Journal of Universal Mathematics
Vol.1 No.2 pp.88-103 (2018)
ISSN-2618-5660

TIME-DEPENDENT NEUTRAL STOCHASTIC DELAY PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY ROSENBLATT PROCESS IN HILBERT SPACE

E. LAKHEL AND A. TLIDI

Abstract. In this paper, we investigate a class of time-dependent neutral stochastic functional differential equations with finite delay driven by Rosenblatt process in a real separable Hilbert space. We prove the existence of unique mild solution by the well-known Banach fixed point principle. At the end we provide a practical example in order to illustrate the viability of our result.

1. Introduction

The theory of the stochastic evolution equations have attracted great interest due to its many real applications in several areas such as biology, medicine, physics, finance, electrical engineering, telecommunication networks. For further details, the reader may refer to the works of [5]. As many phenomena exhibit a memory effect or aftereffect, there has been a real need for developing stochastic evolution systems with delay which incorporate the effect of delay on state equations. The neutral functional differential equations are often used to fulfill this aim, specially in natural phenomena such as extreme weather and natural disasters which often display long-term memory as well as in many stochastic dynamical systems which depend not only on present and past, but also contain the derivatives with delays.

Recently, there has been a growing interest on the stochastic functional differential equations driven by fractional Brownian motion (here after, fBm). The reader is referred to the works of [3, 4, 6], among others. The literature concerning the existence and qualitative properties of solutions of time-dependent functional stochastic differential equations is very restricted.

The fBm has several properties such as self-similarity, stationarity of increments and long-range dependence. Due to these nice properties, the fBm is of interest in real application and it is generally prefered among other processes because it is Gaussian and the calculus is easier. However, in some situations specially when the gaussianity property is not satisfied, the Rosenblatt process is often used instead. Although introduced during the 60s and 70s [13, 15] in the literature, the Rosenblatt processes has been developed only during the last decade due to their

Date: July 8, 2018, accepted.
2010 *Mathematics Subject Classification.* 60H15, 60G15, 60H20.
Key words and phrases. Neutral stochastic evolution equations, Evolution operator, Rosenblatt process, Wiener integral, Banach fixed-point theorem.

appearance in the Non-Central Limit Theorem and to its desirable properties cited above i.e self-similarity, stationarity of increments and long-range dependence. The Rosenblatt processes can also be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian. In the literature, there exists a numerous studies that focuses on different theoretical aspects of the Rosenblatt processes. Leonenko and Ahn [8] studied the rate of convergence to the Rosenblatt process in the Non Central Limit Theorem. Tudor [16] analysed the Rosenblatt process. Maejima & Tudor [9] gave the distribution of the Rosenblatt process. Lakhel [7] established the existence of the unique solution for a class of neutral stochastic differential equation with delay and Poisson jumps driven by Rosenblatt process in Hilbert space.

To the best of our knowledge, there are no studies on time-dependent neutral stochastic functional differential equations with delays driven by Rosenblatt process. The aim of this paper is to fill this gap by providing the existence and uniqueness of mild solutions for a class of time-dependent neutral functional stochastic differential equations driven by non-Gaussian noises. This class is described as follow:

(1.1)
$$\begin{cases} d[x(t)+g(t,x(t-\tau))] = [A(t)x(t)+f(t,x(t-\tau))]dt+\sigma(t)dZ_H(t),\ 0\leq t\leq T, \\ x(t)=\varphi(t),\ -\tau\leq t\leq 0, \end{cases}$$

In a real Hilbert space X with inner product $< .,. >$ and norm $\|.\|$, where $\{A(t),\ t\in[0,T]\}$ is a family of linear closed operators from a space X into X that generates an evolution system of operators $\{U(t,s),\ 0\leq s\leq t\leq T\}$. Z_H is a Rosenblatt process on a real and separable Hilbert space Y, and $f,g:[0,+\infty)\times X\to X$, $\sigma:[0,+\infty)\to\mathcal{L}_2^0(Y,X)$, are appropriate functions. Here $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X.

The rest of the paper is structured as follows: Section 2 is devoted to basic notations and concepts and results about Rosenblatt process as well as Wiener integral with respect to Hilbert space and recall some results about evolution operator. New technical lemma for the $\mathbb{L}^2-$estimate of stochastic convolution integral is proved. Section 3 gives sufficient conditions for the existence and uniqueness of the problem (1.1). Section 4 gives an example to illustrate the efficiency of the obtained result. Section 5 concludes.

2. Preliminaries

In this section we recall some basic results about evolution family, and we introduce the Rosenblatt process as well as the Wiener integral with respect to it. We also establish some important results which will be needed throughout the paper. At first, we introduce the notion of evolution family.

2.1. **Evolution families.**

Definition 2.1. A set $\{U(t,s):0\leq s\leq t\leq T\}$ of bounded linear operators on a Hilbert space X is called an *evolution family* if

(a) $U(t,s)U(s,r)=U(t,r)$, $U(s,s)=I$ if $r\leq s\leq t$,
(b) $(t,s)\to U(t,s)x$ is strongly continuous for $t>s$.

Let $\{A(t),\ t\in[0,T]\}$ be a family of closed densely defined linear unbounded operators on the Hilbert space X under a domain $D(A(t))$ which is independent of t and satisfies the following conditions introduced by [1].

There exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, L, $K \geq 0$, and μ, $\nu \in (0,1]$ with $\mu + \nu > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|} \tag{2.1}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)\big[R(\lambda_0, A(t)) - R(\lambda_0, A(s))\big]\| \leq L|t-s|^\mu |\lambda|^{-\nu}, \tag{2.2}$$

for t, $s \in \mathbb{R}$, $\lambda \in \Sigma_\theta$ where $\Sigma_\theta := \big\{\lambda \in \mathbb{C} - \{0\} : |\arg \lambda| \leq \theta\big\}$.

