\int_0^s x^2(s-x)f_{x_1,S}(x,s)\mathrm{d}x=s\int_0^s x^2 f_{x_1,x_2}(x,s-x)\mathrm{d}x-\int_0^s x^3 f_{x_1,x_2}(x,s-x)\mathrm{d}x.$$

By the similar calculations, we can derive

$$E^{X_1^2 X_2}=\frac{1}{1-\kappa}\left((1+\theta)\chi(\mathrm{VaR}_\kappa(S);\lambda_1;\lambda_2)-\theta\chi(\mathrm{VaR}_\kappa(S);2\lambda_1;\lambda_2)\right.$$
$$\left.-\theta\chi(\mathrm{VaR}_\kappa(S);\lambda_1;2\lambda_2)+\theta\chi(\mathrm{VaR}_\kappa(S);2\lambda_1;2\lambda_2)\right).$$

The formula of $E^{X_2^2 X_1}$ is also symmetrically given by

$$E^{X_2^2 X_1}=\frac{1}{1-\kappa}\left((1+\theta)\chi(\mathrm{VaR}_\kappa(S);\lambda_2;\lambda_1)-\theta\chi(\mathrm{VaR}_\kappa(S);2\lambda_2;\lambda_1)\right.$$
$$\left.-\theta\chi(\mathrm{VaR}_\kappa(S);\lambda_2;2\lambda_1)+\theta\chi(\mathrm{VaR}_\kappa(S);2\lambda_2;2\lambda_2)\right).$$

Since the formula of $E^{X_1 X_2}$ can be calculated similarly, we also omit the proof. This completes the proof of this theorem.

Remark 3.1. In Theorem 3.1, we assume that $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq 2\lambda_2$, and $\lambda_2 \neq 2\lambda_1$. In fact, one can obtains the adjusted results without these restrictions. Since the first part on the right-hand side of equation (3.6) can be rewritten as

$$(1+\theta)\lambda_1\lambda_2 e^{-\lambda_1 s}\left(\frac{6(e^{(\lambda_1-\lambda_2)s}-1)}{(\lambda_2-\lambda_1)^4} + \frac{6s}{(\lambda_2-\lambda_1)^3} - \frac{3s^2}{(\lambda_2-\lambda_1)^2} + \frac{s^3}{\lambda_2-\lambda_1}\right),$$

where $e^{(\lambda_1-\lambda_2)s}-1$ could be expanded in Talyor series if $\lambda_2 \to \lambda_1$,

$$e^{(\lambda_1-\lambda_2)s}-1 = (\lambda_1-\lambda_2)s + \frac{(\lambda_1-\lambda_2)^2 s^2}{2} + \frac{(\lambda_1-\lambda_2)^3 s^3}{6} + \sum_{n=4}^{\infty}\frac{(\lambda_1-\lambda_2)^n s^n}{n!},$$

then the limit of the first part on the right-hand side of equation (3.6) is

$$\lim_{\lambda_2\to\lambda_1}(1+\theta)\lambda_1\lambda_2 e^{-\lambda_1 s}\left(\frac{6(e^{(\lambda_1-\lambda_2)s}-1)}{(\lambda_2-\lambda_1)^4} + \frac{6s}{(\lambda_2-\lambda_1)^3} - \frac{3s^2}{(\lambda_2-\lambda_1)^2} + \frac{s^3}{\lambda_2-\lambda_1}\right)$$

$$= \lim_{\lambda_2\to\lambda_1}6(1+\theta)\lambda_1\lambda_2 e^{-\lambda_1 s}\sum_{n=4}^{\infty}\frac{(\lambda_1-\lambda_2)^{n-4}s^n}{n!} = \frac{(1+\theta)\lambda_1^2}{4}e^{-\lambda_1 s}s^4. \qquad (3.9)$$

For the case $\lambda_1 = \lambda_2$, the first component of $f_{X_1,X_2}(x_1,x_2)$ can be rewritten as $(1+\theta)\lambda_1^2 e^{-\lambda_1(x_1+x_2)}$, the first part on the right-hand side of equation (3.6) can be simplified as

$$\int_0^s x^3(1+\theta)\lambda_1^2 e^{-\lambda_1(x+(s-x))}\mathrm{d}x = \frac{(1+\theta)\lambda_1^2}{4}e^{-\lambda_1 s}s^4. \qquad (3.10)$$

By the results of (3.9) and (3.10), for the case $\lambda_1 = \lambda_2$, we can apply the similar technique to obtain the adjusted results. For the cases $\lambda_1 = 2\lambda_2$ and $\lambda_2 = 2\lambda_1$, one obtains the similar results.

4. Calculation of the optimal capital allocation for n risks

Explicit formulae for the optimal capital allocation for risks linked by a FGM copula cannot only be derived in the bivariate case but even for an undefined number of risks. In order to derive the explicit formulae for the optimal capital allocation, we first give the following lemma.

Lemma 4.1. Let $\vec{X} = (X_1, X_2, \cdots, X_n)$. Then the optimal allocation solution $\vec{d^*} = (d_1^*, d_2^*, \cdots, d_n^*)$ to the TMV model,

$$\min_{d_1,\cdots,d_n}\left\{\mathbf{E}\left[\sum_{i=1}^{n}(X_i-d_i)^2 | S > \mathrm{VaR}_\kappa(S)\right] + \beta\mathrm{Var}\left(\sum_{i=1}^{n}(X_i-d_i)^2 | S > \mathrm{VaR}_\kappa(S)\right)\right\},$$

$$\text{such that } \sum_{i=1}^{n}d_i = d,$$

is presented as follows:

$$\vec{d^*} = \mathbf{A}^{-1}\vec{z}, \qquad (4.1)$$

where $\mathbf{A}^{-1} = (a_{ij})_{n\times n}$ is the inverse matrix of $\mathbf{A} = 8\beta\Sigma_S + 2\mathbf{I}_n$, Σ_S is the conditional covariance matrix of $(\vec{X}|S > \mathrm{VaR}_\kappa(S))$, and $\vec{z} = (\Lambda + \delta_1, \Lambda + \delta_2, \cdots, \Lambda + \delta_n)$ with $\delta_i = 4\beta\sum_{j=1}^{n}\mathrm{Cov}(X_j^2, X_i|S > \mathrm{VaR}_\kappa(S)) + 2\mathbf{E}[X_i|S > \mathrm{VaR}_\kappa(S)]$, $i = 1, 2, \cdots, n$, where $\Lambda = (d - \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\delta_j)/(\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij})$.

Proof. The idea of proof is also based on the method of Lagrange multiplier. One can refer to Xu and Mao [17] for the full proof.

Note that Lemma 4.1 provides general formulae for optimal allocation based on the TMV model. There are some quantities, such as $\mathbf{E}[X_i|S > \text{Var}_\kappa(S)]$, $\text{Cov}(X_j^2, X_i|S > \text{VaR}_\kappa(S))$, and Σ_S, which are non-trivial to calculate. In order to find $\mathbf{E}[X_i|S > \text{Var}_\kappa(S)]$, $\text{Cov}(X_j^2, X_i|S > \text{VaR}_\kappa(S))$, and Σ_S, for the sake of deriving the explicit formulae for the optimal capital allocation, we need to derive the key quantities $E^{X_i^k} = \mathbf{E}[X_i^k|S > \text{VaR}_\kappa(S)]$, $E^{X_i^2 X_j} = \mathbf{E}[X_i^2 X_j|S > \text{VaR}_\kappa(S)]$, and $E^{X_i X_j} = \mathbf{E}[X_i X_j|S > \text{VaR}_\kappa(S)]$, $k = 1, 2, 3$, $i, j = 1, 2, \cdots, n$, $i \neq j$.

As in the bivariate case, the explicit formulae for the optimal capital allocation can also be explicitly given. In order to find the key quantities, we first introduce the nth order divided difference of the function f presented in Chiragiev and Landsman [6]. Assume that $x_1, x_2, \cdots, x_n, x_{n+1}$ are arbitrary points and $x_i \neq x_j$ for $i \neq j$. The nth order divided difference of the function f on $x_1, x_2, \cdots, x_n, x_{n+1}$ is defined as

$$f(x_1, x_2, \cdots, x_n, x_{n+1}) = \sum_{i=1}^{n+1} \frac{f(x_i)}{\prod_{j \neq i}(x_i - x_j)}.$$

In the follow proposition, we provide the explicit formulae of $E^{X_i^k}$, $E^{X_i^2 X_j}$ and $E^{X_i X_j}$ for n exponentially distributed risks linked by a multivariate FGM copula. Let $\vec{X} = (X_1, X_2, \cdots, X_n)$, X_1, X_2, $\cdots$, X_n are n exponentially distributed with λ_1, λ_2, $\cdots$, λ_n. As in the bivariate case, we assume that $\lambda_i \neq \lambda_j$ and $\lambda_i \neq 2\lambda_j$ for $i \neq j$.

