Journal of Universal Mathematics
Vol.1 No.2 pp.148-154 (2018)
ISSN-2618-5660

FIXED POINT THEOREM USING A NEW CLASS OF FUZZY CONTRACTIVE MAPPINGS

S. MELLIANI AND A. MOUSSAOUI

Abstract. In this paper we introduce a new class of fuzzy contractive mapping and we show that such a class unify and generalize several existing concepts in the literature. We establish fixed point theorem for such mappings in complete strong fuzzy metric spaces and we give an illustrative example .

1. Introduction

One of the important theoretical development in the fuzzy sets theory is the way of defining the concept of fuzzy metric space, In 1975, Kramosil and Michalek [12] introduced the concept of fuzzy metric spaces, which constitutes a reformulation of the notion of probabilistic metric space, in 1988 Grabiec [1] introduced the Banach contraction in a fuzzy metric space in the sense of Kramosil and Michalek and extended the well-known fixed point theorems of Banach and Edelstein[2] to fuzzy metric spaces. In order to define a Hausdorff topology George and Veeramani [3] modified the concept of fuzzy metric space introduced by Kramosil and Michalek. Gregori and sapena [4] reconsidered the Banach contraction principle by initiating a new concept of fuzzy contractive mapping. In this direction, Mihet [10] introduced the notion of fuzzy ψ-contractive mappings and generalized the definitions given in [13]and [4]. Wardowski [11] proposed a New family of contractive mappings in a new sense called $\mathcal{H}$-contractive and proved that the class of fuzzy contractive mappings is included in the class of fuzzy $\mathcal{H}$-contractive mappings and obtained a fixed point result in complete fuzzy metric spaces in the sense of George and Veeramani, Gregori and Minana [6] pointed out some drawbacks on the conditions in Wardowski Theorem, The main Wardowskis theorem is correct and it is different from the ones known in the literature. For other related concepts and results on the development of fixed point theory in fuzzy metric spaces and its applications the reader is referred to [7], [9] ,[13], [14] and [15].

Building on this background and aiming to unifying different classes of fuzzy contractive mappings we introduce a new class of mappings called fuzzy $\mathcal{FZ}$-contractive mappings. Moreover, we show that many existing concepts in the literature can be easily deduced from our definition and we provide a fixed point theorem for this class of contractive mapping in strong fuzzy metric spaces.

Date: July 1, 2018, accepted.
Key words and phrases. Fixed point theory, Contractive mapping, Fuzzy metric space.

2. Preliminaries

For the sake of completeness, we briefly recall some basic concepts used in the following.

Definition 2.1. (Schweizer and Sklar [8])
A binary operation $\star : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a continuous triangular norm (in short, continuous t-norm) if it satisfies the following conditions
(T1) $a \star 1 = a \ \forall a \in [0,1]$,
(T2) $a \star b \leq c \star d \ \forall a \leq c \ , \ b \leq d \ and \ a, b, c, d \in [0,1]$,
(T3) $\star$ is commutative and associative .
(T4) $\star$ is continuous ,

Example 2.2. The following instances are classical examples of continuous t-norm:
a) $a \star b = \min(a, b)$ Zadeh's t-norm
b) $a \star b = \max[0, a + b - 1]$ Lukasiewicz's t-norm
c) $a \star b = a.b$ Probabilistic t-norm

Definition 2.3. (George and Veeramani [3])
The 3-tuple $(X, M, \star)$ is said to be a fuzzy metric space if X is an arbitrary set, $\star$ is a continuous t-norm and M is fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions :
$\forall \ x, y, z \in X \ and \ s, t > 0$:
(GV1) $M(x, y, t) > 0$,
(GV2) $M(x, y, t) = 1 \Longleftrightarrow x = y$,
(GV3) $M(x, y, t) = M(y, x, t)$,
(GV4) $M(x, y, t) \star M(y, z, s) \leq M(x, z, t + s)$,
(GV5) $M(x, y, .) :]0, \infty[\rightarrow [0, 1]$ is continuous .