It is well known, that this assumption implies that there exists a unique evolution family $\{U(t,s) : 0 \leq s \leq t \leq T\}$ on X such that $(t,s) \to U(t,s) \in \mathcal{L}(X)$ is continuous for $t > s$, $U(\cdot, s) \in \mathcal{C}^1((s,\infty), \mathcal{L}(X))$, $\partial_t U(t,s) = A(t)U(t,s)$, and

$$\|A(t)^k U(t,s)\| \leq C(t-s)^{-k} \tag{2.3}$$

for $0 < t - s \leq 1$, $k = 0, 1$, $0 \leq \alpha < \mu$, $x \in D((\lambda_0 - A(s))^\alpha)$, and a constant C depending only on the constants in (2.1)-(2.2). Moreover, $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$. We say that $A(\cdot)$ generates $\{U(t,s) : 0 \leq s \leq t \leq T\}$. Note that $U(t,s)$ is exponentially bounded by (2.3) with $k = 0$.

Remark 2.2. If $\{A(t),\, t \in [0,T]\}$ is a second order differential operator A, that is $A(t) = A$ for each $t \in [0,T]$, then A generates a C_0-semigroup $\{e^{At}, t \in [0,T]\}$.

For further details on evolution system and their properties, the reader may refer to [11].

2.2. **Rosenblatt process.** In this section, we collect some definitions and lemmas on Wiener integrals with respect to an infinite dimensional Rosenblatt process and we recall some basic results about analytical semi-groups and fractional powers of their infinitesimal generators, which will be used throughout the whole of this paper.
For details of this section, we refer the reader to [16, 11] and references therein.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Selfsimilar processes are invariant in distribution under suitable scaling. They are of considerable interest in practice since aspects of the selfsimilarity appear in different phenomena like telecommunications, turbulence, hydrology or economics. A self-similar processes can be defined as limits that appear in the so-called Non-Central Limit Theorem (see [15]). We briefly recall the Rosenblatt process as well as the Wiener integral with respect to it.
Let us recall the notion of Hermite rank. Denote by $H_j(x)$ the Hermite polynomial of degree j given by $H_j = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{\frac{-x^2}{2}}$ and let g be a function on $\mathbb{R}$ such that $\mathbb{E}[g(\zeta_0)] = 0$ and $\mathbb{E}[g(\zeta_0)^2] < \infty$. Assume that g has the following expansion in Hermite polynomials

$$g(x) = \sum_{j \geq 0} c_j H_j(x),$$

where $c_j = \frac{1}{j!}\mathbb{E}(g(\zeta_0 H_j(\zeta_0)))$. The Hermite rank of g is defined by

$$k = min\{j | c_j \neq 0\}.$$

Since $\mathbb{E}[g(\zeta_0)] = 0$, we have $k \geq 1$. Consider $(\zeta_n)_{n\in\mathbb{Z}}$ a stationary Gaussian sequence with mean zero and variance 1 which exhibits long range dependence in the sense that the correlation function satisfies

$$r(n) = \mathbb{E}(\zeta_0\zeta_n) = n^{\frac{2H-2}{k}}L(n),$$

with $H \in (\frac{1}{2}, 1)$ and L is a slowly varying function at infinity. Then the following family of stochastic processes

$$\frac{1}{n^H}\sum_{j=1}^{[nt]} g(\zeta_j)$$

converges as $n \longrightarrow \infty$, in the sense of finite dimensional distributions, to the self-similar stochastic process with stationary increments

$$Z_H^k(t) = c(H,k)\int_{\mathbb{R}^k}\left(\int_0^t \prod_{j=1}^{k}(s-y_j)_+^{-(\frac{1}{2}+\frac{1-H}{k})}ds\right)dB(y_1)...dB(y_k), \tag{2.4}$$

where $x_+ = max(x, 0)$. The above integral is a Wiener-Itô multiple integral of order k with respect to the standard Brownian motion $(B(y))_{y\in\mathbb{R}}$ and the constant $c(H,k)$ is a normalizing constant that ensures $\mathbb{E}(Z_H^k(1))^2 = 1$.

The process $(Z_H^k(t))_{t\geq 0}$ is called the Hermite process. When $k = 1$ the process given by (2.4) is nothing else that the fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$. For $k = 2$ the process is not Gaussian. If $k = 2$ then the process (2.4) is known as the Rosenblatt process. It was introduced by Rosenblatt in [13] and was given its name by Taqqu in [14]. The fractional Brownian motion is of course the most studied process in the class of Hermite processes due to its significant importance in modelling. A stochastic calculus with respect to it has been intensively developed in the last decade. The Rosenblatt process is, after fBm, the most well known Hermite process.

We also recall the following properties of the Rorenblatt process:

- The process Z_H^k is H-selfsimilar in the sense that for any $c > 0$,

$$(Z_H^k(ct)) =^{(d)} (c^H Z_H^k(t)), \tag{2.5}$$

where " $=^{(d)}$ " means equivalence of all finite dimensional distributions. It has stationary increments and all moments are finite.

- From the stationarity of increments and the self-similarity, it follows that, for any $p \geq 1$

$$\mathbb{E}|Z_H(t) - Z_H(s)|^p \leq |\mathbb{E}(Z_H(1))|^p|t-s|^{pH}.$$

As a consequence the Rosenblatt process has Hölder continuous paths of order γ with $0 < \gamma < H$.

