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EXISTENCE OF ENTROPY SOLUTIONS FOR STRONGLY NONLINEAR ANISOTROPIC ELLIPTIC PROBLEM INVOLVING LOWER ORDER TERMS AND HARDY POTENTIAL

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Abstract. In this work, we give an existence result of entropy solutions for the following strongly nonlinear anisotropic elliptic Dirichlet problem

$$\begin{cases} -\sum_{i=1}^{N} D^i(|D^iu|^{p_i-2}D^iu) + h(x,u,\nabla u) + |u|^{p_0-2}u + |u|^{s-1}u \\ \qquad\qquad = f + \mu\frac{|u|^{p_0-2}u}{|x|^{p_0}} - \text{div } \phi(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbf{R}^N$ containing the origin, $\phi \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^N)$. We assume that the datum f belongs to $L^1(\Omega)$, μ is a positive constant and $h(x,u,\nabla u)$ is a nonlinear lower order term with natural growth with respect to $|\nabla u|$, satisfying the sign condition.

1. Introduction

In the last decades one of the topics from the field of calculus of variations and partial differential equations that gained interest is the study of anisotropic problems, as witnessed by a number of researches that have introduced anisotropic Sobolev spaces which are the appropriate framework to deal with a class of problems with non-standard structural conditions, involving a growth exponent $\vec{p}$, where prototype of the differential operator considered is the $\vec{p}$-laplacien

$$\Delta_{\vec{p}}(u) = \sum_{i=1}^{i=N} \partial_{x_i}(|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u),$$

which generalize the p-laplace operator.
Let Ω be a bounded open subset of $\mathbf{R}^N$ $(N \geq 2)$, containing the origin with boundary Ω and let $p_0, p_1, \ldots, p_N$ be $N+1$ exponents, with $1 < p_i < \infty$ for $i = 0, \ldots, N$. Our aim is to prove the existence of entropy solutions for the following anisotropic strongly nonlinear elliptic problem

$$(1.1) \quad \begin{cases} -\sum_{i=1}^{N} D^i(|D^iu|^{p_i-2}D^iu) + h(x,u,\nabla u) + |u|^{p_0-2}u + |u|^{s-1}u \\ \qquad\qquad = f + \mu\frac{|u|^{p_0-2}u}{|x|^{p_0}} - \text{div } \phi(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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with $\mu \geq 0$, $f \in L^1(\Omega)$, $\phi \in \mathcal{C}^0(I\!R, I\!R^N)$ and

$$s(x) > \max\Big(\frac{N(p_0-1)}{N-p_0}, \frac{1}{p_0-1}\Big). \tag{1.2}$$

The nonlinear lower order term $h(x,s,\xi) : \Omega \times I\!R \times I\!R^N \longmapsto I\!R^N$ is Carathéodory functions, (measurable with respect to x in Ω for every (s,ξ) in $I\!R \times I\!R^N$ and continuous with respect to (s,ξ) in $I\!R \times I\!R^N$ for almost every x in Ω), which satisfies the following conditions

$$h(x,s,\xi)s \geq 0, \tag{1.3}$$

$$|h(x,s,\xi)| \leq l(x) + j(|s|)\sum_{i=1}^{N} |\xi_i|^{p_i}, \tag{1.4}$$

where $j : I\!R^+ \to I\!R^+$ is a continuous nondecreasing function and $l \in L^1(\Omega)$. The notion of anisotropic Sobolev spaces were introduced and studied by Nikolskiii [35], and Troisi [38], and later by Trudinger [39] in the framework of Orlicz spaces. This rise of interest for the study of such spaces was motivated by their physical applications in the thermistor problem, flow of electroreological fluids and processes of image restoration (see for example [36], [15] and [5, 28]). It is important to point out the classic result of Boccardo et al. [10] in which they have studied the anisotropic equations with right hand side measures

$$\begin{cases} -\text{div}\,(j(Du)) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $j(\xi)$ is the vector whose components are $|\xi_i|^{p_i-2}\xi_i$ $(i = 1, \dots, N, p_i > 1)$, and also when f is a measurable function such that $\int_\Omega |f|\log(1+|f|) < \infty$. Antonsev et al. have studied the uniqueness of weak solutions for elliptic equations of the following type

$$-\partial_{x_i}(a_i(x,u)|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u) + b(x,u) = f(x)$$

in a bounded domain $\Omega \subset I\!R^N$ with Lipschitz continuous boundary $\Gamma = \partial\Omega$ and particular mixed boundary conditions and they have established a similar result for the parabolic case. Anisotropic elliptic equations have been considered under many other différent aspects, for instance with respect to the maximum principle and to the multiplicity of solutions; see e.g. P. Pucci, V. Radulesco et al. [27] and [32], while the authors in [8] and [7] proved the existence of solutions of some anisotropic elliptic equations for a general class of operators of higher order.For more details concerning the anisotropic problems we refer to [4, 3, 14, 16, 17, 41, 40, 33] and the references therein. It would be interesting to refer to some Embedding theorems for anisotropic Sobolev-Orlicz spaces [26] and a fully anisotropic Sobolev Inequality established by Cianchi in [13]. we can also refer the reader to [24] for some basics properties of anisotropic Orlicz-Musielak spaces, moreover Gwiazda et al. in [25] dealt with anisotropic parabolic problems where the N-function was assumed to be homogeneous in space. we mention also that the author in [37] had treated anisotropic behaviour in a parabolic problem in a framework of maximal monotone graphs, possibly multi-valued with growth conditions formulated with help of an x-dependent N-function.

For the isotropic case, i.e $p_i = p$ attention has been focused on elliptic problems with singularity on its right-hand side particularly the so- called Hardy potential and its effect that give rise to the existence (or nonexistence) of solutions. Abdellaoui and Peral have treated the optimal power in order to find a solution the following equation

$$-\triangle u = \lambda \frac{u}{|x|^2} + |\triangle u|^p + cf(x), \tag{1.6}$$

where Ω is domain containing 0. they assumed that λ and c are positive real numbers and f is nonnegative function under some extra hypotheses. In [30], Mercaldo et al. were interested on existence and nonexistence for positive solutions to the degenerated nonlinear elliptic equations

$$\begin{cases} -\text{div}\,(A(x,u,\nabla u)) = \lambda \dfrac{u^s}{|x|^p} + f(x) & \text{in } \Omega, \\ u(x) \geq 0 & \text{on } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

with Ω be an open bounded subset of $I\!R^N$ $(N \geq 3)$, $1 < p < N$, λ and s are positives numbers, f is nonnegative function in some Lebesgue space, and $A : \Omega \times I\!R \times I\!R^N \longmapsto I\!R^n$ is such that

$$\frac{c_0}{(a(x)+|t|)^\theta}|\xi|^p \leq \langle A(x,t,\xi),\xi\rangle \qquad \text{for some} \qquad 0<\theta<1$$

which provide a non coercive operator when $u \to \infty$. To investigate other problems of this kind we refer the reader [2, 34, 21, 22]. It is meritorious mentioning that this type of problems appear in several contexts. The problem (1.7) could be seen as a reaction model which produces a saturation effect in some solid combustion problems, while (1.6) is the stationary counterpart of some flame propagation models. It should be pointed out that it was used for the resolution of this problem the notion of entropy solutions which was introduced by Bénilan et al. in [9], for the reason that the data f belong to $L^1(\Omega)$. Motivated by the papers [34],[2] and [42], we try to deal with strongly nonlinear anisotropic elliptic Dirichlet problem by using the Galerkin method, and to remove the non-existence effect produced by the singular term $\frac{|u|^{p_0-2}u}{|x|^{p_0}}$ by exploiting the regularizing effect of the term $|u|^{s-1}u$. On the other hand the function $\phi \in \mathcal{C}^0(I\!R, I\!R^N)$ then ϕ does not belongs to $(L^1_{loc}(\Omega))^N$, so that proving existence of a weak solution seems to be an arduous task. to overpass this difficulty we will use some techniques in the framework of entropy solutions. The remaining part of this paper is organized as follows: This paper is organized as follows: Section 2 is devoted to introduce some preliminary results including a brief discussion on the anisotropic Sobolev spaces. In section 3, we recall some technical lemmas and we state and prove our main existence results.

2. Preliminary

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$,
Let $p_0, p_1, \ldots, p_N$ be $N+1$ exponents, with $1 < p_i < \infty$ for $i = 0, \ldots, N$. We denote

$$\vec{p} = (p_0, \ldots, p_N), \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \qquad \text{for } \ i = 1, \ldots, N,$$

and we define

$$\underline{p} = \min\{p_0, p_1, \ldots, p_N\} \qquad \text{then} \qquad \underline{p} > 1. \tag{2.1}$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follow

$$W^{1,\vec{p}}(\Omega) = \{u \in L^{p_0}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \dots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \sum_{i=0}^{N} \|D^i u\|_{p_i}. \tag{2.2}$$

We define also $W_0^{1,\vec{p}}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to the norm (2.2). The space $\big(W_0^{1,\vec{p}}(\Omega), \|u\|_{1,\vec{p}}\big)$ is a reflexive Banach space (cf. [31]).

Lemma 2.1. *We have the following continuous and compact embedding*

- *if* $\underline{p} < N$ *then* $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *for* $q \in [\underline{p}, \underline{p}^*[$, *where* $\underline{p}^* = \dfrac{N\underline{p}}{N - \underline{p}}$,
- *if* $\underline{p} = N$ *then* $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ $\forall q \in [\underline{p}, +\infty[$,
- *if* $\underline{p} > N$ *then* $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$.