$M(x, y, t)$ can be thought of as the degree of nearness betwen x and y with respect to t. In the above definition, if we replace $(GV5)$ by $(GV5^{'}) : M(x, y, t) \star M(y, z, s) \leq M(x, z, \max\{t, s\}) \forall \ x, y, z \in X \ and \ s, t > 0$ Then the triple $(X, M, \star)$ is said to be a non-Archimedean fuzzy metric space. As $(GV5^{'}) \Rightarrow (GV5)$, each non-Archimedean fuzzy metric space is a fuzzy metric space, Further if $M(x, y, t) \star M(y, z, t) \leq M(x, z, t) \ \forall t > 0$ M is said to be a strong fuzzy metric .

Let $(X, M, \star)$ be a fuzzy metric space. The open ball $B_M(x, r, t)$ for $t > 0$ with centre $x \in X$ and radius r, $0 < r < 1$ is defined by:

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$

A subset A of a fuzzy metric space $(X, M, \star)$ is said to be open if given any point $x \in A$, there exists $0 < r < 1$, and $t > 0$ such that $B(x, r, t) \subseteq A$.
The familly :

$$\tau_M = \{A \subset X : x \in A \, if \, only \, if \, there \, exist \, t > 0 \, and \, 0 < r < 1 \, such \, that \, B(x, r, t) \subset A\}$$

is a topology on X , τ_M is called the topology on X induced by the fuzzy metric M . (X, τ_M) is a Housdorff first countable topological space .[3]

Example 2.4. [3]
Let (X, d) be a metric space, Define $a \star b = \min(a, b) \ \forall a, b \in [0, 1]$, Define

$M_d(x,y,t) = t/t + d(x,y)\ \forall\ t > 0$. Then $(X, M_d, \star)$ is a fuzzy metric space. M_d is called the standard fuzzy metric induced by d .
Moreover, The topology τ_{M_d} generated by the induced fuzzy metric M_d coincides with the topology generated by d.

Theorem 2.5. [1]
Let $(X, M, \star)$ be a fuzzy metric space. Then $M(x, y, .)$ is non-decreasing for all $x, y \in X$.

Definition 2.6. [3]
Let $(X, M, \star)$ be a fuzzy metric space. Then:

(1) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if and only if for each ε ,$0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.
(2) A sequence $\{x_n\}$ in X is said to convergent to x in X ,denoted $x_n \to x$, if an only if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$, i.e. for each $r \in]0, 1[$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$.
(3) The fuzzy metric space $(X, M, \star)$ is called complete if every Cauchy sequence is convergent.

Definition 2.7. (Gregori and sapena [4])
Let $(X, M, \star)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be fuzzy contractive if there exists $k \in]0, 1[$ such that

$$\frac{1}{M(T(x), T(y), t)} - 1 \leq k(\frac{1}{M(x, y, t)} - 1)$$

for each $x, y \in X$ and $t > 0$. A sequence $\{x_n\}$ in X is said to be fuzzy contractive if there exists $k \in]0, 1[$ such that

$$\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leq k(\frac{1}{M(x_n, x_{n+1}, t)} - 1)$$

for all $t > 0$ $n \in \mathbb{N}$.

Theorem 2.8. [4]
Let $(X, M, \star)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let $T : X \to X$ be a fuzzy contractive mapping being k the contractive constant. Then T has a unique fixed point.

Definition 2.9. (D.Mihet[10].)
Let Ψ be the class of all mapping $\psi :]0, 1] \longrightarrow]0, 1]$ such that ψ is continuous, nondecreasing and, $\psi(t) > t$ for all $t \in]0, 1[$.
Let $\psi \in \Psi$, A mapping $T : X \longrightarrow X$ is said to be fuzzy ψ-contractive mapping if:

$$M(T(x), T(y), t) \geq \psi(M(x, y, t))\ for\ all\ x, y \in X, t > 0$$

In [11] Wardowski proposed a new type of contraction in a fuzzy metric space. We can read it as follows:

Definition 2.10. (Wardowski[11].)
Let $\mathcal{H}$ be the family of the mappings $\eta :]0, 1] \longrightarrow [0, \infty[$ satisfying the following conditions :

$\mathcal{H}_1$): η transforms $]0, 1]$ onto $[0, \infty[$,
$\mathcal{H}_2$): η is strictly decreasing .