Self-similarity and long-range dependence make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields. Consider a time interval $[0,T]$ with arbitrary fixed horizon T and let $\{Z_H(t), t \in [0,T]\}$ the one-dimensional Rosenblatt process with parameter $H \in (1/2, 1)$. By

Tudor [16], it is well known that Z_H has the following integral representation:

$$Z_H(t) = d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right] dB(y_1) dB(y_2), \tag{2.6}$$

where $B = \{B(t) : t \in [0,T]\}$ is a Wiener process, $H' = \frac{H+1}{2}$ and $K^H(t,s)$ is the kernel given by

$$K^H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(,)$ denotes the Beta function. We put $K^H(t,s) = 0$ if $t \leq s$ and $d(H) = \frac{1}{H+1}\sqrt{\frac{H}{2(2H-1)}}$ is a normalizing constant.

The covariance of the Rosenblatt process $\{Z_H(t), t \in [0,T]\}$ satisfies, for every $s, t \geq 0$,

$$R_H(s,t) := \mathbb{E}(Z_H(t) Z_H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

The basic observation is the fact that the covariance structure of the Rosenblatt process is similar to the one of the fractional Brownian motion and this allows the use of the same classes of deterministic integrands as in the fractional Brownian motion case whose properties are known.

Now, we introduce Wiener integrals with respect to the Rosenblatt process. We refer to [16] for additional details on the Rosenblatt process .
By formula (2.6) we can write

$$Z_H(t) = \int_0^t \int_0^t I(\mathbf{1}_{[0,t]})(y_1, y_2) dB(y_1) dB(y_2),$$

where by I we denote the mapping on the set of functions $f : [0,T] \longrightarrow \mathbb{R}$ to the set of functions $f : [0,T]^2 \longrightarrow \mathbb{R}$

$$I(f)(y_1, y_2) = d(H) \int_{y_1 \vee y_2}^T f(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du.$$

Let us denote by $\mathcal{E}$ the class of elementary functions on R of the form

$$f(.) = \sum_{j=1}^n a_j \mathbf{1}_{(t_j, t_{j+1}]}(.), \qquad 0 \leq t_j < t_{j+1} \leq T, \quad a_j \in \mathbb{R}, \quad i = 1, ..., n.$$

For $f \in \mathcal{E}$ as above, it is natural to define its Wiener integral with respect to the Rosenblatt process Z_H by
(2.7)

$$\int_0^T f(s) dZ_H(s) := \sum_{j=1}^n a_j \left[Z_H(t_{j+1}) - Z_H(t_j) \right] = \int_0^T \int_0^T I(f)(y_1, y_2) dB(y_1) dB(y_2).$$

Let $\mathcal{H}$ be the set of functions f such that

$$\mathcal{H} = \left\{ f : [0,T] \longrightarrow \mathbb{R} : \quad \|f\|_{\mathcal{H}} := \int_0^T \int_0^T \left(I(f)(y_1, y_2) \right)^2 dy_1 dy_2 < \infty \right\}.$$

It hold that (see Maejima and Tudor [10])

$$\|f\|_{\mathcal{H}} = H(2H-1)\int_0^T\int_0^T f(u)f(v)|u-v|^{2H-2}dudv,$$

and, the mapping

$$f \longrightarrow \int_0^T f(u)dZ_H(u) \tag{2.8}$$

provides an isometry from $\mathcal{E}$ to $L^2(\Omega)$. On the other hand, it has been proved in [12] that the set of elementary functions $\mathcal{E}$ is dense in $\mathcal{H}$. As a consequence the mapping (2.8) can be extended to an isometry from $\mathcal{H}$ to $L^2(\Omega)$. We call this extension as the Wiener integral of $f \in \mathcal{H}$ with respect to Z_H.

Let us consider the operator K_H^* from $\mathcal{E}$ to $\mathbb{L}^2([0,T])$ defined by

$$(K_H^*\varphi)(y_1,y_2) = \int_{y_1\vee y_2}^T \varphi(r)\frac{\partial K}{\partial r}(r,y_1,y_2)dr,$$

where $K(.,.,.)$ is the kernel of Rosenblatt process in representation (2.6)

$$K(r,y_1,y_2) = \mathbf{1}_{[0,t]}(y_1)\mathbf{1}_{[0,t]}(y_2)\int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du.$$

We refer to [16] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0,T])$. It follows from [16] that $\mathcal{H}$ contains not only functions but its elements could be also distributions. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|^2_{|\mathcal{H}|} := \alpha_H\int_0^T\int_0^T |\psi(s)||\psi(t)||s-t|^{2H-2}dsdt < \infty,$$

where $\alpha_H = H(2H-1)$. The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$ and we have the following inclusions (see [16]).

Lemma 2.3.

$$\mathbb{L}^2([0,T]) \subseteq \mathbb{L}^{1/H}([0,T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H},$$

and for any $\psi \in \mathbb{L}^2([0,T])$, *we have*

$$\|\psi\|^2_{|\mathcal{H}|} \leq 2HT^{2H-1}\int_0^T |\psi(s)|^2ds.$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathcal{L}(Y,X)$. Let $Q \in \mathcal{L}(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty}\lambda_n < \infty$. where $\lambda_n \geq 0$ $(n = 1,2...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1,2...)$ is a complete orthonormal basis in Y. We define the infinite dimensional $Q-$Rosenblatt process on Y as

$$Z_H(t) = Z_Q(t) = \sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n z_n(t), \tag{2.9}$$

where $(z_n)_{n\geq 0}$ is a family of real independent Rosenblatt process.

Note that the series (2.9) is convergent in $L^2(\Omega)$ for every $t \in [0,T]$, since

$$\mathbb{E}|Z_Q(t)|^2 = \sum_{n=1}^{\infty} \lambda_n \mathbb{E}(z_n(t))^2 = t^{2H} \sum_{n=1}^{\infty} \lambda_n < \infty.$$

Note also that Z_Q has covariance function in the sense that

$$E\langle Z_Q(t), x\rangle \langle Z_Q(s), y\rangle = R(s,t)\langle Q(x), y\rangle \text{ for all } x, y \in Y \text{ and } t, s \in [0,T].$$

In order to define Wiener integrals with respect to the Q-Rosenblatt process, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y,X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi\rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.