The proof of this lemma follows from the fact that the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

Proposition 1. The dual of $W_0^{1,\vec{p}}(\Omega)$ is denote by $W^{-1,\vec{p}'}(\Omega)$, where $\vec{p'} = (p'_0, \dots, p'_N)$ and $\frac{1}{p'_i} + \frac{1}{p_i} = 1$, (cf. [6] for the constant exponent case).
For each $F \in W^{-1,\vec{p}'}(\Omega)$ there exists $F_i \in L^{p'_i}(\Omega)$ for $i = 0, 1, \dots, N$, such that $F = F_0 - \sum_{i=1}^{N} D^i F_i$. Moreover for any $u \in W_0^{1,\vec{p}}(\Omega)$, we have

$$\langle F, u\rangle = \sum_{i=0}^{N} \int_\Omega F_i \, D^i u \, dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p}'} = \inf\Big\{ \sum_{i=0}^{N} \|F_i\|_{p'_i} \quad / \quad F = F_0 - \sum_{i=1}^{N} D^i F_i \quad \text{with} \quad F_i \in L^{p'_i}(\Omega)\Big\}.$$

We set

$$\mathcal{T}_0^{1,\vec{p}}(\Omega) := \{u : \Omega \mapsto I\!R \text{ measurable, such that } T_k(u) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

Note that, a measurable function u verifying $T_k(u) \in W_0^{1,\vec{p}}(\Omega)$ for all $k > 0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. However, for any $u \in \mathcal{T}_0^{1,\vec{p}}(\Omega)$ it is possible to define the weak gradient of u, still denoted ∇u.

Proposition 2. Let $u \in \mathcal{T}_0^{1,\vec{p}}(\Omega)$. For any $i \in \{1, \dots, N\}$, there exists a unique measurable function $v_i : \Omega \mapsto I\!R$ such that

$$\forall k > 0 \qquad D^i T_k(u) = v_i . \chi_{\{|u|<k\}} \qquad \text{a.e.} \quad x \in \Omega,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if

u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u, that is, $v_i = D^i u$.

The proof of the Proposition 2.2 follows the usual techniques developed in [9] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [6] and [17].

3. Main results

Let Ω be a bounded open subset of $\mathbf{R}^N$ ($N \geq 2$), containing the origin. First of all, we can give a simpler definition of an entropy solution of (1.1) as follows.

Definition 3.1. A measurable function u is an entropy solution of the strongly nonlinear anisotropic elliptic Dirichlet problem (1.1) if

$$u \in \mathcal{T}_0^{1,\vec{p}}(\Omega), \quad |u|^{s-1}u \in L^1(\Omega), \quad \frac{|u|^{p_0-2}u}{|x|^{p_0}} \in L^1(\Omega), \quad h(x,u,\nabla u) \in L^1(\Omega), \quad \phi_i(u) \in L^{p_i'}(\Omega)$$

for $i = 1, \ldots, N$ such that

(3.1)
$$\begin{aligned} &\sum_{i=1}^N \int_\Omega |D^i u|^{p_i-2} D^i u D^i T_k(u-\varphi)\,dx + \int_\Omega h(x,u,\nabla u) T_k(u-\varphi)\,dx \\ &+ \int_\Omega |u|^{p_0-2} u T_k(u-\varphi)\,dx + \int_\Omega |u|^{s-1} u T_k(u-\varphi)\,dx \\ &\qquad \leq \int_\Omega f T_k(u-\varphi)\,dx + \lambda \int_\Omega \frac{|u|^{p_0-2}u}{|x|^{p_0}} T_k(u-\varphi)\,dx + \sum_{i=1}^N \int_\Omega \phi_i(u) D^i T_k(u-\varphi)\,dx, \end{aligned}$$

for any $\varphi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$.

Our purpose is to establish the following existence theorem:

Theorem 3.2. *Let $\lambda \geq 0$ and $f \in L^1(\Omega)$, assuming that $\phi \in \mathcal{C}^0(I\!R, I\!R^N)$ and* (1.3) – (1.4) *hold true. Then, the problem* (1.1) *has at least one entropy solution u such that $u \in W_0^{1,\vec{q}}(\Omega)$, with*

$$\vec{q} = (s, q_1, \ldots, q_N) \qquad \text{and} \qquad 1 \leq q_i < \frac{p_i s}{s+1} \quad \text{for } \ i = 1, \ldots, N. \tag{3.2}$$

3.1. Technical Lemmas.

Lemma 3.3. *(see* [23]*, Theorem 13.47) Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that*

- **(i):** *$u_n \to u$ a.e. in Ω,*
- **(ii):** *$u_n \geq 0$ and $u \geq 0$ a.e. in Ω,*
- **(iii):** $\int_\Omega u_n\,dx \to \int_\Omega u\,dx$,

then $u_n \to u$ in $L^1(\Omega)$.

Lemma 3.4. *Let $(u_n)_{n\in I\!N}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$ and*

$$\sum_{i=0}^N \int_\Omega \big(|D^i u_n|^{p_i-2} D^i u_n - |D^i u|^{p_i-2} D^i u\big)(D^i u_n - D^i u)\,dx \to 0, \tag{3.3}$$

then $u_n \longrightarrow u$ in $W_0^{1,\vec{p}}(\Omega)$ for a subsequence.

Proof of Lemma For the proof of (3.4) we exploit some techniques of [1]. Let's remark that

(3.4)
$$\sum_{i=0}^{N}\int_{\Omega}|D^iu_n-D^iu|^{p_i}\,dx=\sum_{i=0}^{N}\int_{\Lambda_i}|D^iu_n-D^iu|^{p_i}\,dx+\sum_{i=0}^{N}\int_{\Omega\setminus\Lambda_i}|D^iu_n-D^iu|^{p_i}\,dx.$$

with $\Lambda_i=\{x\in\Omega/1<p_i<2\}$. Note $\Theta_n=\sum_{i=0}^{N}\int_{\Omega}\big(|D^iu_n|^{p_i-2}D^iu_n-|D^iu|^{p_i-2}D^iu\big)(D^iu_n-D^iu)\,dx$, then by applying the following known inequality

$$\big(|a|^{p-2}a-|b|^{p-2}b\big)(a-b)\geq\begin{cases}2^{2-p}|a-b|^p & \text{if}\quad p\geq 2,\\ (p-1)\dfrac{|a-b|^2}{\big(|a|+|b|\big)^{2-p}} & \text{for}\quad 1<p<2,\end{cases}\qquad\forall a,b\in I\!R.$$

On the one hand, by virtue of the Hölder inequality we get

(3.5)
$$\begin{aligned}\sum_{i=0}^{N}\int_{\Lambda_i}|D^iu_n-D^iu|^{p_i}\,dx&=\sum_{i=0}^{N}\int_{\Lambda_i}\frac{|D^iu_n-D^iu|^{p_i}}{(|D^iu_n|+|D^iu|)^{\frac{p_i(2-p_i)}{2}}}(|D^iu_n|+|D^iu|)^{\frac{p_i(2-p_i)}{2}}\,dx\\
&\leq\sum_{i=0}^{N}\left\|\frac{|D^iu_n-D^iu|^{p_i}}{(|D^iu_n|+|D^iu|)^{\frac{p_i(2-p_i)}{2}}}\right\|_{\frac{2}{p_i},\Lambda_i}\left\|(|D^iu_n|+|D^iu|)^{\frac{p_i(2-p_i)}{2}}\right\|_{\frac{2}{2-p_i},\Lambda_i}\\
&\leq\sum_{i=0}^{N}\max\left\{\left(\int_{\Lambda_i}\frac{|D^iu_n-D^iu|^2}{(|D^iu_n|+|D^iu|)^{2-p_i}}>dx\right)^{\frac12},\int_{\Lambda_i}\frac{|D^iu_n-D^iu|^2}{(|D^iu_n|+|D^iu|)^{2-p_i}}\,dx\right\}\\
&\quad\times\left(\int_{\Lambda_i}(|D^iu_n|+|D^iu|)^{p_i}\,dx\right)^{\frac{2-p_i}{2}}\\
&\leq\sum_{i=0}^{N}\max\{\Theta_n^{\frac12}(p_i-1)^{-\frac12},\Theta_n(p_i-1)^{-1}\}\left(\int_{\Lambda_i}(|D^iu_n|+|D^iu|)^{p_i}\right)^{\frac{2-p_i}{2}}.\end{aligned}$$

On the other hand it'easy to check that

$$\Theta_n\geq\sum_{i=0}^{N}2^{2-p_i}\int_{\Omega\setminus\Lambda_i}|D^iu_n-D^iu|^{p_i}\,dx. \tag{3.6}$$

Passing to the limit as $n\to\infty$ while bearing in mind (3.4) and (3.4) – (3.6), we conclude that $u_n\to u$ in $W_0^{1,\vec{p}}$.

Proof of the Theorem 3.2.

Step 1 : Approximate problems. Let $(f_n)_{n\in I\!N}$ be a sequence of smooth functions such that $f_n\to f$ in $L^1(\Omega)$ and $|f_n|\leq|f|$ (for example $f_n=T_n(f)$). We consider the approximate problem

(3.7)
$$A_nu_n+h_n(x,u_n,\nabla u_n)+|u_n|^{p_0-2}u_n+|T_n(u_n)|^{s-1}T_n(u_n)=f_n+\mu\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac1n}-\text{div }\phi_n(u_n),$$

where

$$A_nv=-\sum_{i=1}^{N}D^i(|D^iv|^{p_i-2}D^iv)+|v|^{p_0-2}v,\qquad\qquad \phi_n(s)=\phi(T_n(s)$$

and

$$h_n(x,s,\xi) = \frac{h(x,s,\xi)}{1+\frac{1}{n}h(x,s,\xi)}, \qquad \forall i=1,\ldots,N.$$

let's mention that

$$|h_n(x,s,\xi)| \leq n, \quad |h_n(x,s,\xi)| \leq |h(x,s,\xi)| \quad \text{and} \quad h_n(x,s,\xi)s \geq 0, \quad \forall n \in I\!N^*.$$

We consider the operator $G_n : W_0^{1,\vec{p}}(\Omega) \longrightarrow W^{-1,\vec{p}'}(\Omega)$ by

$$\langle G_n u, v\rangle = \int_\Omega h_n(x,u,\nabla u)v dx + \int_\Omega |T_n(u)|^{s-1}T_n(u)v dx - \mu \int_\Omega \frac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0}+\frac{1}{n}} v dx,$$

for any $u,v \in W_0^{1,\vec{p}}(\Omega)$. Using the Hölder's type inequality, we deduce that