Proposition 4.2. Let $\vec{X} = (X_1, X_2, \cdots, X_n)$, X_1, X_2, $\cdots$,X_n are n exponentially distributed with parameters λ_1, λ_2, $\cdots$, λ_n. The dependence structure for $(X_1, X_2, \cdots, X_n)$ is defined by a multivariate FGM copula. Then quantities $E^{X_i^k}$, $E^{X_i^2 X_j}$ and $E^{X_i X_j}$, $i \neq j$, $i, j = 1, 2, \cdots, n$, $k = 1, 2, 3$, are presented as follows:

$$E^{X_i^v X_j} = \frac{(-1)^{n-2}\eta}{1-\kappa}\{\bar{L}_{X_i^v X_j}(\text{VaR}_\kappa(S); \lambda_1; \cdots; \lambda_n) + \sum_{q=2}^{n} \sum_{1 \leq p_1 < \cdots < p_q \leq n \bigcap i \bigcap j \in \{p_1, \cdots, p_q\}}$$

$$\theta_{p_1 \cdots p_q}(\sum_{l=0}^{q} \sum_{(\epsilon_1, \cdots, \epsilon_q)-i-j \in \tau_{l, q-2}} [(-1)^l 2^{q-2-l} \times$$

$$\bar{L}_{X_i^v X_j}(\text{VaR}_\kappa(S); 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}}; \cdots; \lambda_{i_n})$$

$$+(-1)^{l+1}2^{q-2-l}\bar{M}_{X_i^v X_j}(\text{VaR}_\kappa(S); 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}}; \cdots; \lambda_{i_n})$$

$$+(-1)^{l+1}2^{q-2-l}\bar{N}_{X_i^v X_j}(\text{VaR}_\kappa(S); 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}}; \cdots; \lambda_{i_n})$$

$$+ (-1)^l 2^{q-2-l}\bar{Q}_{X_i^v X_j}(\text{VaR}_\kappa(S); 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}}; \cdots; \lambda_{i_n})])$$

$$+\sum_{q=2}^{n} \sum_{\substack{1 \leq p_1 < \cdots < p_q \leq n \bigcap i \in \{p_1, \cdots, p_q\} \\ 1 \leq p_1 < \cdots < p_q \leq n \bigcap j \notin \{p_1, \cdots, p_q\}}} \theta_{p_1 \cdots p_q}(\sum_{l=0}^{q} \sum_{(\epsilon_1, \cdots, \epsilon_q)-i \in \tau_{l, q-1}}$$

$$[(-1)^{l+1}2^{q-1-l}\bar{L}_{X_i^v X_j}(\text{VaR}_\kappa(S); 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}}; \cdots; \lambda_{i_n})$$

$$+(-1)^l 2^{q-1-l}\bar{M}_{X_i^v X_j}(\text{VaR}_\kappa(S); 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}}; \cdots; \lambda_{i_n})])$$

$$+\sum_{q=2}^{n} \sum_{\substack{1 \leq p_1 < \cdots < p_q \leq n \bigcap i \notin \{p_1, \cdots, p_q\} \\ 1 \leq p_1 < \cdots < p_q \leq n \bigcap j \in \{p_1, \cdots, p_q\}}} \theta_{p_1 \cdots p_q}(\sum_{l=0}^{q} \sum_{(\epsilon_1, \cdots, \epsilon_q)-j \in \tau_{l, q-1}}$$

$$[(-1)^{l+1}2^{q-1-l}\bar{L}_{X_i^v X_j}(\text{VaR}_\kappa(S); 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}}; \cdots; \lambda_{i_n})$$

$$+(-1)^l 2^{q-1-l} \bar{N}_{X_i^v X_j}(\text{VaR}_\kappa(S); 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}}; \cdots; \lambda_{i_n})])$$

$$+\sum_{q=2}^n \sum_{\substack{1 \le p_1 < \cdots < p_q \le n \bigcap i \notin \{p_1,\cdots,p_q\} \\ 1 \le p_1 < \cdots < p_q \le n \bigcap j \notin \{p_1,\cdots,p_q\}}} \theta_{p_1\cdots p_q}(\sum_{l=0}^q \sum_{(\epsilon_1,\cdots,\epsilon_q)\in\tau_{l,q}} (-1)^l 2^{q-l}$$

$$\bar{L}_{X_i^v X_j}(\text{VaR}_\kappa(S); 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}}; \cdots; \lambda_{i_n}))\}, \ v = 1, 2, \ i \neq j, \qquad (4.2)$$

$$E^{X_i^k} = \frac{(-1)^{n-1}\eta}{1-\kappa}\{\bar{L}_{X_i^k}(\text{VaR}_\kappa(S); \lambda_1; \cdots; \lambda_n) + \sum_{q=2}^n \sum_{1 \le p_1 < \cdots < p_q \le n \bigcap i \in \{p_1,\cdots,p_q\}} \theta_{p_1\cdots p_q} \times$$

$$[\sum_{l=0}^{q-1} \sum_{(\epsilon_1,\cdots,\epsilon_q)-i\in\tau_{l,q-1}} \{(-1)^{l+1}2^{q-1-l}\bar{L}_{X_i^k}(\text{VaR}_\kappa(S); 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}} \cdots; \lambda_{i_n})$$

$$+(-1)^l 2^{q-1-l}\bar{M}_{X_i^k}(\text{VaR}_\kappa(S); 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}} \cdots; \lambda_{i_n})\}]$$

$$+\sum_{q=2}^n \sum_{1 \le p_1 < \cdots < p_q \le n \bigcap i \notin \{p_1,\cdots,p_q\}} \theta_{p_1\cdots p_q}[\sum_{l=0}^q \sum_{(\epsilon_1,\cdots,\epsilon_q)\in\tau_{l,q}} (-1)^l 2^{q-l} \times$$

$$\bar{L}_{X_i^k}(\text{VaR}_\kappa(S); 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}} \cdots; \lambda_{i_n})]\}, \quad k = 1, 2, 3, \qquad (4.3)$$

where $\eta = \lambda_1 \times \cdots \times \lambda_n$, $i_{q+1}, \cdots, i_n$ are the missing indexes of $p_1, \cdots, p_q$ to complete $1, \cdots, n$, $\tau_{l,q}$ are the sets of q-tuples composed of l zeros and $q-l$ ones, for $l = 0, 1, \cdots, q$ and $q = 2, \cdots, n$, $\bar{L}_{X_i^k}(x; \alpha_1; \cdots; \alpha_{i-1}; \alpha_{i+1}; \cdots; \alpha_n)$ and $\bar{M}_{X_i^k}(x; \alpha_1; \cdots; \alpha_{i-1}; \alpha_{i+1}; \cdots; \alpha_n)$ are the $(n-2)$th order divided differences of $\bar{L}_{X_i^k}(x; \alpha)$ and $\bar{M}_{X_i^k}(x; \alpha)$, respectively, and $\bar{L}_{X_i^v X_j}(x; \alpha_1; \cdots; \alpha_{i-1}; \alpha_{i+1}; \cdots; \alpha_{j-1}; \alpha_{j+1}; \cdots; \alpha_n)$, $\bar{M}_{X_i^v X_j}(x; \alpha_1; \cdots; \alpha_{i-1};$ $\alpha_{i+1}; \cdots; \alpha_{j-1}; \alpha_{j+1}; \cdots; \alpha_n)$, $\bar{N}_{X_i^v X_j}(x; \alpha_1; \cdots; \alpha_{i-1}; \alpha_{i+1}; \cdots; \alpha_{j-1}; \alpha_{j+1}; \cdots; \alpha_n)$, and $\bar{Q}_{X_i^v X_j}(x; \alpha_1; \cdots; \alpha_{i-1}; \alpha_{i+1}; \cdots; \alpha_{j-1}; \alpha_{j+1}; \cdots; \alpha_n)$ are the $(n-3)$th order divided differences of $\bar{L}_{X_i^v X_j}(x; \alpha)$, $\bar{M}_{X_i^v X_j}(x; \alpha)$, $\bar{N}_{X_i^v X_j}(x; \alpha)$ and $\bar{Q}_{X_i^v X_j}(x; \alpha)$, respectively. The expressions of $\bar{L}_{X_i^k}(x; \alpha)$ and $\bar{M}_{X_i^k}(x; \alpha)$ are given in Eqs. (4.27)-(4.32) and the expressions of $\bar{L}_{X_i^v X_j}(x; \alpha)$, $\bar{M}_{X_i^v X_j}(x; \alpha)$, $\bar{N}_{X_i^v X_j}(x; \alpha)$ and $\bar{Q}_{X_i^v X_j}(x; \alpha)$ are given in Eqs. (4.14)-(4.21).

Proof. We first calculate the explicit the formulae of $E^{X_i^2 X_j}$, $i \neq j$, $i, j = 1, 2, \cdots, n$.

$$E^{X_i^2 X_j} = \mathbf{E}\left[X_i^2 X_j | S > \text{VaR}_\kappa(S)\right] = \int_{\text{VaR}_\kappa(S)}^\infty \mathbf{E}\left[X_i^2 X_j | S = s\right] f_{S | S > \text{VaR}_\kappa(S)}(s)\mathrm{d}s$$

$$= \frac{\int_{\text{VaR}_\kappa(S)}^\infty \int_0^s \int_0^{s-x_i} x_i^2 x_j f_{X_i, X_j, S}(x_i, x_j, s)\mathrm{d}x_j \mathrm{d}x_i \mathrm{d}s}{1-\kappa}. \qquad (4.4)$$