A mapping $T : X \longrightarrow X$. is said to be fuzzy $\mathcal{H}$-contractive with respect to $\eta \in \mathcal{H}$ if $\exists k \in]0,1[$ satisfying the following condition :

$$\eta(M(T(x),T(y),t)) \leq k\eta((M(x,y,t)) \ for\ all\ x,y \in X, t > 0$$

Remark 2.11. (Gregori and Miñana [6])
If $\eta \in \mathcal{H}$ then the mapping $\eta.k$ $:]0,1] \longrightarrow [0,\infty[$ and η^{-1} : $[0,\infty[\longrightarrow]0,1]$, defined in its obvious sense, are two bijective continuous mappings which are strictly decreasing.

3. Main results

Definition 3.1.
Let $\phi :]0,1] \times]0,1] \longrightarrow \mathbb{R}$ be a mapping satisfying the following conditions :

(i) $\phi(1,1) = 0$,
(ii) $\phi(t,s) < \frac{1}{s} - \frac{1}{t}$ for all $t,s < 1$,
(iii) if $\{t_n\}, \{s_n\}$ are sequence in $[0,1]$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n < 1$ then $\lim_{n\to\infty} \sup \phi(t_n, s_n) < 0$.

We denote by $\mathcal{FZ}$ the class of all functions which satisfies the above conditions.

Definition 3.2.
Let $(X, M, \star)$ be a fuzzy metric space. We say that a mapping $T : X \to X$ is a $\mathcal{FZ}$-contractive mapping with respect to ϕ if the following condition is satisfied

$$\phi(M(T(x),T(y),t), M(x,y,t)) \geq 0 \ for\ all\ x,y \in X.$$

A simple example of $\mathcal{FZ}$-contraction is the fuzzy contraction given by Gregori and Sapena which can obtained by taking $k \in]0,1[$ and $\phi(t,s) = k(\frac{1}{s}-1) - \frac{1}{t} + 1$ for all $t, s \in]0,1]$. Consequently, The class of fuzzy contractive mappings in the sense of Gregori and Sapena is included in the class of $\mathcal{FZ}$-contractive mappings.

Remark 3.3. It is clear from the definition that $\phi(t,s) < 0$ for all $s \geq t$. Therefore, if T is a $\mathcal{FZ}$-contraction mapping with respect to ϕ then $M(x,y,t) < M(T(x),T(y),t)$.

Proposition 1. The class of fuzzy ψ-contractive mappings are included in the class of $\mathcal{FZ}$-contractive mappings.

Proof. Suppose that $T : X \to X$ is ψ-contractive with respect to $\psi \in \Psi$,
Define $\phi_\psi :]0,1] \times]0,1] \longrightarrow \mathbb{R}$ by

$$\phi_\psi = \frac{1}{\psi(s)} - \frac{1}{t} \ for\ all\ s,t \in]0,1]$$

T is $\mathcal{FZ}$-contractive mapping with respect to $\phi_\psi \in \mathcal{FZ}$. □

Remark 3.4. Taking in account the remark 2.11, every $\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ is a fuzzy $\mathcal{FZ}$-contraction with respect to the function $\phi \in \mathcal{FZ}$ defined by $\phi_\eta(t,s) = \frac{1}{\eta^{-1}(k.\eta(s))} - \frac{1}{t}\ for\ all\ s,t \in]0,1]$.

Lemma 3.5. *Let $(X, M, \star)$ be a fuzzy metric space and T be a $\mathcal{FZ}$-contraction with respect to $\phi \in \mathcal{FZ}$. Then the fixed point of T in X is unique, provided it exists* .

proof. Suppose $u \in X$ be a fixed point of T, If possible, let $v \in X$ be another fixed point of T and it is distinct from u, that is, $Tv = v$ and $u \neq v$. Now it follows from the definition that

$$0 \leq \phi(M(T(u), T(v), t), M(u, v, t)) = \phi(M(u, v, t), M(u, v, t)).$$

The remark 3.3 yields a contradiction, so $u = v$.

□

Theorem 3.6. *Let $(X, M, \star)$ be a complete strong fuzzy metric space and T be a $\mathcal{FZ}$-contraction with respect to $\phi \in \mathcal{FZ}$. Then the fixed point of T in X is unique .*

proof. Let $x_0 \in X$ be any arbitrary point in X. Now we Construct a sequence $\{x_n\} \in X$ Such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.
Without loss of generality we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$ Trivially, if there exists n_0 such that $x_{n_0} = x_{n_0+1}$,then the equalities $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ implies that x_{n_0} is a fixed point of T.
we prove that $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$, Suppose, to the contrary, that there exists some t_0 such that

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t_0) < 1$$

Now, by (GV2) we have that $M(x_n, x_{n+1}, t_0) < 1$ for all $n \in N$.