Now, let $\phi(s);\ s \in [0,T]$ be a function with values in $\mathcal{L}_2^0(Y,X)$, such that $\sum_{n=1}^{\infty} \|K^*\phi Q^{\frac{1}{2}} e_n\|^2_{\mathcal{L}_2^0} < \infty$. The Wiener integral of ϕ with respect to Z_Q is defined by

(2.10)

$$\int_0^t \phi(s) dZ_Q(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s) e_n dz_n(s) = \sum_{n=1}^{\infty} \int_0^t \int_0^t \sqrt{\lambda_n} K_H^*(\phi e_n)(y_1, y_2) dB(y_1) dB(y_2).$$

Now, we end this subsection by stating the following result which is fundamental to prove our result.

Lemma 2.4. *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$ then the above sum in* (2.10) *is well defined as a X-valued random variable and we have*

$$\mathbb{E}\| \int_0^t \psi(s) dZ_H(s)\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

Proof. By Lemma 2.3, we have

$$\begin{aligned}
\mathbb{E}\| \int_0^t \psi(s) dZ_H(s)\|^2 &= \sum_{n=1}^{\infty} \mathbb{E}\| \int_0^t \int_0^t \sqrt{\lambda_n} K_H^*(\psi e_n)(y_1, y_2) dB_n(y_1) dB_n(y_2)\|^2 \\
&\leq \sum_{n=1}^{\infty} 2Ht^{2H-1} \int_0^t \lambda_n \|\psi(s) e_n\|^2 ds \\
&= 2Ht^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.
\end{aligned}$$

□

2.3. **Definition and assumption.** Henceforth we will assume that the family $\{A(t),\, t \in [0,T]\}$ of linear operators generates an evolution system of operators $\{U(t,s),\, 0 \leq s \leq t \leq T\}$.

Definition 2.5. An X-valued stochastic process $\{x(t),\ t \in [-\tau, T]\}$, is called a mild solution of equation (1.1) if

i) $x(.) \in \mathcal{C}([-\tau,T], \mathbb{L}^2(\Omega, X))$,

ii) $x(t) = \varphi(t)$, $-\tau \leq t \leq 0$.

iii) For arbitrary $t \in [0,T]$, $x(t)$ satisfies the following integral equation:

$$\begin{aligned} x(t) &= U(t,0)(\varphi(0)+g(0,\varphi(-\tau))) - g(t,x(t-\tau)) \\ &- \int_0^t U(t,s)A(s)g(s,x(s-\tau))ds + \int_0^t U(t,s)f(s,x(s-\tau))ds \\ &+ \int_0^t U(t,s)\sigma(s)dZ_Q(s) \quad \mathbb{P}-a.s \end{aligned}$$

We introduce the following assumptions:

($\mathcal{H}.1$) *i)* The evolution family is exponentially stable, that is, there exist two constants $\beta > 0$ and $M \geq 1$ such that

$$\|U(t,s)\| \leq Me^{-\beta(t-s)}, \qquad for\,all \quad t \geq s,$$

ii) There exist a constant $M_* > 0$ such that

$$\|A^{-1}(t)\| \leq M_* \qquad for\ all \quad t \in [0,T].$$

($\mathcal{H}.2$) The maps $f, g : [0,T] \times X \to X$ are continuous functions and there exist two positive constants C_1 and C_2, such that for all $t \in [0,T]$ and $x, y \in X$:

i) $\|f(t,x)-f(t,y)\| \vee \|g(t,x)-g(t,y)\| \leq C_1\|x-y\|$.

ii) $\|f(t,x)\|^2 \vee \|A^k(t)g(t,x)\|^2 \leq C_2(1+\|x\|^2), \quad k = 0,1$.

($\mathcal{H}.3$) *i)* There exists a constant $0 < L_* < \frac{1}{M_*}$ such that

$$\|A(t)g(t,x) - A(t)g(t,y)\| \leq L_*\|x-y\|,$$

for all $t \in [0,T]$ and $x, y \in X$.

ii) The function g is continuous in the quadratic mean sense: for all $x(.) \in \mathcal{C}([0,T], L^2(\Omega,X))$, we have

$$\lim_{t \longrightarrow s} \mathbb{E}\|g(t,x(t)) - g(s,x(s))\|^2 = 0.$$

($\mathcal{H}.4$) *i)* The map $\sigma : [0,T] \longrightarrow \mathcal{L}_2^0(Y,X)$ is bounded, that is : there exists a positive constant L such that $\|\sigma(t)\|_{\mathcal{L}_2^0(Y,X)} \leq L$ uniformly in $t \in [0,T]$.

ii) Moreover, we assume that the initial data $\varphi = \{\varphi(t) : -\tau \leq t \leq 0\}$ satisfies $\varphi \in \mathcal{C}([-\tau,0], \mathbb{L}^2(\Omega,X))$.

3. Existence and Uniqueness of Mild Solutions

In this section we study the existence and uniqueness of mild solutions of equation (1.1). First, it is of great importance to establish the basic properties of the stochastic convolution integral of the form

$$X(t) = \int_0^t U(t,s)\sigma(s)dZ_Q(s), \qquad t \in [0,T],$$

where $\sigma(s) \in \mathcal{L}_2^0(Y,X)$ and $\{U(t,s),\, 0 \leq s \leq t \leq T\}$ is an evolution system of operators.

The properties of the process X are crucial when regularity of the mild solution to stochastic evolution equation is studied, see [5] for asystematic account of the theory of mild solutions to infinite-dimensional stochastic equations. Unfortunately, the process X is not a martingale, and standard tools of the martingale theory, yielding e.g. continuity of the trajectories or $\mathbb{L}^2-$estimates are not available. The following result on the stochastic convolution integral X holds.