In order to evaluate Eq.(4.4), we need to derive a recursive formula for $f_{X_i, X_j, S}(x_i, x_j, s) = f_{X_i, X_j, S-X_i-X_j}(x_i, x_j, s-x_i-x_j)$. Given that the random variables $X_1, X_1, \cdots, X_n$ are not independent, we cannot separate $f_{X_i, X_j, S-X_i-X_j}(x_i, x_j, s-x_i-x_j)$ into the product of $f_{X_i}(x_i)$, $f_{X_j}(x_j)$ and $f_{S-X_i-X_j}(s-x_i-x_j)$. Now, we derive the recursive formula for $f_{X_i, X_j, S-X_i-X_j}(x_i, x_j, s-x_i-x_j)$. First, we rewrite the $f_{X_1,\cdots,X_n}(x_1, x_2, \cdots, x_n)$ defined in Eq.(2.2) as

$$f_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = f_{X_i}(x_i)f_{X_j}(x_j)\{\omega_{-i-j}(x_1; \cdots; x_n; \lambda_1; \cdots; \lambda_n)$$

$$+\sum_{q=2}^n \sum_{\substack{1 \le p_1 < \cdots < p_q \le n \bigcap \\ i \bigcap j \in \{p_1,\cdots,p_q\}}} \theta_{p_1\cdots p_q}(1 - 2F_{X_i}(x_i))(1 - 2F_{X_j}(x_j))[\sum_{l=0}^q \sum_{(\epsilon_1,\cdots,\epsilon_q)-i-j\in\tau_{l,q-2}}$$

$$(-1)^l \omega_{-i-j}(x_{p_1}; \cdots; x_{p_q}; x_{i_{q+1}}; \cdots; x_{i_n}; 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}} \cdots; \lambda_{i_n})]$$

$$+\sum_{q=2}^{n}\sum_{\substack{1\le p_1<\cdots<p_q\le n\bigcap i\in\{p_1,\cdots,p_q\}\\1\le p_1<\cdots<p_q\le n\bigcap j\notin\{p_1,\cdots,p_q\}}}\theta_{p_1\cdots p_q}(1-2F_{X_i}(x_i))[\sum_{l=0}^{q}\sum_{(\epsilon_1,\cdots,\epsilon_q)-i\in\tau_{l,q-1}}$$

$$(-1)^l\omega_{-i-j}(x_{p_1};\cdots;x_{p_q};x_{i_{q+1}};\cdots;x_{i_n};2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}}\cdots;\lambda_{i_n})]$$

$$+\sum_{q=2}^{n}\sum_{\substack{1\le p_1<\cdots<p_q\le n\bigcap i\notin\{p_1,\cdots,p_q\}\\1\le p_1<\cdots<p_q\le n\bigcap j\in\{p_1,\cdots,p_q\}}}\theta_{p_1\cdots p_q}(1-2F_{X_j}(x_j))[\sum_{l=0}^{q}\sum_{(\epsilon_1,\cdots,\epsilon_q)-j\in\tau_{l,q-1}}$$

$$(-1)^l\omega_{-i-j}(x_{p_1};\cdots;x_{p_q};x_{i_{q+1}};\cdots;x_{i_n};2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}}\cdots;\lambda_{i_n})]$$

$$+\sum_{q=2}^{n}\sum_{\substack{1\le p_1<\cdots<p_q\le n\bigcap i\notin\{p_1,\cdots,p_q\}\\1\le p_1<\cdots<p_q\le n\bigcap j\notin\{p_1,\cdots,p_q\}}}\theta_{p_1\cdots p_q}[\sum_{l=0}^{q}\sum_{(\epsilon_1,\cdots,\epsilon_q)\in\tau_{l,q}}$$

$$(-1)^l\omega_{-i-j}(x_{p_1};\cdots;x_{p_q};x_{i_{q+1}};\cdots;x_{i_n};2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}}\cdots;\lambda_{i_n})]\},\quad(4.5)$$

where $\omega_{-i-j}(x_1;x_2;\cdots;x_n;\alpha_1;\alpha_2;\cdots;\alpha_n)=\alpha_1e^{-\alpha_1x_1}\times\cdots\times\alpha_{i-1}e^{-\alpha_{i-1}x_{i-1}}\times\alpha_{i+1}e^{-\alpha_{i+1}x_{i+1}}\times\cdots\times\alpha_{j-1}e^{-\alpha_{j-1}x_{j-1}}\times\alpha_{j+1}e^{-\alpha_{j+1}x_{j+1}}\cdots\alpha_ne^{-\alpha_nx_n}$. Assume that

$$\int_0^{s-x_i-x_j}\int_0^{s-x_1-x_i-x_j}\cdots\int_0^{s-x_1-\cdots-x_{n-1}}\omega_{-i-j}(x_1;\ldots;x_n;\alpha_1;\ldots;\alpha_n)dx_1\cdots dx_{i-1}\times$$

$$dx_{i+1}\cdots dx_{j-1}dx_{j+1}\cdots dx_{n-1}$$

$$=\sum_{l=1,l\neq i,l\neq j}^{n}\left(\prod_{q=1,q\neq l,q\neq i,q\neq j}\frac{\alpha_q}{\alpha_q-\alpha_l}\right)\alpha_le^{-\alpha_l(s-x_i-x_j)}=h_{-i-j}(s-x_i-x_j;\alpha_1;\ldots;\alpha_n).$$
$$(4.6)$$

Note that

$$f_{X_i,X_j,S-X_i-X_j}(x_i,x_j,s-x_i-x_j)$$

$$=f_{X_i,X_j,X_1+\cdots+X_{i-1}+X_{i+1}+\cdots+X_{j-1}+X_{j+1}+\cdots+X_n}(x_i,x_j,s-x_i-x_j)$$

$$=\int_0^{s-x_i-x_j}\int_0^{s-x_1-x_i-x_j}\cdots\int_0^{s-x_1-\cdots-x_{n-1}}f_{X_1,\ldots,X_n}(x_1,\ldots,x_{n-1},s-x_1-\cdots-x_{n-1})$$

$$\times dx_1\cdots dx_{i-1}dx_{i+1}\cdots dx_{j-1}dx_{j+1}\cdots dx_{n-1}.\qquad(4.7)$$

Plugging (4.5) into (4.7), we have

$$f_{X_i,X_j,S-X_i-X_j}(x_i,x_j,s-x_i-x_j)$$

$$=f_{X_i}(x_i)f_{X_j}(x_j)\Big\{h_{-i-j}(s-x_i-x_j;\lambda_1;\cdots;\lambda_{i-1};\lambda_{i+1};\cdots;\lambda_{j-1};\cdots;\lambda_{j+1};\cdots;\lambda_n)$$

$$+\sum_{q=2}^{n}\sum_{1\le p_1<\cdots p_q\le n\bigcap i\bigcap j\in\{p_1,\cdots,p_q\}}\theta_{p_1\cdots p_q}(1-2F_{X_i}(x_i))(1-2F_{X_j}(x_j))\times$$

$$\left[\sum_{l=0}^{q}\sum_{(\epsilon_1,\cdots,\epsilon_q)-i-j\in\tau_{l,q-2}}(-1)^lh_{-i-j}(s-x_i-x_j;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}}\cdots;\lambda_{i_n})\right]$$

$$+\sum_{q=2}^{n}\sum_{\substack{1\le p_1<\cdots<p_q\le n\bigcap i\in\{p_1,\cdots,p_q\}\\1\le p_1<\cdots<p_q\le n\bigcap j\notin\{p_1,\cdots,p_q\}}}\theta_{p_1\cdots p_q}(1-2F_{X_i}(x_i))\times$$

$$\left[\sum_{l=0}^{q}\sum_{(\epsilon_1,\cdots,\epsilon_q)-i\in\tau_{l,q-1}}(-1)^lh_{-i-j}(s-x_i-x_j;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}}\cdots;\lambda_{i_n})\right]$$

$$+\sum_{q=2}^{n}\sum_{\substack{1\leq p_1<\cdots<p_q\leq n\bigcap i\notin\{p_1,\cdots,p_q\}\\1\leq p_1<\cdots<p_q\leq n\bigcap j\in\{p_1,\cdots,p_q\}}}\theta_{p_1\cdots p_q}(1-2F_{X_j}(x_j))\times$$

$$\left[\sum_{l=0}^{q}\sum_{(\epsilon_1,\cdots,\epsilon_q)-j\in\tau_{l,q-1}}(-1)^l h_{-i-j}(s-x_i-x_j;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}}\cdots;\lambda_{i_n})\right]$$

$$+\sum_{q=2}^{n}\sum_{\substack{1\leq p_1<\cdots<p_q\leq n\bigcap i\notin\{p_1,\cdots,p_q\}\\1\leq p_1<\cdots<p_q\leq n\bigcap j\notin\{p_1,\cdots,p_q\}}}\theta_{p_1\cdots p_q}\times$$

$$\left[\sum_{l=0}^{q}\sum_{(\epsilon_1,\cdots,\epsilon_q)\in\tau_{l,q}}(-1)^l h_{-i-j}(s-x_i-x_j;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}}\cdots;\lambda_{i_n})\right]. \quad (4.8)$$

Utilizing the dividend difference presented in Chiragiev and Landsman [6], note that

$$h_{-i-j}(x;\alpha_1;\cdots;\alpha_{i-1};\alpha_{i+1};\cdots;\alpha_{j-1};\alpha_{j+1};\cdots;\alpha_n)=(-1)^{n-3}\times\alpha_1\times\cdots\times\alpha_{i-1}\times$$

$\alpha_{i+1}\times\cdots\times\alpha_{j-1}\times\alpha_{j+1}\times\cdots\times\alpha_n\times\bar{\Lambda}(x;\alpha_1;\cdots;\alpha_{i-1};\alpha_{i+1};\cdots;\alpha_{j-1};\alpha_{j+1};\cdots;\alpha_n),$
where $\bar{\Lambda}(x;\alpha_1;\cdots;\alpha_{i-1};\alpha_{i+1};\cdots;\alpha_{j-1};\alpha_{j+1};\cdots;\alpha_n)$ is the $(n-3)$th order divided difference of $\bar{\Lambda}(x;\alpha)=e^{-\alpha x}$. Then, we get

$$\int_0^s\int_0^{s-x_i}x_i^2 x_j f_{X_i}(x_i)f_{X_j}(x_j)h_{-i-j}(s-x_i-x_j;\lambda_1;\cdots;\lambda_{i-1};$$

$$\lambda_{i+1};\cdots;\lambda_{j-1};\cdots;\lambda_{j+1};\cdots;\lambda_n)\mathrm{d}x_j\mathrm{d}x_i$$

$$=\int_0^s\int_0^{s-x_i}x_i^2 x_j f_{X_i}(x_i)f_{X_j}(x_j)(-1)^{n-3}\eta_{-i-j}\bar{\Lambda}(x;\lambda_1;\cdots;\lambda_{i-1};$$

$$\lambda_{i+1};\cdots;\lambda_{j-1};\lambda_{j+1};\cdots;\lambda_n)\mathrm{d}x_j\mathrm{d}x_i$$

