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NOTES ON IDEALS AND FILTERS IN BCC-ALGEBRAS

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Abstract. In this article we introduce and analyze a new concept of BCC-filters in BCC-algebra. In addition, the relationship of this new concept with BCC-ideals has been analyzed also.

1. Introduction

Y. Komori (cf. [9, 10]) introduced a notion of BCC-algebras, and W. A. Dudek (cf. [2, 3]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [4], W. A. Dudek and X. H. Zhang introduced the concept of ideals in BCC-algebras and described connections between such ideals and congruences. Several authors dealt with these algebras (see for example [5, 6, 8]).

In this article we introduce and analyze the concept of BCC-filters in BCC-algebras. In addition, we identify the relationships between the BCC-filters and the BCC-ideals.

2. Preliminaries: BCC-algebras

The definition of BCC-algebra is taken from the articles [2, 3].

Definition 2.1. An algebra $(A, \cdot, 0)$ is called a *BCC-algebra* if it satisfies the following axioms:
(1) $(\forall x, y, z \in A)(((x \cdot y) \cdot (z \cdot y)) \cdot (x \cdot z) = 0)$,
(2) $(\forall x \in A)(0 \cdot x = 0)$,
(3) $(\forall x \in A)(x \cdot 0 = x)$,
(4) $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \implies x = y)$.

Definition 2.2.
$$(\forall x, y \in A)(x \leq y \iff x \cdot y = 0).$$

Natural BCC-order has ([3], Proposition 2) the following properties
(5) $(\forall x, y \in A)(x \cdot y \leq x)$,
(6) $(\forall x, y, z \in A)((x \cdot y) \cdot (z \cdot y) \leq x \cdot z)$,
(7) $(\forall x, y, z \in A)(x \leq y \implies (x \cdot z \leq y \cdot z \wedge z \cdot y \leq z \cdot x))$.

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3. BCC-ideals

Definition 3.1. ([4]) A non-empty subset J of a BCC-algebra A is called a *BCC-ideal*, if
(8) $0 \in J$,
(9) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in J \ \wedge \ y \in J) \implies x \cdot z \in J)$.

In the following theorem are given two statements of some fundamental characteristics of the BCC-ideals.

Theorem 3.1. *If J is a BCC-ideal of a BCC-algebra A, then hold*
(10) $(\forall x, y \in A)((x \cdot y \in J \wedge y \in J) \implies x \in J)$ *and*
(11) $(\forall y, z \in A)(y \in J \implies y \cdot z \in J)$.

Proof. If we put $z = 0$ in (9) we get (10).
Let $y, z \in A$ be arbitrary elements such that $y \in J$. Thus from $(y \cdot y) \cdot z = 0 \cdot z = 0 \in J$ and $y \in J$, by definition of BCC-ideal J, follows $y \cdot z \in J$. □

The important feature of a BCC-ideal in a BCC-algebra is given by the following statement.

Corollary 3.1. *Let J be BCC-ideal of a BCC-algebra A. Then*
(12) $(\forall x, y \in A)((x \leqslant y \wedge y \in J) \implies x \in J)$.

Proof. Let $x, y \in A$ be arbitrary elements such that $x \leqslant y$ and $y \in J$. Thus, $x \in J$ follows from $x \cdot y = 0 \in J$ and $y \in J$ by (10). □

The statement in the next theorem was proved in analogy with the corresponding statement presented in the texts [4, 5].

Theorem 3.2. *Let J be a BCC-ideal of a BCC-algebra A. The the relation $\prec$ in A, defined by*

$$(\forall x, y \in A)(x \prec y \iff x \cdot y \in J),$$

is a quasi-order in A and the following holds

$$(\forall x, y, z \in A)(x \prec y \implies (x \cdot z \prec y \cdot z \ \wedge \ z \cdot y \prec z \cdot x)).$$

Proof. Let $x \in A$ be an arbitrary element. Then $x \cdot x = 0 \in J$. Thus $x \prec x$. Let $x, y, z \in A$ be arbitrary elements such that $x \prec y$ and $y \prec z$. Thus $x \cdot y \in J$ and $y \cdot z \in J$. If we put $y = z$ and $z = y$ in (1), we get

$$((x \cdot z) \cdot (y \cdot z)) \cdot (x \cdot y) = 0 \in J.$$

Since $x \cdot y \in J$ and $y \cdot z \in$ from the previous formula we get $x \cdot z \in J$ if we apply twice (10). Therefore, $x \prec Z$. So, the relation $\prec$ is a quasi-order in set A.
Let $x, y, z \in A$ arbitrary elements such that $x \prec y$. Thus $x \cdot y \in J$. If we put $y = z$ and $z = y$ u (6), we get $(x \cdot z) \cdot (y \cdot z) \leqslant x \cdot y \in J$. Now, from this obtained formula, we have $(x \cdot z) \cdot (y \cdot z) \in J$ according to (12). Finally, we have $x \cdot z \prec y \cdot z$.
On the other hand, if we put $x = z$ and $z = x$ u (1), we get $((z \cdot y) \cdot (x \cdot y)) \cdot (z \cdot x) = 0 \in J$. From here we get $(z \cdot y) \cdot (z \cdot x) \in J$ according to (9) due $x \cdot y \in J$. So, we have $z \cdot y \prec z \cdot x$.
Therefore, $\prec$ is a right compatible and left anti-compatible relation in A. □

Corollary 3.2. *Let J be a BCC-ideal in BCC-algebra A. The relation $' \sim '$ in A defined by $(\forall x, y \in A)(x \sim y \iff (x \prec y \wedge y \prec x))$ is a BCC-congruence in A.*

Corollary 3.3. *Any BCC-ideal in a BCC-algebra A is determined by some quasi-order relation in A.*

Proof. Let J be a BCC-ideal in A. Then the relation defined by the ideal J as in Theorem 3.2, is a quasi-order relation in A.