Lemma 3.1. *Suppose that $\sigma : [0,T] \rightarrow \mathcal{L}_2^0(Y,X)$ satisfies $\sup_{t\in[0,T]} \|\sigma(t)\|^2_{\mathcal{L}_2^0} < \infty$, and suppose that $\{U(t,s),\ 0 \le s \le t \le T\}$ is an evolution system of operators satisfying $\|U(t,s)\| \le Me^{-\beta(t-s)}$, for some constants $\beta > 0$ and $M \ge 1$ for all $t \ge s$. Then, we have*

1. *The stochastic integral $X: \quad t \longrightarrow \int_0^t U(t,s)\sigma(s)dZ_Q(s)$ is well-defined and we have*
$$\mathbb{E}\|\int_0^t U(t,s)\sigma(s)dZ_Q(s)\|^2 \le C_H M^2 t^{2H} (\sup_{t\in[0,T]} \|\sigma(t)\|_{\mathcal{L}_2^0})^2.$$
2. *The stochastic integral $X: \quad t \longrightarrow \int_0^t U(t,s)\sigma(s)dZ_Q(s)$ is continuous.*

Proof. 1. Let $\{e_n\}_{n\in\mathbb{N}}$ be the complete orthonormal basis of Y and $\{z_n\}_{n\in\mathbb{N}}$ is a sequence of independent, real-valued Rosenblatt process each with the same parameter $H \in (\frac{1}{2}, 1)$. Thus, using isometry property one can write

$$\begin{aligned}
\mathbb{E}\|\int_0^t U(t,s)\sigma(s)dZ_Q(s)\|^2 &= \sum_{n=1}^{\infty} \mathbb{E}\|\int_0^t U(t,s)\sigma(s)e_n dz_n(s)\|^2 \\
&= H(2H-1)\int_0^t \{\|U(t,s)\sigma(s)\| \\
&\times \int_0^t \|U(t,r)\sigma(r)\||s-r|^{2H-2}dr\}ds \\
&\le H(2H-1)M^2 \int_0^t \{e^{-\beta(t-s)}\|\sigma(s)\|_{\mathcal{L}_2^0} \\
&\times \int_0^t e^{-\beta(t-r)}|s-r|^{2H-2}\|\sigma(r)\|_{\mathcal{L}_2^0} dr\}ds.
\end{aligned}$$

Since σ is bounded, one can then conclude that

$$\begin{aligned}
\mathbb{E}\|\int_0^t U(t,s)\sigma(s)dZ_H(s)\|^2 &\le H(2H-1)M^2(\sup_{t\in[0,T]} \|\sigma(t)\|_{\mathcal{L}_2^0})^2 \int_0^t \{e^{-\beta(t-s)} \\
&\times \int_0^t e^{-\beta(t-r)}|s-r|^{2H-2}dr\}ds.
\end{aligned}$$

Make the following change of variables, $v = t-s$ for the first integral and $u = t-r$ for the second. One can write

$$\begin{aligned}
\mathbb{E}\|\int_0^t U(t,s)\sigma(s)dZ_H(s)\|^2 &\le H(2H-1)M^2(\sup_{t\in[0,T]} \|\sigma(t)\|_{\mathcal{L}_2^0})^2 \int_0^t \{e^{-\beta v} \\
&\times \int_0^t e^{-\beta u}|u-v|^{2H-2}du\}dv \\
&\le H(2H-1)M^2(\sup_{t\in[0,T]} \|\sigma(t)\|_{\mathcal{L}_2^0})^2 \int_0^t\int_0^t |u-v|^{2H-2}dudv.
\end{aligned}$$

By using the equality,

$$R_H(t,s) = H(2H-2)\int_0^t\int_0^s |u-v|^{2H-2}dudv,$$

we get that

$$\mathbb{E}\|\int_0^t U(t,s)\sigma(s)dZ_Q(s)\|^2 \leq C_H M^2 t^{2H}(\sup_{t\in[0,T]}\|\sigma(t)\|_{\mathcal{L}_2^0})^2.$$

2. Let $h>0$ small enough, we have

$$\begin{aligned}\mathbb{E}\|\int_0^{t+h}U(t+h,s)\sigma(s)dZ_Q(s)-\int_0^t U(t,s)\sigma(s)dZ_Q(s)\|^2 &\leq 2\|\int_0^t(U(t+h,s)-U(t,s))\sigma(s)dZ_Q(s)\|^2\\ &+ 2\|\int_t^{t+h}U(t+h,s)\sigma(s)dZ_H(s)\|^2\\ &\leq 2[\mathbb{E}\|I_1(h)\|^2+\mathbb{E}\|I_2(h)\|^2].\end{aligned}$$

By Lemma 2.4, we get that

$$E\|I_1(h)\|^2 \quad \leq \quad 2Ht^{2H-1}\int_0^t\|[U(t+h,s)-U(t,s)]\sigma(s)\|^2_{\mathcal{L}_2^0}ds.$$

Since

$$\lim_{h\to 0}\|[U(t+h,s)-U(t,s)]\sigma(s)\|^2_{\mathcal{L}_2^0}=0,$$

and

$$\|(U(t+h,s)-U(t,s))\sigma(s)\|_{\mathcal{L}_2^0}\leq MLe^{-\beta(t-s)}e^{-\beta h+1}\in\mathbb{L}^1([0,T],\,ds),$$

we conclude, by the dominated convergence theorem that,

$$\lim_{h\to 0}\mathbb{E}\|I_1(h)\|^2=0.$$

Again by Lemma 2.4, we get that

$$\mathbb{E}\|I_2(h)\|^2\leq\frac{2Ht^{2H-1}LM^2(1-e^{-2\beta h})}{2\beta}.$$

Thus,

$$\lim_{h\to 0}\mathbb{E}\|I_2(h)\|^2=0.$$

□

Remark 3.2. Thanks to Lemma 3.1, the stochastic integral $X(t)$ is well-defined and it belongs to the space $\mathcal{C}([-\tau,0],\mathbb{L}^2(\Omega,X))$.