(3.8)

$$\begin{aligned} |\langle G_n u, v\rangle| &\leq \|h_n(x,u,\nabla u)\|_{p_0'}\|v\|_{p_0} + \int_\Omega |T_n(u)|^s\ |v|\ dx + \mu \int_\Omega \frac{|T_n(u)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|v|\ dx \\ &\leq \|h_n(x,u,\nabla u)\|_{p_0'}\|v\|_{p_0} + n^s \int_\Omega |v|\ dx + \mu n^{p_0}\int_\Omega |v|\ dx \\ &\leq C_0\|v\|_{1,\vec{p}}\,. \end{aligned}$$

Moreover,we define the operator $R_n : W_0^{1,\vec{p}}(\Omega) \longrightarrow W^{-1,\vec{p}'}(\Omega)$ by

$$\langle R_n(u), v\rangle = \langle \text{div}\ \phi_n(u), v\rangle = -\int_\Omega \phi_n(u)\nabla v\, dx, \quad \text{ for any } \quad u,v \in W_0^{1,\vec{p}}(\Omega),$$

with $\phi_n(u) = (\phi_{i,n}(u),\ldots,\phi_{N,n}(u))$. Thanks to the Hölder's type inequality, we have

(3.9)

$$\begin{aligned} |\int_\Omega \phi_n(u)\nabla v\, dx| &\leq \sum_{i=1}^N \int_\Omega |\phi_{i,n}(u)||D^i v|\, dx \\ &\leq \sum_{i=1}^N \|\phi_{i,n}(u)\|_{p_i'}\|D^i v\|_{p_i} \\ &\leq \sum_{i=1}^N \sup_{|\sigma|\leq n}|\phi_i(\sigma)|\big(meas(\Omega)\big)^{\frac{1}{p_i'}}\|v\|_{1,\vec{p}} \\ &\leq C_1\|v\|_{1,\vec{p}}\,. \end{aligned}$$

Lemma 3.5. *The bounded operator $B_n = A_n + G_n + R_n$ acting from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p}'}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense:*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{1,\vec{p}}} \longrightarrow +\infty \qquad \textit{as} \quad \|v\|_{1,\vec{p}} \to \infty \quad \textit{for} \quad v \in W_0^{1,\vec{p}}(\Omega).$$

Proof of the Lemma 3.5

In view of the Hölder's inequality and by (3.8) and (3.9), it's easy to see that the operator B_n is bounded.

For the coercivity, we have for any $u \in W_0^{1,\vec{p}}(\Omega)$,

$$\begin{aligned}\langle B_n u, u\rangle &= \langle A_n u, u\rangle + \langle G_n u, u\rangle + \langle R_n(u), v\rangle \\ &= \sum_{i=0}^{N}\int_\Omega D^i u_n|^{p_i}\,dx + \int_\Omega h_n(x,u,\nabla u)u\,dx + \int_\Omega |T_n(u)|^s|u|\,dx - \mu\int_\Omega \frac{|T_n(u)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|u|\,dx \\ &\quad -\sum_{i=1}^{N}\int_\Omega |\phi_{i,n}(u)||D^i u|\,dx \\ &\geq \|u\|_{1,\vec{p}}^{\underline{p}} + \int_\Omega |T_n(u)|^{s+1}\,dx - 2\mu n^{p_0}\|1\|_{p_0'}\|u\|_{1,\vec{p}} - C_1\|u\|_{1,\vec{p}} \\ &\geq \|u\|_{1,\vec{p}}^{\underline{p}} - C_2\|u\|_{1,\vec{p}}\end{aligned}$$

it follows that

$$\frac{\langle B_n u, u\rangle}{\|u\|_{1,\vec{p}}} \longrightarrow +\infty \quad \text{as} \quad \|u\|_{1,\vec{p}} \to \infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in I\!N}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } W_0^{1,\vec{p}}(\Omega), \\ B_n u_k \rightharpoonup \chi_n & \text{in } W^{-1,\vec{p}'}(\Omega), \\ \limsup\limits_{k\to\infty}\langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle. \end{cases} \tag{3.10}$$

We will prove that

$$\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \ \text{ as } \ k\to+\infty.$$

We have

$$|D^i u_k|^{p_i-2}D^i u_k \rightharpoonup |D^i u|^{p_i-2}D^i u \quad \text{in} \quad L^{p_i'}(\Omega) \quad \text{as } k\to\infty \tag{3.11}$$

$$|u_k|^{p_0-2}u_k \rightharpoonup |u|^{p_0-2}u \quad \text{in} \quad L^{p_0'}(\Omega) \quad \text{as } k\to\infty \tag{3.12}$$

In view of Lebesgue's dominated convergence theorem, we obtain

$$|T_n(u_k)|^{s-1}T_n(u_k) \to |T_n(u)|^{s-1}T_n(u) \quad \text{in} \quad L^{p_0'}(\Omega), \tag{3.13}$$

and

$$\frac{|T_n(u_k)|^{p_0-2}T_n(u_k)}{|x|^{p_0}+\frac{1}{n}} \to \frac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0}+\frac{1}{n}} \quad \text{in} \quad L^{p_0'}(\Omega). \tag{3.14}$$

It easy to check that $(h_n(x,u_k,\nabla u_k))_k$ is bounded in $L^{\underline{p}'}(\Omega)$, then there exists a function φ_n such that

$$h_n(x,u_k,\nabla u_k) \rightharpoonup \varphi_n \quad \text{in} \quad L^{\underline{p}'}(\Omega) \quad \text{as } k\to\infty \tag{3.15}$$

Furthermore, since $\phi_n = \phi\circ T_n$ is a bounded continuous function and $u_k \to u$ in $L^{\underline{p}}(\Omega)$, by using the Lebesgue dominated convergence theorem, we deduce that

$$\phi_{i,n}(u_k) \rightharpoonup \phi_{i,n}(u) \quad \text{in} \quad L^{p_i'}(\Omega) \quad \text{for } \ i=1,\ldots,N. \tag{3.16}$$

For any $v \in W_0^{1,\vec{p}}(\Omega)$, we get

(3.17)

$$\begin{aligned}\langle \chi_n, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty} \sum_{i=0}^{N} \int_\Omega |D^i u_k|^{p_i-2} D^i u_k D^i v\, dx + \lim_{k\to\infty} \int_\Omega h_n(x, u_k, \nabla u_k) v\, dx \\ &\quad + \lim_{k\to\infty} \int_\Omega |T_n(u_k)|^{s-1} T_n(u_k) v\, dx - \lim_{k\to\infty} \mu \int_\Omega \frac{|T_n(u_k)|^{p_0-2} T_n(u_k)}{|x|^{p_0} + \frac{1}{n}} v\, dx \\ &\quad - \lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u_k) D^i v\, dx \\ &= \sum_{i=0}^{N} \int_\Omega |D^i u|^{p_i-2} D^i u D^i v\, dx + \int_\Omega \varphi_n v\, dx + \int_\Omega |T_n(u)|^{s-1} T_n(u) v\, dx \\ &\quad -\mu \int_\Omega \frac{|T_n(u)|^{p_0-2} T_n(u)}{|x|^{p_0} + \frac{1}{n}} v\, dx - \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u) D^i v\, dx.\end{aligned}$$

Having in mind (3.10) and (3.17), we obtain

(3.18)

$$\begin{aligned}&\limsup_{k\to\infty} \langle B_n(u_k), u_k\rangle \\ &= \limsup_{k\to\infty} \Big\{ \sum_{i=0}^{N} \int_\Omega |D^i u_k|^{p_i}\, dx + \int_\Omega h_n(x, u_k, \nabla u_k) u_k\, dx + \int_\Omega |T_n(u_k)|^{s-1} T_n(u_k) u_k\, dx \\ &\qquad -\mu \int_\Omega \frac{|T_n(u_k)|^{p_0-2} T_n(u_k)}{|x|^{p_0} + \frac{1}{n}} u_k\, dx - \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u_k) D^i u_k\, dx \Big\} \\ &\le \sum_{i=0}^{N} \int_\Omega |D^i u|^{p_i}\, dx + \int_\Omega \varphi_n u\, dx + \int_\Omega |T_n(u)|^{s-1} T_n(u) u\, dx \\ &\qquad -\mu \int_\Omega \frac{|T_n(u)|^{p_0-2} T_n(u)}{|x|^{p_0} + \frac{1}{n}} u\, dx - \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u) D^i u\, dx.\end{aligned}$$

Thanks to (3.13) and (3.14), we have

$$\int_\Omega |T_n(u_k)|^{s-1} T_n(u_k) u_k\, dx \longrightarrow \int_\Omega |T_n(u)|^{s-1} T_n(u) u\, dx, \tag{3.19}$$

and

$$\int_\Omega \frac{|T_n(u_k)|^{p_0-2} T_n(u_k)}{|x|^{p_0} + \frac{1}{n}} u_k\, dx \longrightarrow \int_\Omega \frac{|T_n(u)|^{p_0-2} T_n(u)}{|x|^{p_0} + \frac{1}{n}} u\, dx. \tag{3.20}$$

Due to (3.15) and (3.16), it yields

$$\int_\Omega h_n(x, u_k, \nabla u_k) u_k\, dx \longrightarrow \int_\Omega \varphi_n u\, dx, \tag{3.21}$$

$$\sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u_k) D^i u_k\, dx \longrightarrow \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u) D^i u\, dx. \tag{3.22}$$

Therefore

$$\limsup_{k\to\infty} \sum_{i=0}^{N} \int_\Omega |D^i u_k|^{p_i}\, dx \le \sum_{i=0}^{N} \int_\Omega |D^i u|^{p_i}\, dx. \tag{3.23}$$

On the other hand, we have

$$\sum_{i=0}^{N}\int_{\Omega}(|D^iu_k|^{p_i-2}D^iu_k-|D^iu|^{p_i-2}D^iu)(D^iu_k-D^iu)\,dx\geq 0, \tag{3.24}$$

then

$$\sum_{i=0}^{N}\int_{\Omega}|D^iu_k|^{p_i}\,dx\geq\sum_{i=0}^{N}\int_{\Omega}|D^iu_k|^{p_i-2}D^iu_kD^iu\,dx+\sum_{i=0}^{N}\int_{\Omega}(|D^iu|^{p_i-2}D^iu(D^iu_k-D^iu)\,dx.$$