Then, by using (i) and (ii), taking $x = x_{n-1}$ and $y = x_n$, we have

$$0 \leq \phi(M(x_n, x_{n+1}, t_0), M(x_{n-1}, x_n, t_0)) < \frac{1}{M(x_{n-1}, x_n, t_0)} - \frac{1}{M(x_n, x_{n+1}, t_0)}$$

for all $n \in \mathbb{N}$, this implies that $\{M(x_{n-1}, x_n, t_0), n \in N\}$ is nondecreasing sequence of positives reals numbers. therefore, there exists $l \leq 1$ such that $\lim_{n\to\infty} M(x_{n-1}, x_n, t_0) = l$

We shall show that $l = 1$, we suppose that $l < 1$ and using (iii)

$$t_n = M(x_n, x_{n+1}, t_0) \textit{ and } s_n = M(x_{n-1}, x_n, t_0)$$

we conclude that :

$$0 \leq \phi(M(x_n, x_{n+1}, t_0), M(x_{n-1}, x_n, t_0)) < 0$$

which is a contradiction, Hence $l = 1$, That is :

$$\lim_{n\to\infty} M(x_{n-1}, x_n, t_0) = 1$$

The crucial point of the proof is in establishing that the sequence $\{x_n\}$ is Cauchy in X. Assuming it is not true, Then there exist $0 < \epsilon < 1$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which $n_k > m_k \geq k$

$$M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \epsilon \ (1)$$

and

$$M(x_{m_k}, x_{n_k-1}, t_0) > 1 - \epsilon \ (2)$$

Using (1) and (2) and triangular inequality we obtain

$$\begin{aligned} 1 - \epsilon \geq M(x_{m_k}, x_{n_k}, t_0) &\geq M(x_{m_k}, x_{n_k-1}, t_0) \star M(x_{n_k-1}, x_{n_k}, t_0) \\ &\geq (1 - \epsilon) \star M(x_{n_k-1}, x_{n_k}, t_0) \end{aligned}$$

and by taking limit as $k \to \infty$

$$1-\epsilon \geq \lim_{n\to\infty} M(x_{m_k}, x_{n_k}, t_0) \geq (1-\epsilon)$$

We deduce that $\lim_{n\to\infty} M(x_{m_k}, x_{n_k}, t_0) = 1-\epsilon$

Applying the same reasoning as above, we obtain

$$1-\epsilon \geq M(x_{m_k}, x_{n_k}, t_0) \geq M(x_{m_k}, x_{m_k-1}, t_0) \star M(x_{m_k-1}, x_{n_k-1}, t_0) \star M(x_{n_k-1}, x_{n_k}, t_0)$$

and

$$M(x_{m_k-1}, x_{n_k-1}, t_0) \geq M(x_{m_k-1}, x_{m_k}, t_0) \star M(x_{m_k}, x_{n_k}, t_0) \star M(x_{n_k}, x_{n_k-1}, t_0)$$

Taking limit as $k \to \infty$, we get $\lim_{n\to\infty} M(x_{m_k-1}, x_{n_k-1}, t_0) = 1-\epsilon$ hence using (iii) with $\tau_k = M(x_{m_k}, x_{n_k}, t_0)$ *and* $\delta_k = M(x_{m_k-1}, x_{n_k-1}, t_0)$ we obtain

$$0 \leq \lim_{n\to\infty} \sup \phi(M(x_{m_k}, x_{n_k}, t_0, M(x_{m_k-1}, x_{n_k-1}, t_0)) < 0$$

ie.

$$0 \leq \phi(1-\epsilon, 1-\epsilon) < 0$$

Obviously, this inequality is not true and $\{x_n\}$ is a cauchy sequence in X .