$$=(-1)^{n-2}\eta L_{X_i^2 X_j}(s;\lambda_1;\cdots;\lambda_{i-1};\lambda_{i+1};\cdots;\lambda_{j-1};\lambda_{j+1};\cdots;\lambda_n), \quad (4.9)$$

where $\eta_{-i-j}=\lambda_1\times\cdots\times\lambda_{i-1}\times\lambda_{i+1}\times\cdots\times\lambda_{j-1}\times\lambda_{j+1}\times\cdots\times\lambda_n$, $L_{X_i^2 X_j}(s;\lambda)=-\int_0^s\int_0^{s-x_i}x_i^2 x_j e^{-\lambda_i x_i}e^{-\lambda_j x_j}\bar{\Lambda}(s-x_i-x_j;\lambda)\mathrm{d}x_j\mathrm{d}x_i$. Let also $M_{X_i^2 X_j}(s;\lambda)=-\int_0^s\int_0^{s-x_i}2x_i^2 x_j e^{-2\lambda_i x_i}e^{-\lambda_j x_j}\bar{\Lambda}(s-x_i-x_j;\lambda)\mathrm{d}x_j\mathrm{d}x_i$, $N_{X_i^2 X_j}(s;\lambda)=-\int_0^s\int_0^{s-x_i}2x_i^2 x_j e^{-\lambda_i x_i}e^{-2\lambda_j x_j}\bar{\Lambda}(s-x_i-x_j;\lambda)\mathrm{d}x_j\mathrm{d}x_i$, and $O_{X_i^2 X_j}(s;\lambda)=-\int_0^s\int_0^{s-x_i}4x_i^2 x_j e^{-2\lambda_i x_i}e^{-2\lambda_j x_j}\bar{\Lambda}(s-x_i-x_j;\lambda)\mathrm{d}x_j\mathrm{d}x_i$, $i\neq j$, $i,j=1,2,\cdots,n$.

Given that the risks are exponentially distributed, then $L_{X_i^2 X_j}$, $M_{X_i^2 X_j}$, $N_{X_i^2 X_j}$ and $O_{X_i^2 X_j}$ can be rewritten as

$$L_{X_i^2 X_j}(s;\lambda)=\frac{-2e^{-\lambda s}}{(\lambda_i-\lambda)^3(\lambda_j-\lambda)^2}+\frac{2(\lambda_i-4\lambda_j+3\lambda)e^{-\lambda_j s}}{(\lambda_i-\lambda_j)^4(\lambda_j-\lambda)^2}+\frac{2s(3\lambda_i-\lambda_j-2\lambda))e^{-\lambda_i s}}{(\lambda_i-\lambda_j)^3(\lambda_i-\lambda)^2}$$

$$+\frac{2se^{-\lambda_j s}}{(\lambda_i-\lambda_j)^3(\lambda_j-\lambda)}+\frac{s^2 e^{-\lambda_i s}}{(\lambda_i-\lambda_j)^2(\lambda_i-\lambda)}+\frac{2(6\lambda_i^2+\lambda_j^2+2\lambda\lambda_j+3\lambda^2-4\lambda_i(\lambda_j+2\lambda))e^{-\lambda_i s}}{(\lambda_i-\lambda_j)^4(\lambda_i-\lambda)^3},$$

$$(4.10)$$

$$M_{X_i^2 X_j}(s;\lambda)=-2(\frac{2e^{-\lambda s}}{(2\lambda_i-\lambda)^3(\lambda_j-\lambda)^2}-\frac{2(2\lambda_i-4\lambda_j+3\lambda)e^{-\lambda_j s}}{(2\lambda_i-\lambda_j)^4(\lambda_j-\lambda)^2}-\frac{2se^{-\lambda_j s}}{(2\lambda_i-\lambda_j)^3(\lambda_j-\lambda)}$$

$$-\frac{s^2 e^{-2\lambda_i s}}{(2\lambda_i-\lambda_j)^2(2\lambda_i-\lambda)}-\frac{2s(3(2\lambda_i)-\lambda_j-2\lambda))e^{-2\lambda_i s}}{(2\lambda_i-\lambda_j)^3(2\lambda_i-\lambda)^2}$$

$$-\frac{2(6(2\lambda_i)^2+\lambda_j^2+2\lambda\lambda_j+3\lambda^2-4(2\lambda_i)(\lambda_j+2\lambda))e^{-2\lambda_i s}}{(2\lambda_i-\lambda_j)^4(2\lambda_i-\lambda)^3}), \quad (4.11)$$

$$N_{X_i^2 X_j}(s;\lambda)=-2(\frac{2e^{-\lambda s}}{(\lambda_i-\lambda)^3(2\lambda_j-\lambda)^2}-\frac{2(\lambda_i-4(2\lambda_j)+3\lambda)e^{-2\lambda_j s}}{(\lambda_i-2\lambda_j)^4(2\lambda_j-\lambda)^2}$$

$$-\frac{2se^{-2\lambda_j s}}{(\lambda_i-2\lambda_j)^3(2\lambda_j-\lambda)}-\frac{s^2e^{-\lambda_i s}}{(\lambda_i-2\lambda_j)^2(\lambda_i-\lambda)}-\frac{2s(3\lambda_i-2\lambda_j-2\lambda))e^{-\lambda_i s}}{(\lambda_i-2\lambda_j)^3(\lambda_i-\lambda)^2}$$

$$-\frac{2(6\lambda_i^2+(2\lambda_j)^2+2\lambda 2\lambda_j+3\lambda^2-4\lambda_i(2\lambda_j+2\lambda))e^{-\lambda_i s}}{(\lambda_i-2\lambda_j)^4(\lambda_i-\lambda)^3}), \tag{4.12}$$

$$O_{X_i^2 X_j}(s;\lambda)=-4(\frac{2e^{-\lambda s}}{(2\lambda_i-\lambda)^3(2\lambda_j-\lambda)^2}-\frac{2(2\lambda_i-4(2\lambda_j)+3\lambda)e^{-2\lambda_j s}}{(2\lambda_i-2\lambda_j)^4(2\lambda_j-\lambda)^2}$$

$$-\frac{2se^{-2\lambda_j s}}{(2\lambda_i-2\lambda_j)^3(2\lambda_j-\lambda)}-\frac{s^2e^{-2\lambda_i s}}{(2\lambda_i-2\lambda_j)^2(2\lambda_i-\lambda)}-\frac{2s(3(2\lambda_i)-2\lambda_j-2\lambda))e^{-2\lambda_i s}}{(2\lambda_i-2\lambda_j)^3(2\lambda_i-\lambda)^2}$$

$$-\frac{2(6(2\lambda_i)^2+(2\lambda_j)^2+2\lambda 2\lambda_j+3\lambda^2-4(2\lambda_i)(2\lambda_j+2\lambda))e^{-2\lambda_i s}}{(2\lambda_i-2\lambda_j)^4(2\lambda_i-\lambda)^3}). \tag{4.13}$$

Utilizing the results of (4.8) and (4.10)-(4.13), one can rewrite $\int_0^s\int_0^{s-x_i}x_i^2 x_j f_{x_i,x_j,S}\mathrm{d}x_j\mathrm{d}x_i$ as

$$\int_0^s\int_0^{s-x_i}x_i^2 x_j f_{x_i,x_j,S}\mathrm{d}x_j\mathrm{d}x_i=(-1)^{n-2}\eta\{L_{X_i^2 X_j}(s;\lambda_1;\cdots;\lambda_n)$$

$$+\sum_{q=2}^n\sum_{\substack{1\le p_1<\cdots<p_q\le n\bigcap\\i\bigcap j\in\{p_1,\cdots,p_q\}}}\theta_{p_1\cdots p_q}(\sum_{l=0}^q\sum_{(\epsilon_1,\cdots,\epsilon_q)-i-j\in\tau_{l,q-2}}(-1)^l 2^{q-2-l}\times$$

$$[L_{X_i^2 X_j}(s;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}};\cdots;\lambda_{i_n})$$

$$-M_{X_i^2 X_j}(s;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}};\cdots;\lambda_{i_n})$$

$$-N_{X_i^2 X_j}(s;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}};\cdots;\lambda_{i_n})$$

$$+Q_{X_i^2 X_j}(s;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}};\cdots;\lambda_{i_n})])$$

$$-\sum_{q=2}^n\sum_{\substack{1\le p_1<\cdots<p_q\le n\bigcap i\in\{p_1,\cdots,p_q\}\\1\le p_1<\cdots<p_q\le n\bigcap j\notin\{p_1,\cdots,p_q\}}}\theta_{p_1\cdots p_q}(\sum_{l=0}^q\sum_{(\epsilon_1,\cdots,\epsilon_q)-i\in\tau_{l,q-1}}(-1)^l 2^{q-1-l}\times$$

$$[L_{X_i^2 X_j}(s;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}};\cdots;\lambda_{i_n})$$

$$-M_{X_i^2 X_j}(s;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}};\cdots;\lambda_{i_n})])$$

$$-\sum_{q=2}^n\sum_{\substack{1\le p_1<\cdots<p_q\le n\bigcap i\notin\{p_1,\cdots,p_q\}\\1\le p_1<\cdots<p_q\le n\bigcap j\in\{p_1,\cdots,p_q\}}}\theta_{p_1\cdots p_q}(\sum_{l=0}^q\sum_{(\epsilon_1,\cdots,\epsilon_q)-j\in\tau_{l,q-1}}(-1)^l 2^{q-1-l}\times$$

$$[L_{X_i^2 X_j}(s;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}};\cdots;\lambda_{i_n})$$

$$-N_{X_i^2 X_j}(s;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}};\cdots;\lambda_{i_n})])$$

$$+\sum_{q=2}^n\sum_{\substack{1\le p_1<\cdots<p_q\le n\bigcap i\notin\{p_1,\cdots,p_q\}\\1\le p_1<\cdots<p_q\le n\bigcap j\notin\{p_1,\cdots,p_q\}}}\theta_{p_1\cdots p_q}\times$$

$$(\sum_{l=0}^q\sum_{(\epsilon_1,\cdots,\epsilon_q)\in\tau_{l,q}}(-1)^l 2^{q-l}L_{X_i^2 X_j}(s;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}};\cdots;\lambda_{i_n})).$$