In view of (3.11) and (3.12) we get

$$\liminf_{k\to\infty}\sum_{i=0}^{N}\int_{\Omega}|D^iu_k|^{p_i}\,dx\geq\sum_{i=0}^{N}\int_{\Omega}|D^iu|^{p_i}\,dx.$$

Having in mind (3.23), we conclude that

$$\lim_{k\to\infty}\sum_{i=0}^{N}\int_{\Omega}|D^iu_k|^{p_i}\,dx=\sum_{i=0}^{N}\int_{\Omega}|D^iu|^{p_i}\,dx. \tag{3.25}$$

Therefore, by combining (3.17), (3.19) – (3.22) and (3.25) we obtain

$$\langle B_nu_k,u_k\rangle\longrightarrow\langle\chi_n,u\rangle\quad\text{as}\quad k\to\infty.$$

Now, by (3.25) we can prove that

$$\lim_{k\to+\infty}\Big(\sum_{i=0}^{N}\int_{\Omega}\big(|D^iu_k|^{p_i-2}D^iu_k-|D^iu|^{p_i-2}D^iu\big)\Big)(D^iu_k-D^iu)\,dx=0.$$

and so, by virtue of Lemma 3.4, we get

$$u_k\to u\quad\text{in}\quad W_0^{1,\vec{p}}(\Omega)\qquad\text{and}\qquad D^iu_k\to D^iu\quad\text{a.e. in}\quad\Omega,$$

then

$$h_n(x,u_k,\nabla u_k)\rightharpoonup h_n(x,u,\nabla u)\quad\text{in}\quad L^{p_0'}(\Omega),$$

which implies $\chi_n=B_nu$. Finally, in view of Lemma 3.5, there exists at least one weak solution $u_n\in W_0^{1,\vec{p}}(\Omega)$ of the problem (3.7) (cf. [29], Theorem 8.2).

Step 2 : A priori estimates.

Lemma 3.6. *Let u_n be a weak solution of the approximate problem (3.7), then the following regularity results hold true*

$$u\in W_0^{1,\vec{q}}(\Omega)\qquad\textit{with}\quad\vec{q}=(s,q_1,\ldots,q_N) \tag{3.26}$$

where the exponent s verify the condition $s>\max\Big(\frac{N(p_0-1)}{N-p_0},\frac{1}{p_0-1}\Big)$ and $1\leq q_i<\frac{p_is}{s+1}$, then

$$\sum_{i=1}^{N}\int_{\Omega}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta}}dx\leq C\qquad\textit{for all}\quad 1<\theta<\frac{s(p_i-q_i)}{q_i}, \tag{3.27}$$

$$\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i}dx\leq C(1+k)^{\theta}\qquad\textit{for all}\quad k>0, \tag{3.28}$$

with C is a positive constant that doesn't depend on k and n.

Proof of Lemma 3.6

Let $\theta > 1$ which will be chosen later, we consider the two functions $\varphi(t) : I\!R \longmapsto I\!R$ defined by

$$\varphi(t) = \Big(1 - \frac{1}{(1+|t|)^{\theta-1}}\Big) sign(t) \quad \text{and} \quad J(t) = \int_0^t j(|\rho|)d\rho$$

It's clear that $\varphi(u_n)\exp(J(|u_n|)) \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, and $0 \le J(\infty) < \infty$.
By taking $\varphi(u_n)\exp(J(|u_n|))$ as test function in (3.7) we get

$$\begin{aligned}
&(\theta-1)\sum_{i=1}^N \int_\Omega \frac{(|D^i u_n|^{p_i}}{(1+|u_n|)^\theta} \exp(J(|u_n|))\,dx + \sum_{i=1}^N \int_\Omega (|D^i u_n|^{p_i} j(|u_n|)|\varphi(u_n)|\exp(J(|u_n|))\,dx \\
&\quad + \int_\Omega h_n(x,u_n,\nabla u_n)\varphi(u_n)\exp(J(|u_n|))\,dx + \int_\Omega |u_n|^{p_0-2}u_n\varphi(u_n)\exp(J(|u_n|))\,dx \\
&\quad + \int_\Omega |T_n(u_n)|^{s-1}T_n(u_n)\varphi(u_n)\exp(J(|u_n|))\,dx \\
&= \int_\Omega f_n\varphi(u_n)\exp(J(|u_n|))\,dx + \mu\int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}\varphi(u_n)\exp(J(|u_n|))\,dx \\
&\quad + \sum_{i=1}^N \int_\Omega \phi_{i,n}(u_n)D^i(\varphi(u_n)\exp(J(|u_n|)))\,dx.
\end{aligned}$$

Since $\varphi(u_n)$ have the same sign of u_n, thus the fourth term on the left-hand side of the previous inequality is positive. Also, we have

(3.29)
$$\begin{aligned}
\sum_{i=1}^N \int_\Omega \phi_{i,n}(u_n)D^i(\varphi(u_n)\exp(J(|u_n|)))\,dx &\le \sum_{i=1}^N \int_\Omega |\phi_{i,n}(u_n)|\ |D^i(\varphi(u_n)\exp(J(|u_n|)))|\,dx \\
&\le \sum_{i=1}^N \sup_{|s|\le n}|\phi_i(s)| \int_\Omega |D^i(\varphi(u_n)\exp(J(|u_n|)))|\,dx \le C_3.
\end{aligned}$$

Seeing that, $|\varphi(\cdot)| \le 1$ and in view of (1.4) and (3.29) we obtain

(3.30)
$$\begin{aligned}
&(\theta-1)\sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^\theta}\exp(J(|u_n|))\,dx + \int_\Omega |T_n(u_n)|^s|\varphi(u_n)|\exp(J(|u_n|))\,dx \\
&\qquad \le \mu\int_\Omega \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\exp(J(|u_n|))\,dx + \int_\Omega (|f|+|l|)\exp(J(|u_n|))\,dx + C_3.
\end{aligned}$$

It is clear that

$$\frac{1}{2} \le 1 - \frac{1}{(1+|u_n|)^{\theta-1}} \qquad \text{for} \quad |u_n| \ge R = \max(2^{\frac{1}{\theta-1}}-1, 1).$$

Thus, we have

$$\begin{aligned}
\frac{1}{2}\int_{\{|u_n|\ge R\}} |T_n(u_n)|^s\,dx &\le \int_{\{|u_n|\ge R\}} |T_n(u_n)|^s\Big(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\Big)\,dx \\
&\le \int_\Omega |T_n(u_n)|^s\Bigg(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\Bigg)\,dx,
\end{aligned}$$

which implies

$$\begin{aligned}\frac{1}{2}\int_\Omega |T_n(u_n)|^s\,dx &= \frac{1}{2}\int_{\{|u_n|<R\}} |T_n(u_n)|^s\,dx + \frac{1}{2}\int_{\{|u_n|\geq R\}} |T_n(u_n)|^s\,dx \\ &\leq \frac{1}{2}R^s|\Omega| + \int_\Omega |T_n(u_n)|^s\Big(1-\frac{1}{(1+|u_n|)^{\theta-1}}\Big)\,dx.\end{aligned}$$

Using (3.30), we deduce that

(3.31)

$$\begin{aligned}&(\theta-1)\sum_{i=1}^N\int_\Omega \frac{|D^iu_n|^{p_i}}{(1+|u_n|)^\theta}\,dx + \frac{1}{2}\int_\Omega |T_n(u_n)|^s\,dx \\ &\quad\leq \frac{1}{2}R^s|\Omega| + \mu\int_\Omega \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}}\exp(J(\infty))\,dx + \int_\Omega (|f|+|l|)\exp(J(\infty))\,dx + C_3.\end{aligned}$$

Inasmuch as $s > p_0 - 1$, the Young inequality enables us to obtain

$$\mu\int_\Omega \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}}\exp(J(\infty))\,dx \leq \frac{1}{4}\int_\Omega |T_n(u_n)|^s\,dx + C_4\int_\Omega \frac{dx}{|x|^{\frac{sp_0}{s-p_0+1}}},$$

with C_2 is a positive constant depending only on s, p_0, $\exp(J(\infty))$ and μ. Thus, we obtain

(3.32)

$$\begin{aligned}&(\theta-1)\sum_{i=1}^N\int_\Omega \frac{|D^iu_n|^{p_i}}{(1+|u_n|)^\theta}\,dx + \frac{1}{4}\int_\Omega |T_n(u_n)|^s\,dx \\ &\quad\leq \frac{1}{2}R^s|\Omega| + C_4\int_\Omega \frac{dx}{|x|^{\frac{sp_0}{s-p_0+1}}} + \exp(J(\infty))\int_\Omega (|f|+|l|)\,dx + C_3.\end{aligned}$$

Under the assumption $s > \dfrac{N(p_0-1)}{N-p_0}$, the integral $\displaystyle\int_\Omega \frac{dx}{|x|^{\frac{sp_0}{s-p_0+1}}}$ is finite. Therefore (3.27) is deduced. Moreover, we have

$$\int_\Omega |T_n(u_n)|^s\,dx \leq C. \tag{3.33}$$

Taking q_i such that $1 \leq q_i < p_i$ for $i = 1, \ldots, N$. By virtue of the generalized Hölder's inequality we get

(3.34)

$$\begin{aligned}\sum_{i=1}^N\int_\Omega |D^iu_n|^{q_i}\,dx &\leq \sum_{i=1}^N \Big\|\frac{|D^iu_n|^{q_i}}{(1+|u_n|)^{\frac{\theta q_i}{p_i}}}\Big\|_{\frac{p_i}{q_i}} \Big\|(1+|u_n|)^{\frac{\theta q_i}{p_i}}\Big\|_{\frac{p_i}{p_i-q_i}} \\ &\leq \sum_{i=1}^N \Big(\int_\Omega \frac{|D^iu_n|^{p_i}}{(1+|u_n|)^\theta}\,dx\Big)^{\frac{q_i}{p_i}} \Big(\int_\Omega (1+|u_n|)^{\frac{q_i\theta}{p_i-q_i}}\,dx + 1\Big)^{1-\frac{q_i}{p_i}}.\end{aligned}$$