We have the following theorem on the existence and uniqueness of mild solutions of equation (1.1).

Theorem 3.3. *Suppose that* $(\mathcal{H}.1)$-$(\mathcal{H}.4)$ *hold. Then, for all* $T>0$*, the equation (1.1) has a unique mild solution on* $[-\tau,T]$.

Proof. Fix $T>0$ and let $B_T:=\mathcal{C}([-\tau,T],\mathbb{L}^2(\Omega,X))$ be the Banach space of all continuous functions from $[-\tau,T]$ into $\mathbb{L}^2(\Omega,X)$, equipped with the supremum norm

$$\|x\|^2_{B_T}=\sup_{-\tau\leq t\leq T}\mathbb{E}\|x(t,\omega)\|^2.$$

Let us consider the set

$$S_T(\varphi)=\{x\in B_T: x(s)=\varphi(s),\ \text{for}\ \ s\in[-\tau,0]\}.$$

$S_T(\varphi)$ is a closed subset of B_T provided with the norm $\|.\|_{B_T}$.
We transform (1.1) into a fixed-point problem. Consider the operator ψ on $S_T(\varphi)$ defined by $\psi(x)(t) = \varphi(t)$ for $t \in [-\tau, 0]$ and for $t \in [0, T]$

$$\begin{aligned}
\psi(x)(t) &= U(t,0)(\varphi(0)+g(0,\varphi(-\tau))) - g(t,x(t-\tau)) \\
&- \int_0^t U(t,s)A(s)g(s,x(s-\tau))ds + \int_0^t U(t,s)f(s,x(s-\tau))ds \\
&+ \int_0^t U(t,s)\sigma(s)dZ_Q(s) \\
&= \sum_{i=1}^{5} I_i(t).
\end{aligned}$$

Clearly, the fixed points of the operator ψ are mild solutions of (1.1). The fact that ψ has a fixed point will be proved in several steps. We will first prove that the function ψ is well defined.
Step 1: For arbitrary $x \in S_T(\varphi)$, we are going to show that each function $t \to I_i(t)$ is continuous on $[0, T]$ in the $\mathbb{L}^2(\Omega, X)$-sense.

For the first term $I_1(h)$, by Definition 2.1, we obtain

$$\lim_{h\longrightarrow 0}(U(t+h,0) - U(t,0))(\varphi(0)+g(0,\varphi(-\tau))) = 0.$$

From ($\mathcal{H}$.1), we have

$$\|(U(t+h,0)-U(t,0))(\varphi(0)+g(0,\varphi(-\tau)))\| \leq Me^{-\beta t}(e^{-\beta h}+1)\|\varphi(0)+g(0,\varphi(-\tau))\| \in L^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_1(t+h) - I_1(t)\|^2 = 0.$$

For the second term $I_2(h)$, assumption ($\mathcal{H}$.2) ensures that

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_2(t+h) - I_2(t)\|^2 = 0.$$

To show that the third term $I_3(h)$ is continuous, we suppose $h > 0$ (similar calculus for $h < 0$). We have

$$\begin{aligned}
\|I_3(t+h) - I_3(t)\| &\leq \left\|\int_0^t (U(t+h,s) - U(t,s))A(s)g(s,x(s-\tau))ds\right\| \\
&\quad + \left\|\int_t^{t+h} U(t,s)g(s,x(s-\tau))ds\right\| \\
&\leq I_{31}(h) + I_{32}(h).
\end{aligned}$$

By Hölder's inequality, we have

$$\mathbb{E}\|I_{31}(h)\| \leq t\mathbb{E}\int_0^t \|(U(t+h,s) - U(t+h,s))A(s)g(s,x(s-\tau)\|^2 ds.$$

By Definition 2.1, we obtain

$$\lim_{h\longrightarrow 0}(U(t+h,s) - U(t,s))A(s)g(s,x(s-\tau)) = 0.$$

From ($\mathcal{H}$.1) and ($\mathcal{H}$.2), we have

$$\|(U(t+h,s)-U(t,s))A(s)g(s,x(s-\tau))\| \leq C_2Me^{-\beta(t-s)}(e^{-\beta h}+1)\|A(s)g(s,x(s-\tau))\| \in L^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_{31}(h)\|^2=0.$$

So, estimating as before. By using $(\mathcal{H}.1)$ and $(\mathcal{H}.2)$, we get

$$\mathbb{E}\|I_{32}(h)\|^2\le\frac{M^2C_2(1-e^{-2\beta h})}{2\beta}\int_t^{t+h}(1+\mathbb{E}\|x(s-\tau)\|^2)ds.$$

Thus,

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_{32}(h)\|^2=0.$$

For the fourth term $I_4(h)$, we suppose $h>0$ (similar calculus for $h<0$). We have

$$\begin{aligned}\|I_4(t+h)-I_4(t)\| &\le \left\|\int_0^t(U(t+h,s)-U(t,s))f(s,x(s-\tau))ds\right\|\\ &\quad+\left\|\int_t^{t+h}U(t,s)f(s,x(s-\tau))ds\right\|\\ &\le \|I_{41}(h)\|+\|I_{42}(h)\|.\end{aligned}$$

By Hölder's inequality, we have

$$\mathbb{E}\|I_{41}(h)\|\le t\mathbb{E}\int_0^t\|(U(t+h,s)-U(t,s))f(s,x(s-\tau))\|^2ds.$$