In order to calculate the explicit the formulae of $E^{X_i^2 X_j}$, $i\ne j$, as in (4.4), the $L_{X_i^2 X_j}(s;\lambda)$, $M_{X_i^2 X_j}(s;\lambda)$, $N_{X_i^2 X_j}(s;\lambda)$ and $O_{X_i^2 X_j}(s;\lambda)$ terms in the equation above must be integrated on s as follows.

$$\bar{L}_{X_i^2 X_j}(V;\lambda)=\int_V^\infty L_{X_i^2 X_j}(s;\lambda)\mathrm{d}s=\frac{2(\lambda_i-4\lambda_j+3\lambda)e^{-\lambda_j V}}{\lambda_j(\lambda_i-\lambda_j)^4(\lambda_j-\lambda)^2}+\frac{2(1+V\lambda_j)e^{-\lambda_j V}}{\lambda_j^2(\lambda_i-\lambda_j)^3(\lambda_i-\lambda)}$$

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$$+\frac{(2+\lambda_i V(2+\lambda_i V))e^{-\lambda_i V}}{\lambda_i^3(\lambda_i-\lambda_j)^2(\lambda_i-\lambda)}+\frac{2(1+V\lambda_i)(3\lambda_i-\lambda_j-2\lambda))e^{-\lambda_i V}}{\lambda_i^2(\lambda_i-\lambda_j)^3(\lambda_i-\lambda)^2}$$

$$+\frac{2(6\lambda_i^2+\lambda_j^2+2\lambda\lambda_j+3\lambda^2-4\lambda_i(\lambda_j+2\lambda))e^{-\lambda_i V}}{\lambda_i(\lambda_i-\lambda_j)^4(\lambda_i-\lambda)^3}-\frac{2e^{-\lambda V}}{\lambda(\lambda_i-\lambda)^3(\lambda_j-\lambda)^2}, \qquad (4.14)$$

$$\bar{M}_{X_i^2 X_j}(V;\lambda)=2\Big(\frac{2(2\lambda_i-4\lambda_j+3\lambda)e^{-\lambda_j V}}{\lambda_j(2\lambda_i-\lambda_j)^4(\lambda_j-\lambda)^2}+\frac{2(1+V\lambda_j)e^{-\lambda_j V}}{\lambda_j^2(2\lambda_i-\lambda_j)^3(\lambda_j-\lambda)}$$

$$+\frac{(2+2\lambda_i V(2+2\lambda_i V))e^{-2\lambda_i V}}{(2\lambda_i)^3(2\lambda_i-\lambda_j)^2(2\lambda_i-\lambda)}+\frac{2(1+V2\lambda_i)(3(2\lambda_i)-\lambda_j-2\lambda))e^{-2\lambda_i V}}{(2\lambda_i)^2(2\lambda_i-\lambda_j)^3(2\lambda_i-\lambda)^2}$$

$$+\frac{2(6(2\lambda_i)^2+\lambda_j^2+2\lambda\lambda_j+3\lambda^2-4(2\lambda_i)(\lambda_j+2\lambda))e^{-2\lambda_i V}}{2\lambda_i(2\lambda_i-\lambda_j)^4(2\lambda_i-\lambda)^3}-\frac{2e^{-\lambda V}}{\lambda(2\lambda_i-\lambda)^3(\lambda_j-\lambda)^2}\Big),$$
$$(4.15)$$

$$\bar{N}_{X_i^2 X_j}(V;\lambda)=2\Big(\frac{2(\lambda_i-4(2\lambda_j)+3\lambda)e^{-2\lambda_j V}}{2\lambda_j(\lambda_i-2\lambda_j)^4(2\lambda_j-\lambda)^2}+\frac{2(1+V2\lambda_j)e^{-2\lambda_j V}}{(2\lambda_j)^2(\lambda_i-2\lambda_j)^3(2\lambda_j-\lambda)}$$

$$+\frac{(2+\lambda_i V(2+\lambda_i V))e^{-\lambda_i V}}{\lambda_i^3(\lambda_i-2\lambda_j)^2(\lambda_i-\lambda)}+\frac{2(1+V\lambda_i)(3\lambda_i-2\lambda_j-2\lambda))e^{-\lambda_i V}}{\lambda_i^2(\lambda_i-2\lambda_j)^3(\lambda_i-\lambda)^2}$$

$$+\frac{2(6\lambda_i^2+(2\lambda_j)^2+2\lambda2\lambda_j+3\lambda^2-4\lambda_i(2\lambda_j+2\lambda))e^{-\lambda_i V}}{\lambda_i(\lambda_i-2\lambda_j)^4(\lambda_i-\lambda)^3}-\frac{2e^{-\lambda V}}{\lambda(\lambda_i-\lambda)^3(2\lambda_j-\lambda)^2}\Big),$$
$$(4.16)$$

$$\bar{O}_{X_i^2 X_j}(s;\lambda)=4\Big(\frac{2(2\lambda_i-4(2\lambda_j)+3\lambda)e^{-2\lambda_j V}}{2\lambda_j(2\lambda_i-2\lambda_j)^4(2\lambda_j-\lambda)^2}+\frac{2(1+V2\lambda_j)e^{-2\lambda_j V}}{(2\lambda_j)^2(2\lambda_i-2\lambda_j)^3(2\lambda_j-\lambda)}$$

$$+\frac{(2+2\lambda_i V(2+2\lambda_i V))e^{-2\lambda_i V}}{(2\lambda_i)^3(2\lambda_i-2\lambda_j)^2(2\lambda_i-\lambda)}+\frac{2(1+V2\lambda_i)(3(2\lambda_i)-\lambda_j-2\lambda))e^{-2\lambda_i V}}{(2\lambda_i)^2(2\lambda_i-2\lambda_j)^3(2\lambda_i-\lambda)^2}$$

$$+\frac{2(6(2\lambda_i)^2+(2\lambda_j)^2+2\lambda2\lambda_j+3\lambda^2-4(2\lambda_i)(2\lambda_j+2\lambda))e^{-2\lambda_i V}}{2\lambda_i(2\lambda_i-2\lambda_j)^4(2\lambda_i-\lambda)^3}-\frac{2e^{-\lambda V}}{\lambda(2\lambda_i-\lambda)^3(2\lambda_j-\lambda)^2}\Big).$$
$$(4.17)$$

Finally, we can derive the explicit the formulae for $E^{X_i^2 X_j}$, $i\neq j$, $i,j=1,2,\cdots,n$.

Since the formula of $E^{X_i X_j}$ can be calculated similarly, we omit the proof. We give the expressions of $\bar{L}_{X_i X_j}(V;\lambda)$, $\bar{M}_{X_i X_j}(V;\lambda)$, $\bar{N}_{X_i X_j}(V;\lambda)$ and $\bar{O}_{X_i X_j}(V;\lambda)$ directly,

$$\bar{L}_{X_i X_j}(V;\lambda)=-\Big(\frac{e^{-\lambda V}}{\lambda(\lambda_i-\lambda)^2(\lambda_j-\lambda)^2}-\frac{(\lambda_i-3\lambda_j+2\lambda)e^{-\lambda_j V}}{\lambda_j(\lambda_i-\lambda_j)^3(\lambda_j-\lambda)^2}-\frac{(1+V\lambda_j)e^{-\lambda_j V}}{\lambda_j^2(\lambda_i-\lambda_j)^2(\lambda_j-\lambda)}$$

$$-\frac{2(1+V\lambda_i)e^{-\lambda_i V}}{\lambda_i^2(\lambda_i-\lambda_j)^2(\lambda_i-\lambda)}-\frac{(3\lambda_i-\lambda_j-2\lambda)e^{-\lambda_i V}}{\lambda_i(\lambda_i-\lambda_j)^3(\lambda_i-\lambda)^3}\Big), \qquad (4.18)$$

$$\bar{M}_{X_i X_j}(V;\lambda)=-2\Big(\frac{e^{-\lambda V}}{\lambda(2\lambda_i-\lambda)^2(\lambda_j-\lambda)^2}-\frac{(2\lambda_i-3\lambda_j+2\lambda)e^{-\lambda_j V}}{\lambda_j(2\lambda_i-\lambda_j)^3(\lambda_j-\lambda)^2}$$

$$-\frac{(1+V\lambda_j)e^{-\lambda_j V}}{\lambda_j^2(2\lambda_i-\lambda_j)^2(\lambda_j-\lambda)}-\frac{2(1+V2\lambda_i)e^{-2\lambda_i V}}{(2\lambda_i)^2(2\lambda_i-\lambda_j)^2(2\lambda_i-\lambda)}-\frac{(6\lambda_i-\lambda_j-2\lambda)e^{-2\lambda_i V}}{2\lambda_i(2\lambda_i-\lambda_j)^3(2\lambda_i-\lambda)^3}\Big),$$
$$(4.19)$$

$$\bar{N}_{X_i X_j}(V;\lambda)=-2\Big(\frac{e^{-\lambda V}}{\lambda(\lambda_i-\lambda)^2(2\lambda_j-\lambda)^2}-\frac{(\lambda_i-6\lambda_j+2\lambda)e^{-2\lambda_j V}}{2\lambda_j(\lambda_i-2\lambda_j)^3(2\lambda_j-\lambda)^2}$$