We now choose $\theta > 1$ such that $\dfrac{q_i\theta}{p_i-q_i} < s$, such a real number θ exists if

$$1 < \frac{s(p_i-q_i)}{q_i} \qquad \text{that is} \qquad q_i < \frac{p_i s}{s+1}.$$

Combining $(3.32)-(3.34)$, we obtain the desired estimates (3.26).
To get (3.28), we have thanks to (3.27) that

$$\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i}\,dx=\sum_{i=1}^{N}\int_{\{|u_n|<k\}}|D^iu_n|^{p_i}\,dx\leq(1+k)^{\theta}\sum_{i=1}^{N}\int_{\Omega}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta}}\,dx.$$

Step 3 : The weak convergence of $(T_k(u_n))_n$ ***in*** $W_0^{1,\vec{p}}(\Omega)$. In order to establish the weak convergence of $(T_k(u_n))_n$ in $W_0^{1,\vec{p}}(\Omega)$, we begin by proving that $(u_n)_n$ is a Cauchy sequence. In fact, thanks to (3.28), we can obtain

$$\sum_{i=0}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i}\,dx\leq C(1+k)^{\theta}+k^{p_0}|\Omega|\qquad\text{for}\quad k\geq 1,$$

Therefore, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}}(\Omega)$, and there exists a subsequence still denoted $(T_k(u_n))_n$ such that

$$\left\{\begin{array}{lll} T_k(u_n)\rightharpoonup\eta_k & \text{in} & W_0^{1,\vec{p}}(\Omega),\\ T_k(u_n)\to\eta_k & \text{in} & L^{\underline{p}}(\Omega)\quad\text{and a.e. in}\quad\Omega.\end{array}\right.\tag{3.35}$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i}\,dx &\geq\sum_{i=1}^{N}\int_{\Omega}(|D^iT_k(u_n)|^{\underline{p}}-1)\,dx\\ &=\|\nabla T_k(u_n)\|_{\underline{p}}^{\underline{p}}-N|\Omega|,\end{aligned}$$

Thanks to (3.28), we deduce that there exists a constant C_5 that does not depend on k and n, such that

$$\|\nabla T_k(u_n)\|_{\underline{p}}\leq C_5k^{\frac{\theta}{\underline{p}}}\qquad\text{for}\quad k\geq 1.\tag{3.36}$$

Thanks to the Poincaré type inequality, we obtain

$$\begin{aligned}k\ \text{meas}\{|u_n|>k\} &=\int_{\{|u_n|>k\}}|T_k(u_n)|\,dx\leq\int_{\Omega}|T_k(u_n)|\,dx\\ &\leq C_6\|T_k(u_n)\|_{\underline{p}}\\ &\leq C_7\|\nabla T_k(u_n)\|_{\underline{p}}\\ &\leq C_8k^{\frac{\theta}{\underline{p}}},\end{aligned}\tag{3.37}$$

Choosing θ small enough $(1<\theta<\underline{p})$, we conclude that

$$\text{meas}\big\{|u_n|>k\big\}\leq C_8\frac{1}{k^{1-\frac{\theta}{\underline{p}}}}\longrightarrow 0\quad\text{as}\quad k\to+\infty.\tag{3.38}$$

For all $\delta>0$, we have

$$\text{meas}\{|u_n-u_m|>\delta\}\leq\text{meas}\{|u_n|>k\}+\text{meas}\{|u_m|>k\}+\text{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}.$$

Let $\varepsilon>0$, using (3.38) we can choose $k=k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n|>k\}\leq\frac{\varepsilon}{3}\qquad\text{and}\qquad\text{meas}\{|u_m|>k\}\leq\frac{\varepsilon}{3}.\tag{3.39}$$

On the other hand, thanks to (3.35) we can assume that $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Thus, for any $k>0$ and $\delta,\varepsilon>0$, there exists $n_0=n_0(k,\delta,\varepsilon)$ such that

$$\text{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}\leq\frac{\varepsilon}{3}\qquad\text{for all } m,n\geq n_0(k,\delta,\varepsilon).\tag{3.40}$$

In view of (3.39) and (3.40), we deduce that

$\forall \delta, \varepsilon > 0$ there exists $n_0 = n_0(\delta, \varepsilon)$ such that $\text{meas}\{|u_n - u_m| > \delta\} \le \varepsilon \quad \forall n, m \ge n_0(\delta, \varepsilon)$,

which proves that the sequence $(u_n)_n$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u. Consequently, we have

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in} \quad W_0^{1,\vec{p}}(\Omega), \tag{3.41}$$

and in view Lebesgue's dominated convergence theorem, we obtain

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{in} \quad L^{p_0}(\Omega) \quad \text{and a.e in} \quad \Omega. \tag{3.42}$$

Step 4 : Strong convergence of truncations. In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \dots$, various real-valued functions of real variables that converge to 0 as n tends to infinity.
Let $h > k > 0$, taking $z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u)$, $M = 4k + h$ and $\omega_n := T_{2k}(z_n)$.
We also consider $\psi_k(s) = s.\exp(\lambda s^2)$ where $\lambda = (b(k)/(2))^2$. It is simple to see that ([11], Lemma 1)

$$\psi_k'(s) - b(k)|\psi_k(s)| \ge \frac{1}{2} \qquad \text{for any } s \in I\!R.$$

By using $\psi_k(\omega_n)$ as a test function in the approximate problem (3.7) we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i-2} D^i u_n \psi_k'(\omega_n) D^i \omega_n \, dx + \int_\Omega h_n(x, u_n, \nabla u_n) \psi_k(\omega_n) \, dx \\
&\qquad + \int_\Omega |u_n|^{p_0-2} u_n \psi_k(\omega_n) \, dx + \int_\Omega |T_n(u_n)|^{s-1} T_n(u_n) \psi_k(\omega_n) \, dx \\
&= \mu \int_\Omega \frac{|T_n(u_n)|^{p_0-2} T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \psi_k(\omega_n) \, dx + \int_\Omega f_n \psi_k(\omega_n) \, dx + \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u_n) \psi_k'(\omega_n) D^i \omega_n \, dx.
\end{aligned}$$

For $M = 4k + h$, it's clear that $D^i \omega_n = 0$ on the set $\{|u_n| \ge M\}$, and since $h_n(x, u_n, \nabla u_n)\psi_k(\omega_n) \ge 0$ on the set $\{|u_n| > k\}$, therefore

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n| \le M\}} |D^i T_M(u_n)|^{p_i-2} D^i T_M(u_n) \psi_k'(\omega_n) D^i \omega_n \, dx + \int_{\{|u_n| \le k\}} h_n(x, u_n, \nabla u_n) \psi_k(\omega_n) \, dx \\
&\qquad + \int_{\{|u_n| \le k\}} |T_k(u_n)|^{p_0-2} T_k(u_n) \psi_k(\omega_n) \, dx + \int_\Omega |T_n(u_n)|^{s-1} T_n(u_n) \psi_k(\omega_n) \, dx \\
&\le \mu \int_\Omega \frac{|T_n(u_n)|^{p_0-2} T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \psi_k(\omega_n) \, dx + \int_\Omega f_n \psi_k(\omega_n) \, dx \\
&\qquad + \sum_{i=1}^{N} \int_{\{|u_n| \le M\}} \phi_{i,n}(u_n) \psi_k'(\omega_n) D^i \omega_n \, dx.
\end{aligned}$$

Thanks to the Young inequality, we have

$$\mu \int_{\{|u_n| > k\}} \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0} + \frac{1}{n}} |\psi_k(\omega_n)| dx \le \int_{\{|u_n| > k\}} |T_n(u_n)|^s |\psi_k(\omega_n)| dx + C_7 \int_{\{|u_n| > k\}} \frac{|\psi_k(\omega_n)|}{|x|^{\frac{p_0 s}{s - p_0 + 1}}} dx,$$

and since $\omega_n = T_k(u_n) - T_k(u)$ on the set $\{|u_n| \leq k\}$, then

(3.43)

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n| \leq M\}} |D^i T_M(u_n)|^{p_i-2} D^i T_M(u_n) \psi_k'(\omega_n) D^i \omega_n \, dx + \int_{\{|u_n| \leq k\}} h_n(x, u_n, \nabla u_n) \psi_k(\omega_n) \, dx \\
&\quad + \int_{\{|u_n| \leq k\}} |T_k(u_n)|^{p_0-2} T_k(u_n) \psi_k(\omega_n) \, dx + \int_{\{|u_n| \leq k\}} |T_n(u_n)|^{s-1} T_n(u_n) \psi_k(\omega_n) \, dx \\
&\leq \mu \int_{\Omega} \frac{|T_n(u_n)|^{p_0-2} T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \psi_k(\omega_n) \, dx + \int_{\Omega} f_n \psi_k(\omega_n) \, dx + C_9 \int_{\{|u_n| > k\}} \frac{|\psi_k(\omega_n)|}{|x|^{\frac{p_0 s}{s - p_0 + 1}}} \, dx \, dx \\
&\quad + \sum_{i=1}^{N} \int_{\{|u_n| \leq M\}} \phi_{i,n}(u_n) \psi_k'(\omega_n) D^i \omega_n \, dx.
\end{aligned}$$

Now, we will study each terms in the previews inequality.
Firstly, we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n| \leq M\}} |D^i T_M(u_n)|^{p_i-2} D^i T_M(u_n) \psi_k'(\omega_n) D^i \omega_n \, dx \\
&= \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} |D^i T_k(u_n)|^{p_i-2} D^i T_k(u_n) \psi_k'(\omega_n) D^i T_{2k}(u_n - T_k(u)) \, dx \qquad (3.44) \\
&\quad + \sum_{i=1}^{N} \int_{\{k < |u_n| \leq M\}} D^i T_M(u_n)|^{p_i-2} D^i T_M(u_n) \psi_k'(\omega_n) D^i \omega_n \, dx.
\end{aligned}$$

on one hand, since $|u_n - T_k(u)| \leq 2$ on $\{|u_n| \leq k\}$, then

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} |D^i T_k(u_n)|^{p_i-2} D^i T_k(u_n) \psi_k'(\omega_n) D^i T_{2k}(u_n - T_k(u)) \, dx \\
&= \sum_{i=1}^{N} \int_{\Omega} |D^i T_k(u_n)|^{p_i-2} D^i T_k(u_n) \psi_k'(\omega_n) (D^i T_k(u_n) - D^i T_k(u)) \, dx \qquad (3.45) \\
&\quad + \sum_{i=1}^{N} \int_{\{k < |u_n|\}} |D^i T_k(u_n)|^{p_i-2} D^i T_k(u_n) \psi_k'(\omega_n) D^i T_k(u) \, dx.
\end{aligned}$$