Again exploiting properties of Definition 2.1, we obtain

$$\lim_{h\longrightarrow 0}(U(t+h,s)-U(t,s))f(s,x(s-\tau))=0,$$

and

$$\|(U(t+h,s)-U(t,s))f(s,x(s-\tau))\|\le Me^{-\beta(t-s)}(e^{-\beta h}+1)\|f(s,x(s-\tau))\|\in L^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_{41}(h)\|^2=0.$$

On the other hand, by $(\mathcal{H}.1)$, $(\mathcal{H}.2)$, and the Hölder's inequality, we have

$$\mathbb{E}\|I_{42}(h)\|\le\frac{M^2C_2(1-e^{-2\beta h})}{2\beta}\int_t^{t+h}(1+\mathbb{E}\|x(s-\tau)\|^2)ds.$$

Thus

$$\lim_{h\to 0}\mathbb{E}\|I_{42}(h)\|^2=0.$$

Now, for the term $I_5(h)$, we have

$$\begin{aligned}\mathbb{E}\|I_5(t+h)-I_5(t)\|^2 &\le 2\mathbb{E}\|\int_0^t(U(t+h,s)-U(t,s))\sigma(s)dZ_Q(s)\|^2\\ &+ 2\mathbb{E}\|\int_t^{t+h}U(t+h,s)\sigma(s)dZ_Q(s)\|^2.\end{aligned}$$

By Lemma 3.1 we get

$$\lim_{h\to 0}\|I_5(t+h)-I_5(t)\|^2=0.$$

The above arguments show that $\lim\limits_{h\to 0}\mathbb{E}\|\psi(x)(t+h)-\psi(x)(t)\|^2=0$. Hence, we conclude that the function $t\to\psi(x)(t)$ is continuous on $[0,T]$ in the $\mathbb{L}^2$-sense.

Step 2: Now, we are going to show that ψ is a contraction mapping in $S_{T_1}(\varphi)$ with some $T_1 \leq T$ to be specified later. Let $x, y \in S_T(\varphi)$, by using the inequality

$$(a+b+c)^2 \leq \frac{1}{\nu}a^2 + \frac{2}{1-\nu}b^2 + \frac{2}{1-\nu}c^2,$$

where $\nu := L_* M_* < 1$, we obtain for any fixed $t \in [0, T]$

$$\begin{aligned}
\|\psi(x)(t) \quad &- \quad \psi(y)(t)\|^2 \\
&\leq \frac{1}{\nu}\|g(t, x(t-\tau)) - g(t, y(t-\tau))\|^2 \\
&\quad + \frac{2}{1-\nu}\|\int_0^t U(t,s)A(s)(g(s, x(s-\tau)) - g(s, y(s-\tau)))ds\|^2 \\
&\quad + \frac{2}{1-\nu}\|\int_0^t U(t,s)(f(s, x(s-\tau)) - f(s, y(s-\tau)))ds\|^2 \\
&= \sum_{k=1}^{3} J_k(t).
\end{aligned}$$

By using the fact that the operator $\|(A^{-1}(t))\|$ is bounded, combined with the condition $(\mathcal{H}.3)$, we obtain that

$$\begin{aligned}
\mathbb{E}\|J_1(t)\| &\leq \frac{1}{\nu}\|A^{-1}(t)\|^2 \mathbb{E}|A(t)g(t, x(t-\tau)) - A(t)g(t, y(t-\tau))\|^2 \\
&\leq \frac{L_*^2 M_*^2}{\nu}\mathbb{E}\|x(t-\tau) - y(t-\tau)\|^2 \\
&\leq \nu \sup_{s\in[-\tau,t]} \mathbb{E}\|x(s) - y(s)\|^2.
\end{aligned}$$

By hypothesis $(\mathcal{H}.3)$ combined with Hölder's inequality, we get that

$$\begin{aligned}
\mathbb{E}\|J_2(t)\| &\leq \mathbb{E}\|\int_0^t U(t,s)\left[A(t)g(t, x(t-\tau)) - A(t)g(t, y(t-\tau))\right]ds\| \\
&\leq \frac{2}{1-\nu}\int_0^t M^2 e^{-2\beta(t-s)}ds \int_0^t \mathbb{E}\|x(s-\tau) - y(s-\tau)\|^2 ds \\
&\leq \frac{2M^2 L_*^2}{1-\nu}\frac{1-e^{-2\beta t}}{2\beta} t \sup_{s\in[-\tau,t]} \mathbb{E}\|x(s) - y(s)\|^2.
\end{aligned}$$

Moreover, by hypothesis $(\mathcal{H}.2)$ combined with Hölder's inequality, we can conclude that

$$\begin{aligned}
E\|J_3(t)\| &\leq E\|\int_0^t U(t,s)\left[f(s, x(s-\tau)) - f(s, y(s-\tau))\right]ds\|^2 \\
&\leq \frac{2C_1^2}{1-\nu}\int_0^t M^2 e^{-2\beta(t-s)}ds \int_0^t \mathbb{E}\|x(s-\tau) - y(s-\tau)\|^2 ds \\
&\leq \frac{2M^2 C_1^2}{1-\nu}\frac{1-e^{-2\beta t}}{2\beta} t \sup_{s\in[-\tau,t]} \mathbb{E}\|x(s) - y(s)\|^2.
\end{aligned}$$

Hence

$$\sup_{s\in[-\tau,t]} \mathbb{E}\|\psi(x)(s)-\psi(y)(s)\|^2 \leq \gamma(t) \sup_{s\in[-\tau,t]} \mathbb{E}\|x(s)-y(s)\|^2,$$

where

$$\gamma(t) = \nu + [L_*^2 + C_1^2]\frac{2M^2}{1-\nu}\frac{1-e^{-2\beta t}}{2\beta}t$$