$$-\frac{(1+V2\lambda_j)e^{-2\lambda_j V}}{(2\lambda_j)^2(\lambda_i-2\lambda_j)^2(2\lambda_j-\lambda)}-\frac{2(1+V\lambda_i)e^{-\lambda_i V}}{\lambda_i^2(\lambda_i-2\lambda_j)^2(\lambda_i-\lambda)}-\frac{(3\lambda_i-2\lambda_j-2\lambda)e^{-\lambda_i V}}{\lambda_i(\lambda_i-2\lambda_j)^3(\lambda_i-\lambda)^3}\Big), \quad (4.20)$$

$$\bar{O}_{X_i X_j}(V;\lambda)=4\Big(\frac{(3\lambda_i-\lambda_j-\lambda)e^{-2\lambda_i V}}{\lambda_i(2\lambda_i-2\lambda_j)^3(2\lambda_i-\lambda)^3}+\frac{(\lambda_i-3\lambda_j+\lambda)e^{-2\lambda_j V}}{\lambda_j(2\lambda_i-2\lambda_j)^3(2\lambda_j-\lambda)^2}$$

$$+\frac{(1+V2\lambda_j)e^{-2\lambda_j V}}{(2\lambda_j)^2(2\lambda_i-2\lambda_j)^2(2\lambda_j-\lambda)}+\frac{2(1+V2\lambda_i)e^{-2\lambda_i V}}{(2\lambda_i)^2(2\lambda_i-2\lambda_j)^2(2\lambda_i-\lambda)}-\frac{e^{-\lambda V}}{\lambda(2\lambda_i-\lambda)^2(2\lambda_j-\lambda)^2}\Big).$$
$$(4.21)$$

Now, we give the formulae of $E^{X_i^3}$, $i = 1, 2, \cdots, n$.

$$E^{X_i^3} = \mathbf{E}\left[X_i^3 | S > \mathrm{VaR}_\kappa(S)\right] = \frac{\int_{\mathrm{VaR}_\kappa(S)}^\infty \int_0^s x_i^3 f_{X_i,S}(x_i, s) \mathrm{d}x_i \mathrm{d}s}{1 - \kappa}. \qquad (4.22)$$

Now, we rewrite the $f_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n)$ defined in Eq.(2.2) as

$$f_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n) = f_{X_i}(x_i) \Big\{ \omega_{-i}(x_1; x_2; \cdots; x_n; \lambda_1; \lambda_2; \cdots; \lambda_n)$$

$$+ \sum_{q=2}^n \sum_{1 \le p_1 < \cdots < p_q \le n \bigcap i \in \{p_1, \cdots, p_q\}} \theta_{p_1 \cdots p_q}(1 - 2F_{X_i}(x_i))[\sum_{l=0}^q \sum_{(\epsilon_1, \cdots, \epsilon_q) - i \in \tau_{l,q-1}}$$

$$(-1)^l \omega_{-i}(x_{p_1}; x_{p_2}; \cdots; x_{p_q}; x_{i_{q+1}}; \cdots; x_{i_n}; 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}} \cdots; \lambda_{i_n})]$$

$$+ \sum_{q=2}^n \sum_{1 \le p_1 < \cdots < p_q \le n \bigcap i \notin \{p_1, \cdots, p_q\}} \theta_{p_1 \cdots p_q}[\sum_{l=0}^q \sum_{(\epsilon_1, \cdots, \epsilon_q) \in \tau_{l,q}}$$

$$(-1)^l \omega_{-i}(x_{p_1}; x_{p_2}; \cdots; x_{p_q}; x_{i_{q+1}}; \cdots; x_{i_n}; 2^{\epsilon_1}\lambda_{p_1}; \cdots; 2^{\epsilon_q}\lambda_{p_q}; \lambda_{i_{q+1}} \cdots; \lambda_{i_n})], \quad (4.23)$$

where $\omega_{-i}(x_1; x_2; \cdots; x_n; \alpha_1; \alpha_2; \cdots; \alpha_n) = \alpha_1 e^{-\alpha_1 x_1} \times \cdots \times \alpha_{i-1} e^{-\alpha_{i-1} x_{i-1}} \times \alpha_{i+1} \times e^{-\alpha_{i+1} x_{i+1}} \times \cdots \times \alpha_n e^{-\alpha_n x_n}$.

Assume that

$$\int_0^{s-x_i} \int_0^{s-x_1-x_i} \cdots \int_0^{s-x_1-\cdots-x_{n-1}} \omega_{-i}(x_1; \ldots; x_n; \alpha_1; \ldots; \alpha_n) \mathrm{d}x_1 \cdots \mathrm{d}x_{i-1} \times$$

$$\mathrm{d}x_{i+1} \cdots \mathrm{d}x_{n-1}$$

$$= \sum_{l=1, l \ne i}^n \left(\prod_{q=1, q \ne l, q \ne i} \frac{\alpha_q}{\alpha_q - \alpha_l} \right) \alpha_l e^{-\alpha_l(s-x_i)} = h_{-i}(s - x_i; \alpha_1; \alpha_2; \cdots; \alpha_n). \qquad (4.24)$$

Utilizing the dividend difference presented in Chiragiev and Landsman [6], note that

$$h_{-i}(x; \alpha_1; \cdots; \alpha_{i-1}; \alpha_{i+1}; \cdots; \alpha_n)$$

$$= (-1)^{n-2} \times \alpha_1 \times \cdots \times \alpha_{i-1} \times \alpha_{i+1} \times \cdots \times \alpha_n \times \bar{\Lambda}(x; \alpha_1; \cdots; \alpha_{i-1}; \alpha_{i+1}; \cdots; \alpha_n),$$

where $\bar{\Lambda}(x; \alpha_1; \cdots; \alpha_{i-1}; \alpha_{i+1}; \cdots; \alpha_n)$ is the $(n-2)$th order divided difference of $\bar{\Lambda}(x; \alpha)$. Then, we get

$$\int_0^s x_i^3 f_{X_i}(x_i) h_{-i}(s - x_i; \lambda_1; \cdots; \lambda_{i-1}; \lambda_{i+1}; \cdots; \lambda_n) \mathrm{d}x_i$$

$$= (-1)^{n-1} \eta L_{X_i^3}(s; \lambda_1; \cdots; \lambda_{i-1}; \lambda_{i+1}; \cdots; \lambda_{j-1}; \lambda_{j+1}; \cdots; \lambda_n),$$

where $\eta_{-i} = \lambda_1 \times \cdots \times \lambda_{i-1} \times \lambda_{i+1} \times \cdots \times \lambda_n$, $L_{X_i^3}(s; \lambda) = -\int_0^s x_i^3 e^{-\lambda_i x_i} \bar{\Lambda}(s - x_i; \lambda) \mathrm{d}x_i$. Let also $M_{X_i^3}(s; \lambda) = -\int_0^s 2x_i^3 e^{-2\lambda_i x_i} \bar{\Lambda}(s - x_i; \lambda) \mathrm{d}x_i$. Given that the risks are exponentially distributed, then $L_{X_i^3}$ and $M_{X_i^3}$ can be rewritten as

$$L_{X_i^3}(s; \lambda) = \frac{s^3 e^{-s\lambda_i}}{\lambda_i - \lambda} + \frac{3s^2 e^{-s\lambda_i}}{(\lambda_i - \lambda)^2} + \frac{6s e^{-s\lambda_i}}{(\lambda_i - \lambda)^3} + \frac{6e^{-s\lambda_i}}{(\lambda_i - \lambda)^4} - \frac{6e^{-s\lambda}}{(\lambda_i - \lambda)^4}, \qquad (4.25)$$

$$M_{X_i^3}(s; \lambda) = 2\left(\frac{s^3 e^{-2s\lambda_i}}{2\lambda_i - \lambda} + \frac{3s^2 e^{-s\lambda_i}}{(\lambda_i - \lambda)^2} + \frac{6s e^{-2s\lambda_i}}{(2\lambda_i - \lambda)^3} + \frac{6e^{-s2\lambda_i}}{(2\lambda_i - \lambda)^4} - \frac{6e^{-s\lambda}}{(2\lambda_i - \lambda)^4} \right). \qquad (4.26)$$

Then, utilizing the results of (4.23), (4.25) and (4.26), we can express $\int_0^s x_i^3 f_{X_i,S}(x_i, s) \mathrm{d}x_i$ as

$$\int_0^s x^3 f_{X_i,S}(x, s) \mathrm{d}x = (-1)^{n-1} \eta \{ L_{X_i^3}(s; \lambda_1; \lambda_2; \cdots; \lambda_n) + \sum_{q=2}^n \sum_{1 \le p_1 < \cdots < p_q \le n \bigcap i \in \{p_1, \cdots, p_q\}}$$

$$\theta_{p_1\cdots p_q}[\sum_{l=0}^{q-1}\sum_{(\epsilon_1,\cdots,\epsilon_q)-i\in\tau_{l,q-1}}\{(-1)^{l+1}2^{q-1-l}L_{X_i^3}(s;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}}\cdots;\lambda_{i_n})$$

$$+(-1)^l2^{q-1-l}M_{X_i^3}(s;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}}\cdots;\lambda_{i_n})\}]+\sum_{q=2}^{n}\sum_{1\leq p_1<\cdots<p_q\leq n\bigcap i\notin\{p_1,\cdots,p_q\}}$$

$$\theta_{p_1\cdots p_q}\times[\sum_{l=0}^{q}\sum_{(\epsilon_1,\cdots,\epsilon_q)\in\tau_{l,q}}(-1)^l2^{q-l}L_{X_i^3}(s;2^{\epsilon_1}\lambda_{p_1};\cdots;2^{\epsilon_q}\lambda_{p_q};\lambda_{i_{q+1}}\cdots;\lambda_{i_n})]\}].$$

In order to calculate the explicit the formulae of $E^{X_i^3}$, $i=1,2,\cdots,n$, as in (4.22), the $L_{X_i^3}(s;\lambda)$ and $M_{X_i^3}(s;\lambda)$ terms in the equation above must be integrated on s as follows.