Seeing that $1 \leq \psi_k'(\omega_n) \leq \psi_k'(2k)$, then

$$\begin{aligned}
&\Big| \sum_{i=1}^{N} \int_{\{k < |u_n|\}} |D^i T_k(u_n)|^{p_i-2} D^i T_k(u_n) \psi_k'(\omega_n) D^i T_k(u) \, dx \Big| \\
&\leq \psi_k'(2k) \sum_{i=1}^{N} \int_{\{k < |u_n|\}} |D^i T_k(u_n)|^{p_i-1} |D^i T_k(u)| \, dx,
\end{aligned}$$

and since $|D^i T_k(u_n)|^{p_i-1}$ is bounded in $L^{p_i'}(\Omega)$, then there exists $\zeta \in L^{p_i'}(\Omega)$ such that $|D^i T_k(u_n)|^{p_i-1} \rightharpoonup \zeta$ in $L^{p_i'}(\Omega)$. Thus,

$$\sum_{i=1}^{N} \int_{\{k < |u_n|\}} |D^i T_k(u_n)|^{p_i-1} |D^i T_k(u)| \, dx \to \int_{\{k < |u|\}} \zeta |D^i T_k(u)| \, dx = 0.$$

subsequently

$$\sum_{i=1}^{N} \int_{\{k < |u_n|\}} |D^i T_k(u_n)|^{p_i-2} D^i T_k(u_n) \psi_k'(\omega_n) D^i T_k(u) \, dx = \varepsilon_0(n). \qquad (3.46)$$

Taking $z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u)$, in the second term on the right hand side of (3.44), we get

$$\begin{aligned}
&\int_{\{k<|u_n|\le M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx\\
&= \int_{\{k<|u_n|\le M\}\cap\{|z_n|\le 2k\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)D^i(u_n - T_h(u_n)+T_k(u_n)-T_k(u))\,dx\\
&\ge= \int_{\{k<|u_n|\le M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i(u_n-T_k(u)).\chi_{|u_n|>h}\,dx\\
&\quad -\int_{\{k<|u_n|\le M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^iT_k(u).\chi_{|u_n|\le h}\,dx\\
&\ge -\psi_k'(2k)\int_{\{k<|u|\le M\}} |D^iT_M(u_n)|^{p_i-1}\;|D^iT_k(u)|\,dx = 0.
\end{aligned}$$

Similarly to (3.46), we can prove that

$$\psi_k'(2k)\int_{\{k<|u|\le M\}} |D^iT_M(u_n)|^{p_i-1}\;|D^iT_k(u)|\,dx = \varepsilon_1(n) \tag{3.47}$$

By combining (3.44) – (3.47), we obtain
(3.48)

$$\begin{aligned}
&\sum_{i=1}^N\int_{\{|u_n|\le M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx\\
&\ge \sum_{i=1}^N\int_\Omega |D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\psi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx+\varepsilon_2(n).
\end{aligned}$$

Therefore, we have
(3.49)

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega \big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)-|D^iT_k(u)|^{p_i-2}D^iT_k(u)\big).(D^iT_k(u_n)-D^iT_k(u))\psi_k'(\omega_n)\,dx\\
&\le \sum_{i=1}^N\int_{\{|u_n|\le M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx\\
&\quad -\sum_{i=1}^N\int_\Omega |D^iT_k(u)|^{p_i-2}D^iT_k(u)(D^iT_k(u_n)-D^iT_k(u))\psi_k'(\omega_n)\,dx-\varepsilon_2(n)\\
&\le \sum_{i=1}^N\int_{\{|u_n|\le M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx\\
&\quad +\psi_k'(2k)\int_\Omega |D^iT_k(u)|^{p_i-1}\;|D^iT_k(u_n)-D^iT_k(u)|\,dx-\varepsilon_2(n).
\end{aligned}$$

For the second term on the right-hand side of (3.49), since $|D^iT_k(u)|^{p_i-1}$ is bounded in $L^{p_i'}(\Omega)$ and $D^iT_k(u_n) \rightharpoonup D^iT_k(u)$ in $L^{p_i}(\Omega)$, then

$$\psi_k'(2k)\int_\Omega |D^iT_k(u)|^{p_i-1}\;|D^iT_k(u_n)-D^iT_k(u)|\,dx = \varepsilon_3(n) \quad \text{as} \quad n\to 0. \tag{3.50}$$

Consequently, we deduce that

(3.51)

$$\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)-|D^iT_k(u)|^{p_i-2}D^iT_k(u)\big).(D^iT_k(u_n)-D^iT_k(u))\psi_k'(\omega_n)\,dx$$
$$\leq \sum_{i=1}^{N}\int_{\{|u_n|\leq M\}}|D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx+\varepsilon_4(n).$$

Secondly, we deal with the second term on the left-hand side of (3.43). In view of (1.4) we have

$$\begin{aligned}
&\Big|\int_{\{|u_n|\leq k\}}h_n(x,u_n,\nabla u_n)\psi_k(\omega_n)\,dx\Big|\\
&\leq \int_{\{|u_n|\leq k\}}l(x)|\psi_k(\omega_n)|\,dx+j(k)\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i}|\psi_k(\omega_n)|\,dx\\
&= \int_{\{|u_n|\leq k\}}l(x)|\psi_k(\omega_n)|\,dx+j(k)\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}|D^iT_k(u_n)\big)D^iT_k(u_n)|\psi_k(\omega_n)|\,dx\\
&\leq \int_{\{|u_n|\leq k\}}l(x)|\psi_k(\omega_n)|\,dx\\
&\quad +j(k)\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)-|D^iT_k(u)|^{p_i-2}D^iT_k(u)\big)\\
&\quad .(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx\\
&\quad +j(k)\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u)|^{p_i-2}D^iT_k(u)(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx\\
&\quad +j(k)\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\big)D^iT_k(u).
\end{aligned}$$

It yields

(3.52)

$$\begin{aligned}
&j(k)\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)-|D^iT_k(u)|^{p_i-2}D^iT_k(u)\big)\\
&.(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx\\
&\geq \Big|\int_{\{|u_n|\leq k\}}h_n(x,u_n,\nabla u_n)\psi_k(\omega_n)\,dx\Big|-\int_{\{|u_n|\leq k\}}l(x)|\psi_k(\omega_n)|\,dx\\
&\quad -j(k)\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u)|^{p_i-2}D^iT_k(u)(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx\\
&\quad -j(k)\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\big)D^iT_k(u)|\psi_k(\omega_n)|\,dx.
\end{aligned}$$

Regarding the third term on the right-hand side of (3.52), due to (3.50), we obtain

(3.53)

$$\Big|\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u)|^{p_i-2}D^iT_k(u)(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\Big|\,dx$$
$$\leq \psi_k(2k)\sum_{i=1}^{N}\int_{\Omega}\big||D^iT_k(u)|^{p_i-2}D^iT_k(u)\big|\;\big|D^iT_k(u_n)-D^iT_k(u)\big|\to 0\quad\text{as}\quad n\to\infty.$$

Concerning the fourth term of the right-hand side of (3.52), knowing that $|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)$ is bounded in $L^{p_i'}(\Omega)$, then there exists $\gamma \in L^{p_i'}(\Omega)$ such that $|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n) \rightharpoonup \gamma$ in $L^{p_i'}(\Omega)$ and, by applying

$$D^iT_k(u)|\psi_k(\omega_n)| \rightharpoonup D^iT_k(u)|\psi_k(T_{2k}(u-T_h(u)))| \qquad \text{in} \qquad L^{p_i'}(\Omega),$$

we deduce that

$$\begin{aligned}&\int_\Omega \big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\big)D^iT_k(u)|\psi_k(\omega_n)|\,dx\\&\qquad\to \int_\Omega \gamma D^iT_k(u)|\psi_k(T_{2k}(u-T_h(u)))| = 0.\end{aligned} \tag{3.54}$$

For the second term of the right-hand side of (3.52), using the fact that $\psi_k(\omega_n) \rightharpoonup \psi_k(T_{2k}(u-T_h(u)))$ weak-* in $L^\infty(\Omega)$ as $n \to +\infty$, then

$$\int_{\{|u_n|\le k\}} l(x)|\psi_k(\omega_n)|\,dx \to \int_{\{|u|\le k\}} l(x)|\psi_k(T_{2k}(u-T_h(u)))|\,dx = 0 \tag{3.55}$$

By combining (3.52) – (3.55) we conclude that

$$\begin{aligned}&\Big|\int_{\{|u_n|\le k\}} h_n(x,u_n,\nabla u_n)\psi_k(\omega_n)\,dx\Big| + \varepsilon_5(n)\\&\le j(k)\sum_{i=1}^N \int_\Omega \big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n) - |D^iT_k(u)|^{p_i-2}D^iT_k(u)\big)\\&\qquad .(D^iT_k(u_n) - D^iT_k(u))|\psi_k(\omega_n)|\,dx.\end{aligned} \tag{3.56}$$

As a third estimate, we have

$$\begin{aligned}&\int_{\{|u_n|\le k\}} |u_n|^{p_0-2}u_n\psi_k(\omega_n)\,dx\\&= \int_\Omega \big(|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u)\big)(T_k(u_n)-T_k(u))\exp(\lambda\omega_n^2)dx\\&\quad + \int_\Omega |T_k(u)|^{p_0-2}T_k(u)(T_k(u_n)-T_k(u))\exp(\lambda\omega_n^2)dx\\&\quad - \int_{\{|u_n|\ge k\}} |T_k(u_n)|^{p_0-2}T_k(u_n)(T_k(u_n)-T_k(u))\exp(\lambda\omega_n^2)dx\\&\ge \int_\Omega \big(|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u)\big)(T_k(u_n)-T_k(u))dx\\&\quad - \exp(\lambda(2k)^2)\int_\Omega |T_k(u)|^{p_0-1}|(T_k(u_n)-T_k(u))|dx\\&\quad - \exp(\lambda(2k)^2)\int_{\{|u_n|\ge k\}} k^{p_0-1}(T_k(u_n)-T_k(u))|dx.\end{aligned}$$