By condition ($\mathcal{H}$.3), we have $\gamma(0) = \nu = L_* M_* < 1$. Then there exists $0 < T_1 \leq T$ such that $0 < \gamma(T_1) < 1$ and ψ is a contraction mapping on $S_{T_1}(\varphi)$ and therefore has a unique fixed point, which is a mild solution of equation (1.1) on $[-\tau, T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[-\tau, T]$ in finitely many steps. This completes the proof. $\square$

4. An Example

In recent years, the interest in neutral systems has been growing rapidly due to their successful applications in practical fields such as physics, chemical technology, bioengineering, and electrical networks. We consider the following stochastic partial neutral functional differential equation with finite delay τ $(0 \leq \tau < \infty,)$, driven by a Rosenblatt process

(4.1)
$$\begin{cases} d\left[u(t,\zeta) + G(t,u(t-\tau,\zeta))\right] & = \left[\frac{\partial^2}{\partial^2\zeta}u(t,\zeta) + b(t,\zeta)u(t,\zeta) + F(t,u(t-\tau,\zeta))\right]dt \\ & + \sigma(t)dZ_H(t),\ 0 \leq t \leq T,\ 0 \leq \zeta \leq \pi, \\ u(t,0) = u(t,\pi) = 0, \quad 0 \leq t \leq T & \\ u(t,\zeta) = \varphi(t,\zeta), \quad t \in [-\tau,0],\ 0 \leq \zeta \leq \pi, & \end{cases}$$

where Z_H is a Rosenblatt process, $b(t,\zeta)$ is a continuous function and is uniformly Hölder continuous in t, F, $G : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.
To study this system, we consider the space $X = L^2([0,\pi])$, $Y = \mathbb{R}$ and the operator $A : D(A) \subset X \longrightarrow X$ given by $Ay = y''$ with

$$D(A) = \{y \in X : y'' \in X, \quad y(0) = y(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on X. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$ and the corresponding normalized eigenfunctions given by

$$e_n := \sqrt{\frac{2}{\pi}}\sin nx,\ n = 1, 2, \ldots.$$

In addition $(e_n)_{n\in\mathbb{N}}$ is a complete orthonormal basis in X and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} < x, e_n > e_n,$$

for $x \in X$ and $t \geq 0$.

Now, we define an operator $A(t) : D(A) \subset X \longrightarrow X$ by

$$A(t)x(\zeta) = Ax(\zeta) + b(t,\zeta)x(\zeta).$$

By assuming that $b(.,.)$ is continuous and that $b(t,\zeta) \leq -\gamma$ $(\gamma > 0)$ for every $t \in \mathbb{R}$, $\zeta \in [0,\pi]$, it follows that the system

$$\begin{cases} u'(t) & = A(t)u(t), \quad t \geq s, \\ u(s) & = x \in X, \end{cases}$$

has an associated evolution family given by

$$U(t,s)x(\zeta) = \left[T(t-s)\exp^{\int_s^t b(\tau,\zeta)d\tau} x\right](\zeta).$$

From this expression, it follows that $U(t,s)$ is a compact linear operator and that for every $s,t \in [0,T]$ with $t > s$

$$\|U(t,s)\| \leq e^{-(\gamma+1)(t-s)}$$

In addition, $A(t)$ satisfies the assumption $\mathcal{H}_1$ (see [2]).
To rewrite the initial-boundary value problem (4.1) in the abstract form we assume the following:

i) The substitution operator $f : [0,T] \times X \longrightarrow X$ defined by $f(t,u)(.) = F(t,u(.))$ is continuous and we impose suitable conditions on F to verify assumption $\mathcal{H}_2$.

ii) The substitution operator $g : [0,T] \times X \longrightarrow X$ defined by $g(t,u)(.) = G(t,u(.))$ is continuous and we impose suitable conditions on G to verify assumptions $\mathcal{H}_2$ and $\mathcal{H}_3$.

iii) The function $\sigma : [0,T] \longrightarrow \mathcal{L}_2^0(L^2([0,\pi]),\mathbb{R})$ is bounded, that is, there exists a positive constant L such that $\|\sigma(t)\|_{\mathcal{L}_2^0} \leq L < \infty$, uniformly in $t \in [0,T]$, where $L := sup_{t\in[0,T]}e^{-t}$.

If we put

$$\begin{cases} u(t)(\zeta) = u(t,\zeta),\ t \in [0,T],\ \zeta \in [0,\pi] \\ u(t,\zeta) = \varphi(t,\zeta),\ t \in [-\tau,0],\ \zeta \in [0,\pi], \end{cases} \tag{4.2}$$

then, the problem (4.1) can be written in the abstract form

$$\begin{cases} d[x(t) + g(t,x(t-\tau))] = [A(t)x(t) + f(t,x(t-\tau))]dt + \sigma(t)dZ_H(t),\ 0 \leq t \leq T, \\ x(t) = \varphi(t),\ -\tau \leq t \leq 0. \end{cases}$$

Furthermore, if we assume that the initial data $\varphi = \{\varphi(t) : -\tau \leq t \leq 0\}$ satisfies $\varphi \in \mathcal{C}([-\tau,0],\mathbb{L}^2(\Omega,X))$, thus all the assumptions of Theorem 3.3 are fulfilled. Therefore, we conclude that the system (4.1) has a unique mild solution on $[-\tau,T]$.

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(E. Lakhel) Cadi Ayyad university, National School of Applied Sciences, 46000 Safi, Morocco

E-mail address, E. Lakhel: `e.lakhel@uca.ma`

(A. Tlidi) Cadi Ayyad university, National School of Applied Sciences, 46000 Safi, Morocco

E-mail address, A. Tlidi: `mtlidi2010@gmail.com`