$$\bar{L}_{X_i^3}(V;\lambda)=\int_V^\infty L_{X_i^3}(s;\lambda)\mathrm{d}s=\frac{e^{-\lambda_i V}(6+V\lambda_i(6+V\lambda_i(3+V\lambda_i)))}{\lambda_i^4(\lambda_i-\lambda)}+\frac{6e^{-V\lambda_i}}{\lambda_i(\lambda_i-\lambda)^4}$$

$$-\frac{6e^{-V\lambda}}{\lambda(\lambda_i-\lambda)^4}+\frac{3e^{-V\lambda_i}(2+V\lambda_i(2+V\lambda_i))}{\lambda_i^3(\lambda_i-\lambda)^2}+\frac{6e^{-V\lambda_i}(1+V\lambda_i)e^{-V\lambda_i}}{\lambda_i^2(\lambda_i-\lambda)^3}. \qquad (4.27)$$

$$\bar{M}_{X_i^3}(V;\lambda)=\int_V^\infty M_{X_i^3}(s;\lambda)\mathrm{d}s=\frac{e^{-2\lambda_i V}(3+V\lambda_i(6+V2\lambda_i(3+V2\lambda_i)))}{2^2(\lambda_i)^4(2\lambda_i-\lambda)}-\frac{12e^{-V\lambda}}{\lambda(2\lambda_i-\lambda)^4}$$

$$+\frac{3e^{-V2\lambda_i}(1+V\lambda_i(2+V2\lambda_i))}{2(\lambda_i)^3(2\lambda_i-\lambda)^2}+\frac{3e^{-V2\lambda_i}(1+V2\lambda_i)e^{-V2\lambda_i}}{(\lambda_i)^2(2\lambda_i-\lambda)^3}+\frac{6e^{-V2\lambda_i}}{\lambda_i(2\lambda_i-\lambda)^4}. \qquad (4.28)$$

Finally, we can derive the explicit formulae (4.3) for $E^{X_i^3}$, $i=1,2,\cdots,n$.

Since the formulae of $E^{X_i^2}$ and E^{X_i} can be calculated similarly, we also omit the proof. We give the expressions of $\bar{L}_{X_i^2}(V;\lambda)$, $\bar{M}_{X_i^2}(V;\lambda)$, $\bar{L}_{X_i}(V;\lambda)$ and $\bar{M}_{X_i}(V;\lambda)$ directly,

$$\bar{L}_{X_i^2}(V;\lambda)=\frac{e^{-V\lambda_i}(2+V\lambda_i(2+V\lambda_i))}{\lambda_i^3(\lambda_i-\lambda)}+\frac{2e^{-V\lambda_i}(1+V\lambda_i)e^{-V\lambda_i}}{\lambda_i^2(\lambda_i-\lambda)^2}$$

$$+\frac{2e^{-V\lambda_i}}{\lambda_i(\lambda_i-\lambda)^3}-\frac{2e^{-V\lambda}}{\lambda(\lambda_i-\lambda)^3}, \qquad (4.29)$$

$$\bar{M}_{X_i^2}(V;\lambda)=\frac{e^{-V2\lambda_i}(1+V\lambda_i(2+V2\lambda_i))}{2^2(\lambda_i)^3(2\lambda_i-\lambda)}+\frac{e^{-V2\lambda_i}(1+V2\lambda_i)e^{-V2\lambda_i}}{2(\lambda_i)^2(2\lambda_i-\lambda)^2}$$

$$+\frac{e^{-V2\lambda_i}}{\lambda_i(2\lambda_i-\lambda)^3}-\frac{2e^{-V\lambda}}{\lambda(2\lambda_i-\lambda)^3}, \qquad (4.30)$$

$$\bar{L}_{X_i}(V;\lambda)=\frac{e^{-V\lambda_i}(1+V\lambda_i)e^{-V\lambda_i}}{\lambda_i^2(\lambda_i-\lambda)}+\frac{e^{-V\lambda_i}}{\lambda_i(\lambda_i-\lambda)^2}-\frac{e^{-V\lambda}}{\lambda(\lambda_i-\lambda)^2}, \qquad (4.31)$$

$$\bar{M}_{X_i}(V;\lambda)=\frac{e^{-V2\lambda_i}(1+V2\lambda_i)e^{-V2\lambda_i}}{(2\lambda_i)^2(2\lambda_i-\lambda)}+\frac{e^{-V2\lambda_i}}{2\lambda_i(2\lambda_i-\lambda)^2}-\frac{e^{-V\lambda}}{\lambda(2\lambda_i-\lambda)^2}. \qquad (4.32)$$

This completes the proof of this proposition.

5. Numerical illustrations

5.1 Bivariate case

We illustrate our results with an example for the bivariate case. As an example, assume that risk variables X_1 and X_2 are exponentially distributed with parameters $\lambda_1=2/5$ and $\lambda_2=3/4$. We calculate the optimal capital allocations based on Tail Mean-Variance principle for different risk levels κ, FGM copula parameters θ and different β.

The values of $d_i(i=1,2)$ with a total capital $d=40$ for $\theta=-1,0,1$, $\beta=0.1,0.3,0.5$, $0.7,0.9$, and $\kappa=0.5,0.75,0.95,0.99,0.995$ are listed in Table 1-3. We observe that for

fixed κ and θ, d_1 is a decreasing function with respect to β, but d_2 is an increasing function with respect to β. We also find that for fixed β and θ, d_1 is an increasing function with respect to κ, but d_2 is a decreasing function with respect to κ. Moreover, the larger the value of θ, the greater changes in d_1 and d_2 for different κ and β.

5.2 Multivariate case

For multivariate case, we illustrate our results with an example for three risk variables dependent through a trivariate FGM copula. Assume that risk variables X_1, X_2 and X_3 are exponentially distributed with parameters $\lambda_1 = 1/2$, $\lambda_2 = 1/3$ and $\lambda_3 = 1/5$. The trivariate FGM copula with 4 parameters can be written as

$$C(u_1, u_2, u_3) = u_1 u_2 u_3 (1 + \theta_{123}\bar{u}_1\bar{u}_2\bar{u}_3 + \theta_{12}\bar{u}_1\bar{u}_2 + \theta_{13}\bar{u}_1\bar{u}_3 + \theta_{23}\bar{u}_2\bar{u}_3),$$

where $\bar{u}_i = 1 - u_i$, $i = 1, 2, 3$.

We calculate the optimal capital allocations (the amount d_i to X_i, $i = 1, 2, 3$) based on Tail Mean-Variance principle for different risk levels κ, FGM copula parameters and different β.

The values of $d_i(i = 1, 2, 3)$ with a total capital $d = 120$ for $\beta = 0.1, 0.3, 0.5, 0.7, 0.9$, $\kappa = 0.5, 0.75, 0.95, 0.99, 0.995$, $\theta_{123} = 0.25$, $\theta_{12} = -0.2, 0.2$, $\theta_{13} = 0.5$, and $\theta_{23} = -0.6, 0.6$ are listed in Table 4-5. We observe that for fixed κ, d_2 and d_3 are decreasing functions with respect to β, but d_1 is an increasing function with respect to β in both cases. We also find that for fixed β, d_3 is an increasing function with respect to κ, but d_1 and d_2 is decreasing function with respect to κ. Moreover, the changes in d_1, d_2 and d_2 for different κ and β in case $\theta_{123} = 0.25$, $\theta_{12} = 0.2$, $\theta_{13} = 0.5$ and $\theta_{23} = 0.6$ are greater than in case $\theta_{123} = 0.25$, $\theta_{12} = -0.2$, $\theta_{13} = 0.5$ and $\theta_{23} = -0.6$.

6. Concluding Remarks

In this paper, we introduce the use of copulas in optimal capital allocation based on Tail Mean-Variance principle. We obtain explicit expressions for the optimal capital allocation for exponential distributed risks linked by a FGM copula. The handy form of the FGM copula permits a direct calculation of the $\mathbf{E}\left[X_i^k | S > \mathrm{VaR}_\kappa(S)\right]$, $\mathbf{E}[X_i^2 X_j | S > \mathrm{VaR}_\kappa(S)]$, and $\mathbf{E}\left[X_i X_j | S > \mathrm{VaR}_\kappa(S)\right]$ when we assume only two risks. In the multivariate case, the dividend differences presented in Chiragiev and Landsman (2007) are used.

Acknowledgements

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Table 1. Optimal capital allocations for different β and κ with a total capital $d = 40$ and $\theta = -1$.

β	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$\beta = 0.1$	$d_1 = 17.373255$	$d_1 = 18.259265$	$d_1 = 19.786721$	$d_1 = 21.285369$	$d_1 = 22.077165$
	$d_2 = 22.626745$	$d_2 = 21.740735$	$d_2 = 20.213279$	$d_2 = 18.714631$	$d_2 = 17.922835$
$\beta = 0.3$	$d_1 = 16.657079$	$d_1 = 17.760345$	$d_1 = 19.374409$	$d_1 = 20.740284$	$d_1 = 21.438334$
	$d_2 = 23.342921$	$d_2 = 22.239655$	$d_2 = 20.625591$	$d_2 = 19.259716$	$d_2 = 18.561666$
$\beta = 0.5$	$d_1 = 16.479421$	$d_1 = 17.642965$	$d_1 = 19.281445$	$d_1 = 20.615971$	$d_1 = 21.290319$
	$d_2 = 23.520579$	$d_2 = 22.357035$	$d_2 = 20.718555$	$d_2 = 19.384029$	$d_2 = 18.709681$
$\beta = 0.7$	$d_1 = 16.398849$	$d_1 = 17.590521$	$d_1 = 19.240384$	$d_1 = 20.560895$	$d_1 = 21.224463$
	$d_2 = 23.601151$	$d_2 = 22.409479$	$d_2 = 20.759616$	$d_2 = 19.439105$	$d_2 = 18.775537$
$\beta = 0.9$	$d_1 = 16.352859$	$d_1 = 17.560802$	$d_1 = 19.217245$	$d_1 = 20.529814$	$d_1 = 21.187221$
	$d_2 = 23.647141$	$d_2 = 22.439198$	$d_2 = 20.782755$	$d_2 = 19.470186$	$d_2 = 18.812779$

Table 2. Optimal capital allocations for different β and κ with a total capital $d = 40$ and $\theta = 0$.