Thanks to (3.42), the second and the last term on the right-hand side of the previous inequality converges to 0 as $n \to \infty$. Thus, we obtain

$$\begin{aligned}&\int_\Omega \big(|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u)\big)(T_k(u_n)-T_k(u))dx\\&\qquad \le \int_{\{|u_n|\le k\}} |u_n|^{p_0-2}u_n\psi_k(\omega_n)\,dx + \varepsilon_6(n).\end{aligned} \tag{3.57}$$

For the fourth term of the left-hand side of (3.43). The Lebesgue dominated convergence theorem give us

$$|T_k(u_n)|^{s-1}T_k(u_n) \longrightarrow |T_k(u)|^{s-1}T_k(u) \qquad \text{in} \qquad L^1(\Omega),$$

and

$$\begin{aligned}
&\left|\lim_{n\to\infty}\int_{\{|u_n|\le k\}}|T_n(u_n)|^{s-1}T_n(u_n)\psi_k(\omega_n)\,dx\right|dx\\
&\le k^s\lim_{n\to\infty}\int_{\{|u_n|\le k\}}|\psi_k(\omega_n)|\,dx\\
&= k^s\int_{\{|u|\le k\}}\psi_k(T_{2k}(u-T_h(u)))\,dx=0,
\end{aligned}$$

it follows that

$$\lim_{n\to\infty}\int_{\{|u_n|\le k\}}|T_n(u_n)|^{s-1}T_n(u_n)\psi_k(\omega_n)\,dx=0. \tag{3.58}$$

Concerning the first term on the right-hand side of (3.43), by virtue of the Hölder's type inequality and as above we have

(3.59)

$$\begin{aligned}
\varepsilon_7(n) &= \left|\int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}\psi_k(\omega_n)\,dx\right|\\
&\le k^s\left\|\frac{1}{|x|^{p_0(x)}}\right\|_{L^{\frac{s}{s-p+1}}(\Omega)}\left\|\psi_k(\omega_n)\right\|_{L^{\frac{s}{p-1}}(\{|u_n|\le k\})}\longrightarrow 0 \qquad \text{as} \quad n\to\infty,
\end{aligned}$$

also, due to the weak-* convergence of $\psi_k(\omega_n)$ in $L^\infty(\Omega)$ we have

$$\int_\Omega f_n\ \psi_k(\omega_n)\,dx=\int_\Omega f\ \psi_k(T_{2k}(u-T_h(u)))\,dx+\varepsilon_8(n). \tag{3.60}$$

once again the theorem of Lebesgue allows us

$$\lim_{n\to\infty}\int_{\{|u_n|>k\}}\frac{|\psi_k(\omega_n)|}{|x|^{\frac{p_0 s}{s-p_0+1}}}\,dx=\int_{\{|u|>k\}}\frac{|\psi_k(T_{2k}(u-T_h(u))|}{|x|^{\frac{p_0 s}{s-p_0+1}}}\,dx. \tag{3.61}$$

Concerning the last term on the right-hand side of (3.43), we have for n large enough

$$\begin{aligned}
&\int_{\{|u_n|\le M\}}\phi_{i,n}(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx\\
&=\int_\Omega \phi_i(T_M(u))\psi_k'(T_{2k}(u-T_h(u)))D^iT_{2k}(u-T_h(u))\,dx+\varepsilon_9(n)
\end{aligned}$$

By using $W_i(t)=\int_0^t\phi_i(\varsigma)\psi_k'(\varsigma-T_h(\varsigma))d\varsigma$, we have $W_i\in\mathcal{C}^1(I\!R)$ and $W_i(0)=0$. By applying the Green formula, we have

$$\begin{aligned}
&\int_\Omega \phi_i(T_M(u))\psi_k'(T_{2k}(u-T_h(u)))D^iT_{2k}(u-T_h(u))\,dx\\
&=\int_{\{h<|u|\le 2k+h\}}\phi_i(u)\psi_k'(u-T_h(u))D^iu\,dx\\
&=\int_{\{|u|<2k+h\}}\phi_i(T_{2k+h}(u))\psi_k'(T_{2k+h}(u)-T_h(u))D^iT_{2k+h}\,dx\\
&\qquad-\int_{\{|u|<h\}}\phi_i(T_h(u))\psi_k'(T_h(u)-T_h(u))D^iT_h\,dx\\
&=\int_\Omega D^iW_i(T_{2k+h}(u))\,dx-\int_\Omega D^iW_i(T_h(u))\,dx\\
&=\int_{\partial\Omega}W_i(T_{2k+h}(u)).n_i\,dx-\int_{\partial\Omega}W_i(T_h(u)).n_i\,dx=0.
\end{aligned}$$

Consequently, we get

$$\int_{\{|u_n|\leq M\}} \phi_{i,n}(u_n)\psi'_k(\omega_n)D^i\omega_n\,dx = \varepsilon_{10}(n). \tag{3.62}$$

Consequently, taking into account (3.43), (3.51) and (3.56) – (3.62), we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{N}\int_\Omega \big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n) - |D^iT_k(u)|^{p_i-2}D^iT_k(u)\big)\\
&\qquad .(D^iT_k(u_n) - D^iT_k(u))\,dx\\
&\qquad + \int_\Omega \big(|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u)\big)(T_k(u_n) - T_k(u))\,dx\\
&\leq \sum_{i=1}^{N}\int_{\{|u_n|\leq M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi'_k(\omega_n)D^i\omega_n\,dx\\
&\qquad -\Big|\int_{\{|u_n|\leq k\}} h_n(x,u_n,\nabla u_n)\psi_k(\omega_n)\,dx\Big|\\
&\qquad + \int_{\{|u_n|\leq k\}} |u_n|^{p_0-2}u_n\psi_k(\omega_n)\,dx + \varepsilon_{10}(n)\\
&\leq \int_\Omega f\ \psi_k(T_{2k}(u-T_h(u)))\,dx + C_9\int_{\{|u|>h\}} \frac{|\psi_k(T_{2k}(u-T_h(u))|}{|x|^{\frac{p_0 s}{s-p_0+1}}}\,dx + \varepsilon_{11}(n)
\end{aligned} \tag{3.63}$$

therefore

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega \big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n) - |D^iT_k(u)|^{p_i-2}D^iT_k(u)\big)\\
&\qquad .(D^iT_k(u_n) - D^iT_k(u))\,dx\\
&\qquad + \int_\Omega \big(|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u)\big)(T_k(u_n) - T_k(u))\,dx\\
&\leq 2\int_\Omega f\ \psi_k(T_{2k}(u-T_h(u)))\,dx + 2C_9\int_{\{|u|>h\}} \frac{|\psi_k(T_{2k}(u-T_h(u))|}{|x|^{\frac{p_0 s}{s-p_0+1}}}\,dx + \varepsilon_{11}(n)
\end{aligned} \tag{3.64}$$

Since $\dfrac{N(p_0-1)}{N-p_0} < s$, we have $\dfrac{p_0 s}{s-p_0+1} < N$ then $\dfrac{1}{|x|^{\frac{p_0 s}{s-p_0+1}}} \in L^1(\Omega)$.
Finally , we conclude by letting h and n goes to infinity in (3.64)

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega \big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n) - |D^iT_k(u)|^{p_i-2}D^iT_k(u)\big)(D^iT_k(u_n) - D^iT_k(u))\,dx\\
&\qquad + \int_\Omega \big(|T_k(u_n)|^{p_0(x)-2}T_k(u_n) - |T_k(u_n)|^{p_0(x)-2}T_k(u_n)\big)(T_k(u_n) - T_k(u))\,dx\Big) = 0.
\end{aligned} \tag{3.65}$$

In view of Lemma 3.4, we conclude that

$$\begin{cases} T_k(u_n) \longrightarrow T_k(u) & \text{strongly in} \quad W_0^{1,\vec{p}}(\Omega), \\ D^iu_n \longrightarrow D^iu & \text{a.e. in} \quad \Omega \qquad \text{for } i=1,\dots,N. \end{cases} \tag{3.66}$$

Step 5 : The equi-integrability of the nonlinear functions. Thanks to (3.66), we get

$$h_n(x,u_n,\nabla u_n) \to h(x,u,\nabla u) \quad \text{a.e. in} \quad \Omega, \tag{3.67}$$

and we have also

$$|T_n(u_n)|^{s-1}T_n(u_n) \to |u|^{s-1}u \quad \text{a.e. in} \quad \Omega, \tag{3.68}$$

$$\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}} \longrightarrow \frac{|u|^{p_0-2}u}{|x|^{p_0}} \quad \text{a.e. in} \quad \Omega,. \tag{3.69}$$

In order to prove the uniform equi-integrability of these functions, we take $T_1(u_n - T_k(u_n))$ as a test function in (3.7), and since $T_1(u_n - T_k(u_n))$ have the same sign as u_n we can obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{k<|u_n|\le k+1\}}|D^iu_n|^{p_i}\,dx+\int_{\{|u_n|\ge k\}}h_n(x,u_n,\nabla u_n)T_1(u_n-T_k(u_n))\,dx\\
&\qquad+\int_{\{|u_n|\ge k\}}|T_n(u_n)|^s|T_1(u_n-T_k(u_n))|\,dx\\
&\le \mu\int_{\{|u_n|\ge k\}}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|T_1(u_n-T_h(u_n))|\,dx+\int_{\{|u_n|\ge k\}}|f_n|\,dx\\
&\qquad+\sum_{i=1}^{N}\int_{\{k<|u_n|\le k+1\}}\phi_{i,n}(u_n)D^iu_n\,dx.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\{|u_n|\ge k\}}h_n(x,u_n,\nabla u_n)T_1(u_n-T_k(u_n))\,dx\\
&\qquad\ge\int_{\{|u_n|\ge k+1\}}h_n(x,u_n,\nabla u_n)T_1(u_n-T_k(u_n))\,dx=\int_{\{|u_n|\ge k+1\}}|h_n(x,u_n,\nabla u_n)|\,dx.
\end{aligned}$$