The completeness of $(X, M, \star)$ ensures that the sequence $\{x_n\}$ converges to some $u \in X$, that is $\lim_{n\to\infty} M(x_n, u, t) = 1$ $\forall t > 0$ we shall show that the point u is a fixed point of T , suppose that $Tu \neq u$ then $M(u, Tu, t) < 1$,we have

$$\begin{aligned}
0 &\leq \lim_{n\to\infty} \sup \phi(M(Tx_n, Tu, t), M(x_n, u, t)) \\
&\leq \lim_{n\to\infty} \sup[\frac{1}{M(x_n, u, t)} - \frac{1}{M(Tx_n, Tu, t)}] \\
&= \lim_{n\to\infty} \sup[\frac{1}{M(x_n, u, t)} - \frac{1}{M(x_{n+1}, Tu, t)}] \\
&= 1 - \frac{1}{M(u, Tu, t)}
\end{aligned}$$

Finally, from the above we have $1 \leq M(u, Tu, t)$ hence $M(u, Tu, t) = 1$ Which is contradiction, thus u is a fixed point of T.

□

Example 3.7. Let $X =]0, \infty[$, $a \star b = ab$ $\forall a, b \in [0, 1]$ and

$$M(x, y, t) = \frac{min(x, y)}{max(x, y)} \ \forall t \in]0, \infty[\ \forall x, y > 0$$

$(X, M, \star)$ is an complete strong fuzzy metric space .[7]
The mapping $T : X \longrightarrow X$, $T(x) = \sqrt{x}$ is $\mathcal{FZ}$-contractive mapping with respect to the function defined by

$$\Phi(t, s) = \frac{1}{\sqrt{s}} - \frac{1}{t} \ \forall t, s \in]0, 1]$$

Indeed, since $\frac{1}{\sqrt{s}} - \frac{1}{t} < \frac{1}{s} - \frac{1}{t}$ $\forall t, s \in]0, 1[$ with $\Phi(1, 1) = 0$;

Note that , all the condition of the previous Theorem are satisfied and T has a unique fixed point $x = 1 \in X$.

References

[1] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets Syst. 27 (1988) 385389.
[2] M. Edelstein, On fixed and periodic points under contractive mappings, J. Lond. Math. Soc. 37 (1962) 7479.
[3] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst. 64 (1994) 395399.
[4] V. Gregori and A. Sapena, On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 125 (2002), 245252.
[5] V.Gregori, S.Morillas, A.Sapena,Examples of fuzzy metrics and applications,FuzzyS ets and Systems 170(2011)95111..
[6] V. Gregori, J.-J. Minana, Some remarks on fuzzy contractive mappings, Fuzzy Sets Syst. 251 (2014) 101103, https://doi .org /10 .1016 /j .fss .2014 .01 .002.
[7] V.Radu, Some remarks on the probabilistic contractions on fuzzy Menger spaces, Automat.Comput.Appl.Math.11(2002)125131.
[8] B.Schweizer, A.Sklar, Statistical metric spaces,PacificJ.Math.10(1960)313334.
[9] D.Mihet, On fuzzy contractive mappings in fuzzy metric spaces,Fuzzy Sets and Systems 158(2007)915921.
[10] D.Mihet, Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces,Fuzzy Sets and Systems 159(2008)739744.
[11] D. Wardowski, Fuzzy contractive mappings and fixed points in fuzzy metric spaces, Fuzzy Sets Syst. 222 (2013) 108114.
[12] I. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 15 (1975) 326334.
[13] C iric, L.: Some new results for Banach contractions and Edelstein contractive mappings on fuzzy metric spaces. Chaos Solitons Fractals 42, 146154 (2009)
[14] Shen, Y., Qiu, D., Chen,W.: Fixed point theorems in fuzzy metric spaces. Appl. Math. Lett. 25, 138141 (2012)
[15] Roldan, A., Martinez, J., Roldan, C., Cho,Y.J.:Multidimensional coincidence point results for compatible mappings in partially ordered fuzzy metric spaces. Fuzzy Sets Syst. 251, 7182 (2014)

[1,2] Laboratory of Applied Mathematics & Scientific Calculus , LMACS, Sultan Moulay Slimane University, PO Box 523, 23000 Beni Mellal Morocco

Current address: Sultan Moulay Slimane University, BP 523, 23000 Beni Mellal, Morocco

E-mail address: [1,2] s.melliani@usms.ma; a.moussaoui@usms.ma