β	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$\beta = 0.1$	$d_1 = 17.085053$	$d_1 = 18.257321$	$d_1 = 20.322754$	$d_1 = 22.219689$	$d_1 = 23.064009$
	$d_2 = 22.914947$	$d_2 = 21.742679$	$d_2 = 19.677246$	$d_2 = 17.780311$	$d_2 = 16.935991$
$\beta = 0.3$	$d_1 = 16.182965$	$d_1 = 17.688514$	$d_1 = 19.991725$	$d_1 = 21.946054$	$d_1 = 22.801869$
	$d_2 = 23.817035$	$d_2 = 22.311486$	$d_2 = 20.008275$	$d_2 = 18.053946$	$d_2 = 17.198131$
$\beta = 0.5$	$d_1 = 15.951057$	$d_1 = 17.551042$	$d_1 = 19.916478$	$d_1 = 21.885122$	$d_1 = 22.743742$
	$d_2 = 24.048943$	$d_2 = 22.448958$	$d_2 = 20.083522$	$d_2 = 18.114878$	$d_2 = 17.256258$
$\beta = 0.7$	$d_1 = 15.844774$	$d_1 = 17.489157$	$d_1 = 19.883171$	$d_1 = 21.858299$	$d_1 = 22.718181$
	$d_2 = 24.155226$	$d_2 = 22.510843$	$d_2 = 20.116829$	$d_2 = 18.141701$	$d_2 = 17.281819$
$\beta = 0.9$	$d_1 = 15.783795$	$d_1 = 17.453962$	$d_1 = 19.864381$	$d_1 = 21.843206$	$d_1 = 22.703806$
	$d_2 = 24.216205$	$d_2 = 22.546038$	$d_2 = 20.135619$	$d_2 = 18.156794$	$d_2 = 17.296194$

Table 3. Optimal capital allocations for different β and κ with a total capital $d = 40$ and $\theta = 1$.

β	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$\beta = 0.1$	$d_1 = 16.663625$	$d_1 = 18.058854$	$d_1 = 20.322699$	$d_1 = 22.208001$	$d_1 = 23.017989$
	$d_2 = 23.336375$	$d_2 = 21.941146$	$d_2 = 19.677301$	$d_2 = 17.791999$	$d_2 = 16.982011$
$\beta = 0.3$	$d_1 = 15.445589$	$d_1 = 17.357541$	$d_1 = 19.991519$	$d_1 = 21.965215$	$d_1 = 22.791365$
	$d_2 = 24.554411$	$d_2 = 22.642459$	$d_2 = 20.008481$	$d_2 = 18.034785$	$d_2 = 17.208635$
$\beta = 0.5$	$d_1 = 15.115614$	$d_1 = 17.181922$	$d_1 = 19.915242$	$d_1 = 21.911007$	$d_1 = 22.741114$
	$d_2 = 24.884386$	$d_2 = 22.818078$	$d_2 = 20.084758$	$d_2 = 18.088993$	$d_2 = 17.258886$
$\beta = 0.7$	$d_1 = 14.961949$	$d_1 = 17.102057$	$d_1 = 19.881358$	$d_1 = 21.887125$	$d_1 = 22.719015$
	$d_2 = 25.038051$	$d_2 = 22.897943$	$d_2 = 20.118642$	$d_2 = 18.112875$	$d_2 = 17.280985$
$\beta = 0.9$	$d_1 = 14.873078$	$d_1 = 17.056409$	$d_1 = 19.862212$	$d_1 = 21.873684$	$d_1 = 22.706589$
	$d_2 = 25.126922$	$d_2 = 22.943591$	$d_2 = 20.137788$	$d_2 = 18.126316$	$d_2 = 17.293411$

Table 4. Optimal capital allocations with $d = 120$ and $\theta_{123} = 0.25$, $\theta_{12} = -0.2$, $\theta_{13} = 0.5$, $\theta_{23} = -0.6$.

β	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$\beta = 0.1$	$d_1 = 48.9773$	$d_1 = 45.7090$	$d_1 = 40.7692$	$d_1 = 36.6939$	$d_1 = 35.0049$
	$d_2 = 40.7396$	$d_2 = 40.2660$	$d_2 = 39.5750$	$d_2 = 38.9001$	$d_2 = 38.4863$
	$d_3 = 30.2831$	$d_3 = 34.0250$	$d_3 = 39.6558$	$d_3 = 44.4060$	$d_3 = 46.5088$
$\beta = 0.3$	$d_1 = 50.5392$	$d_1 = 46.6406$	$d_1 = 41.2002$	$d_1 = 36.9332$	$d_1 = 35.1863$
	$d_2 = 40.1102$	$d_2 = 39.9010$	$d_2 = 39.4434$	$d_2 = 38.8840$	$d_2 = 38.5102$
	$d_3 = 29.3506$	$d_3 = 33.4584$	$d_3 = 39.3564$	$d_3 = 44.1828$	$d_3 = 46.3035$
$\beta = 0.5$	$d_1 = 50.9125$	$d_1 = 46.8548$	$d_1 = 41.2958$	$d_1 = 36.9856$	$d_1 = 35.2258$
	$d_2 = 39.9472$	$d_2 = 39.8112$	$d_2 = 39.4116$	$d_2 = 38.8784$	$d_2 = 38.5135$
	$d_3 = 29.1403$	$d_3 = 33.3340$	$d_3 = 39.2926$	$d_3 = 44.1360$	$d_3 = 46.2607$
$\beta = 0.7$	$d_1 = 51.0801$	$d_1 = 46.9500$	$d_1 = 41.3378$	$d_1 = 37.0085$	$d_1 = 35.2431$
	$d_2 = 39.8727$	$d_2 = 39.7707$	$d_2 = 39.3973$	$d_2 = 38.8757$	$d_2 = 38.5146$
	$d_3 = 29.0472$	$d_3 = 33.2793$	$d_3 = 39.2649$	$d_3 = 44.1158$	$d_3 = 46.2423$
$\beta = 0.9$	$d_1 = 51.1753$	$d_1 = 47.0038$	$d_1 = 41.3614$	$d_1 = 37.0213$	$d_1 = 35.2527$
	$d_2 = 39.8300$	$d_2 = 39.7476$	$d_2 = 39.3893$	$d_2 = 38.8742$	$d_2 = 38.5154$
	$d_3 = 28.9947$	$d_3 = 33.2486$	$d_3 = 39.2493$	$d_3 = 44.1045$	$d_3 = 46.2319$

Table 5. Optimal capital allocations with $d = 120$ and $\theta_{123} = 0.25$, $\theta_{12} = 0.2$, $\theta_{13} = 0.5$, $\theta_{23} = 0.6$.

β	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$\beta = 0.1$	$d_1 = 53.9283$	$d_1 = 49.0771$	$d_1 = 42.6048$	$d_1 = 38.0363$	$d_1 = 36.2516$
	$d_2 = 38.0905$	$d_2 = 38.1029$	$d_2 = 37.7804$	$d_2 = 37.2270$	$d_2 = 36.8990$
	$d_3 = 27.9812$	$d_3 = 32.8200$	$d_3 = 39.6148$	$d_3 = 44.7367$	$d_3 = 46.8494$
$\beta = 0.3$	$d_1 = 56.0911$	$d_1 = 50.4378$	$d_1 = 43.3266$	$d_1 = 38.5192$	$d_1 = 36.6674$
	$d_2 = 37.0629$	$d_2 = 37.4505$	$d_2 = 37.4379$	$d_2 = 37.0155$	$d_2 = 36.7271$
	$d_3 = 26.8460$	$d_3 = 32.1117$	$d_3 = 39.2355$	$d_3 = 44.4653$	$d_3 = 46.6055$
$\beta = 0.5$	$d_1 = 56.6026$	$d_1 = 50.7502$	$d_1 = 43.4875$	$d_1 = 38.6257$	$d_1 = 36.7588$
	$d_2 = 36.8042$	$d_2 = 37.2938$	$d_2 = 37.3591$	$d_2 = 36.9671$	$d_2 = 36.6877$
	$d_3 = 26.5932$	$d_3 = 31.9560$	$d_3 = 39.1534$	$d_3 = 44.4072$	$d_3 = 46.5535$
$\beta = 0.7$	$d_1 = 56.8317$	$d_1 = 50.8888$	$d_1 = 43.5584$	$d_1 = 38.6725$	$d_1 = 36.7989$
	$d_2 = 36.6865$	$d_2 = 37.2235$	$d_2 = 37.3241$	$d_2 = 36.9457$	$d_2 = 36.6703$
	$d_3 = 26.4818$	$d_3 = 31.8877$	$d_3 = 39.1175$	$d_3 = 44.3818$	$d_3 = 46.5308$
$\beta = 0.9$	$d_1 = 56.9616$	$d_1 = 50.9672$	$d_1 = 43.5983$	$d_1 = 38.6987$	$d_1 = 36.8215$
	$d_2 = 36.6193$	$d_2 = 37.1835$	$d_2 = 37.3043$	$d_2 = 36.9337$	$d_2 = 36.6604$
	$d_3 = 26.4191$	$d_3 = 31.8493$	$d_3 = 39.0974$	$d_3 = 44.3676$	$d_3 = 46.5181$