Let's consider $\Psi_{i,n}=\int_0^t\phi_{i,n}(\varsigma)d\varsigma$, we have $\Psi_n\in\mathcal{C}^1(I\!R)$ and $\Psi_{i,n}(0)=0$. By virtue of Green formula, we obtain

$$\begin{aligned}
&\textstyle\int_{\{k<|u_n|\le k+1\}}\phi_{i,n}(u_n)D^iu_n\,dx\\
&=\int_\Omega\phi_{i,n}(T_{k+1}(u_n))D^iT_{k+1}(u_n)\,dx-\int_\Omega\phi_{i,n}(T_k(u_n))D^iT_k(u_n)\,dx\\
&=\int_\Omega D^i\Psi_{i,n}(T_{k+1}(u_n))\,dx-\int_\Omega D^i\Psi_{i,n}(T_k(u_n))\,dx=0
\end{aligned}$$

Thanks to Young's inequality, we have

$$\begin{aligned}
&\mu\int_{\{|u_n|\ge k\}}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|T_1(u_n-T_k(u_n))|\,dx\\
&\le\frac{1}{3}\int_{\{|u_n|\ge k\}}|T_n(u_n)|^s|T_1(u_n-T_k(u_n))|\,dx+C_{10}\int_{\{|u_n|\ge k\}}\frac{|T_1(u_n-T_k(u_n))|}{|x|^{\frac{sp_0}{s-p_0+1}}}\,dx,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\int_{\{|u_n|\ge k+1\}}|h_n(x,u_n,\nabla u_n)|\,dx+\frac{1}{3}\int_{\{|u_n|\ge k+1\}}|T_n(u_n)|^s\,dx+\mu\int_{\{|u_n|\ge k+1\}}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx\\
&\le 2C_{10}\int_{\{|u_n|\ge k\}}\frac{|T_1(u_n-T_k(u_n))|}{|x|^{\frac{sp_0}{s-p_0+1}}}\,dx+\int_{\{|u_n|\ge k\}}|f_n|\,dx.
\end{aligned}$$

Hence, for any $\delta>0$, there exists $k(\delta)>0$ such that

$$\int_{\{|u_n|\ge k(\delta)\}}|h_n(x,u_n,\nabla u_n)|dx+\int_{\{|u_n|\ge k(\delta)\}}|T_n(u_n)|^sdx+\int_{\{|u_n|\ge k(\delta)\}}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}dx\le\frac{\delta}{2}. \tag{3.70}$$

Now, let E be a measurable subset of Ω, we have

(3.71)
$$\begin{aligned}
&\int_E |h_n(x,u_n,\nabla u_n)|\,dx + \int_E |T_n(u_n)|^s\,dx + \int_E \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx \\
&\le \int_{E\cap\{|u_n|<k(\delta)\}} |h_n(x,T_{k(\delta)}(u_n),\nabla T_{k(\delta)}(u_n))|\,dx + \int_{E\cap\{|u_n|<k(\delta)\}} |T_{k(\delta)}(u_n)|^s\,dx \\
&\quad + \int_{E\cap\{|u_n|<k(\delta)\}} \frac{|T_{k(\delta)}(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx + \int_{\{|u_n|\ge k(\delta)\}} |h_n(x,u_n,\nabla u_n)|\,dx \\
&\quad + \int_{\{|u_n|\ge k(\delta)\}} |T_n(u_n)|^s\,dx + \int_{\{|u_n|\ge k(\delta)\}} \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx.
\end{aligned}$$

On other hand, we have

$$\int_{E\cap\{|u_n|<k(\delta)\}} |h_n(x,T_{k(\delta)}(u_n),\nabla T_{k(\delta)}(u_n)|dx \le \int_{E\cap\{|u_n|<k(\delta)\}} \Big(l(x)+j(|k(\delta)|)\sum_{i=1}^{N}|D_i|^{p_i}\Big)dx$$

Then, thanks to (3.66), there exists $\beta(\delta)>0$ such that : for any $E\subseteq\Omega$ with $\text{meas}(E)\le\beta(\delta)$

(3.72)
$$\begin{aligned}
&\int_{E\cap\{|u_n|<k(\delta)\}} |h_n(x,T_{k(\delta)}(u_n),\nabla T_{k(\delta)}(u_n))|\,dx + \int_{E\cap\{|u_n|<k(\delta)\}} |T_{k(\delta)}(u_n)|^s\,dx \\
&\quad + \int_{E\cap\{|u_n|<k(\delta)\}} \frac{|T_{k(\delta)}(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx \le \frac{\delta}{2}.
\end{aligned}$$

Finally, by combining (3.70), (3.71) and (3.72), one easily has

(3.73)
$$\int_E |h_n(x,u_n,\nabla u_n)|dx + \int_E |T_n(u_n)|^s dx + \int_E \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}dx \le \delta \quad \text{with meas}(E)\le\beta(\delta),$$

We deduce that $(h_n(x,u_n,\nabla u_n))_n$, $(|T_n(u_n)|^{s(x)-1}T_n(u_n))_n$ and $\Big(\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}\Big)_n$ are equi-integrable, and in view of $(3.67)-(3.72)$ and Vitali's theorem, the following convergences are established

$$h_n(x,u_n,\nabla u_n)\to h(x,u,\nabla u) \quad \text{in} \quad L^1(\Omega), \tag{3.74}$$

$$|T_n(u_n)|^{s-1}T_n(u_n)\to |u|^{s-1}u \quad \text{in} \quad L^1(\Omega), \tag{3.75}$$

and

$$\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}} \longrightarrow \frac{|u|^{p_0-2}u}{|x|^{p_0}} \quad \text{in} \quad L^1(\Omega). \tag{3.76}$$

Step 6 : Passage to the limit. By taking $T_k(u_n-\varphi)$ as a test function in (3.7), with $\varphi\in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$, and choosing $M=k+\|\varphi\|_\infty$, we obtain
(3.77)

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega |D^iu_n|^{p_i-2}D^iu_nD^iT_k(u_n-\varphi)\,dx+\int_\Omega h_n(x,u_n,\nabla u_n)T_k(u-\varphi)\,dx\\&\qquad+\int_\Omega |T_n(u_n)|^{s-1}T_n(u_n)T_k(u_n-\varphi)\,dx+\int_\Omega |u_n|^{p_0-2}u_nT_k(u_n-\varphi)\,dx\\&=\mu\int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}T_k(u_n-\varphi)\,dx\\&\qquad+\int_\Omega f_nT_k(u_n-\varphi)\,dx+\sum_{i=1}^N\int_\Omega \phi_{i,n}(u_n)D^iT_k(u-\varphi)\,dx.\end{aligned}$$

On the one hand, as soon as $|u_n|>M$ we get $|u_n-\varphi|\geq|u_n|-\|\varphi\|_\infty>k$, then $\{|u_n-\varphi|\leq k\}\subseteq\{|u_n|\leq M\}$, it follows that

$$\begin{aligned}&\int_\Omega |D^iu_n|^{p_i-2}D^iu_nD^iT_k(u_n-\varphi)\,dx\\&=\int_\Omega |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\,dx\\&=\int_\Omega (|D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)-|D^i\varphi|^{p_i-2}D^i\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\,dx\\&\qquad+\int_\Omega |D^i\varphi|^{p_i-2}D^i\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\,dx.\end{aligned}$$

According to Fatou's Lemma, we obtain
(3.78)

$$\begin{aligned}&\liminf_{n\to\infty}\sum_{i=1}^N\int_\Omega |D^iu_n|^{p_i-2}D^iu_nD^iT_k(u_n-\varphi)\,dx\\&\geq\sum_{i=1}^N\int_\Omega (|D^iT_M(u)|^{p_i-2}D^iT_M(u)-|D^i\varphi|^{p_i-2}D^i\varphi)(D^iT_M(u)-D^i\varphi)\chi_{\{|u_-\varphi|\leq k\}}\,dx\\&\qquad+\liminf_{n\to\infty}\sum_{i=1}^N\int_\Omega |D^i\varphi|^{p_i-2}D^i\varphi)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\&=\sum_{i=1}^N\int_\Omega |D^iT_M(u)|^{p_i-2}D^iT_M(u)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\&=\sum_{i=1}^N\int_\Omega a_i(x,u,\nabla u)D^iT_k(u-\varphi)\,dx.\end{aligned}$$

On the other hand, we have $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weak$-*$ in $L^\infty(\Omega)$ and thanks to (3.74) – (3.76), we deduce that

$$\int_\Omega h_n(x,u_n,\nabla u_n)T_k(u-\varphi)\,dx\longrightarrow\int_\Omega h(x,u,\nabla u)T_k(u-\varphi)\,dx, \tag{3.79}$$

$$\int_\Omega |T_n(u_n)|^{s-1}T_n(u_n)T_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega |u|^{s-1}uT_k(u-\varphi)\,dx, \tag{3.80}$$

$$\int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}T_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega \frac{|u|^{p_0-2}u}{|x|^{p_0}}T_k(u-\varphi)\,dx, \tag{3.81}$$

and

$$\int_\Omega f_n T_k(u_n-\varphi)dx \longrightarrow \int_\Omega f T_k(u-\varphi)dx. \tag{3.82}$$

Moreover, since $T_k(u_n-\varphi) \rightharpoonup T_k(u-\varphi)$ in $W_0^{1,\vec{p}}(\Omega)$ and $\phi_{i,n}(u_n)=\phi_i(T_M(u_n))$ in $\{|u_n-\varphi|\le k\}$ for $n\ge M$, we obtain

$$\int_\Omega \phi_{i,n}(u_n) D^i T_k(u-\varphi)\,dx \longrightarrow \int_\Omega \phi_i(u) D^i T_k(u-\varphi)\,dx \tag{3.83}$$

and

$$\int_\Omega |u_n|^{p_0-1} u_n T_k(u_n-\varphi)\,dx \longrightarrow \int_\Omega |u|^{p_0-1} u T_k(u-\varphi)\,dx. \tag{3.84}$$

Putting all the terms together, the proof of Theorem 3.2 is now complete.

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