Opposite, let $\prec$ be a quasi-order relation in A. Then the relation $\sim = \prec \cap \prec^{-1}$ is a congruence in A. Then, by Lemma 3.2 in [4], the class $[0]_\sim$ of relation $\sim$ generated by 0 is a BCC-ideal in A. □

Corollary 3.4. *The lattice $\mathfrak{Q}_A$ of all quasi-order relations in a BCC-algebra A is isomorphic the the lattice $\mathfrak{J}_A$ of all BCC-ideals in A.*

4. A New Concept of BCC-filters

In this short note, we intend to offer a new concept of BCC-filters F of BCC-algebras A that satisfies the following condition

$$(\forall x, y \in A)((x \leqslant y \wedge x \in F) \implies y \in F)$$

and has a standard attitude towards the BCC-ideal. This formula can be transformed in the following formula

$$(\forall x, y \in A)((x \cdot y = 0 \wedge x \in F) \implies y \in F)$$

Looking at this last formula, a new concept of BCC-filters in the BCC-algebra is introduced by the following definition.

Definition 4.1. A subset F of a BCC-algebra A is called a *BCC-filter*, if
(13) $\neg(0 \in F)$,
(14) $(\forall x, y, z \in A)((\neg((x \cdot y) \cdot z \in F) \wedge x \cdot z \in F) \implies y \in F)$.

The BCC-filter defined on this way has the following properties.

Theorem 4.1. *Let A be BCC-algebra and F a BCC-filter of A. Then*
(15) $(\forall x, y \in A)((\neg(x \cdot y \in F) \wedge x \in F) \implies y \in F)$,
(16) $(\forall x, y \in A)(x \cdot y \in F \implies y \in F)$.

Proof. Putting $z = 0$ in (14) we obtain (15).
If we put y = x and z = y in (14), we have

$$(\neg((x \cdot x) \cdot y = 0 \cdot y = 0 \in F) \wedge x \cdot y \in F) \implies y \in F.$$

Therefore, (16) is proved. □

Corollary 4.1. *Let F be a BCC-filter of BCC-algebra A. Then*
(17) $(\forall x, y \in A)((x \leq y \wedge x \in F) \implies y \in F)$.

Proof. Let $x, y \in A$ be arbitrary elements such that $x \leq y$ and $x \in F$. Thus $\neg(x \cdot y = 0 \in F)$ and $x \in F$. Then by (15) we have $y \in F$. □

Remark 4.1. Let us recall that some authors a subset F of an BCK-algebra A satisfying condition (16) call '*a deductive system*' of that algebra (For example see [7] or [1]). As can be seen, the filter in our sense is a deductive system in a BCC algebra in the sense of the articles [7, 1]. The difference between our concept of deductive system and the concept of a deductive system known in literature is the requirement (13) instead of the demand (8).

Theorem 4.2. *If F is a BCC-filter of BCC-algebra A, then the set $J = A\backslash F$ is a BCC-ideal. Opposite, if J is a BCC-ideal of BCC-algebra A, then the set $F = A\backslash J$ is a BCC-filter of A.*

Proof. It is clear that $0 \in J$. Let $x, y, z \in A$ be arbitrary elements such that $(x \cdot y) \cdot z \in J$ and $y \in J$. Then we have $\neg((x \cdot y) \cdot z \in F)$ and $\neg(y \in F)$. If we suppose that $x \cdot z \in F$ by (13) we will have $y \in F$. So, must to be $\neg(x \cdot z \in F)$.

Opposite, let J be a BCC-ideal of A. It is that $\neg(0 \in A\backslash J)$. Let $x, y, z \in A$ be arbitrary elements such that $\neg((x \cdot y) \cdot z \in A\backslash J)$ and $x \cdot z \in A\backslash J$. Then $y \in A\backslash J$. Indeed. If it were $y\ inJ$ then $x \cdot z \in J$ follows from $(x \cdot y) \cdot z \in J$ and the $y \in J$ which is in a contradiction with the assumption $x \cdot z \in A\backslash J$. □

Theorem 4.3. *The family $\mathfrak{F}_A$ of all BCC-filters in BCC-algebra A forms a completely lattice.*

Proof. Let $\{F_{i\in I}\}$ be a family of BCC-filters in BCC-algebra A. It is clear that $\neg(0 \in \bigcap_{i\in I} F_i)$ and $\neg(0 \in \bigcup_{i\in I} F_i)$.

(a) Let $x, y, z \in A$ arbitrary elements such that $\neg((x \cdot y) \cdot z \in \bigcup_{i\in I} F_i)$ and $x \cdot z \in \bigcup_{i\in I} F_i$. Thus $\neg((x \cdot y) \cdot z \in F_i)$ for all $i \in I$ and there exists an index $j \in I$ such that $x \cdot z \in F_j$. Then $y \in F_j \subseteq \bigcup_{i\in I} F_i$.

(b) Let $\mathfrak{X}$ be the family of all BCC-filters which contained in the intersection $\bigcap_{i\in I} F_i$. The union $\bigcup \mathfrak{X}$ is the maximal BCC-filter contained in the intersection $\bigcap_{i\in I} F_i$.

(c) So, if we choose $\sqcap_{i\in I} F_i = \bigcup \mathfrak{X}$ and $\sqcup_{i\in I} F_i = \bigcup_{i\in I} F_i$, then $(\mathfrak{F}, \sqcap, \sqcup)$ is a completely lattice. □

Corollary 4.2. *The lattice $\mathfrak{F}_A$ is isomorphic to the lattice $\mathfrak{J}_A$.*

5. Future Intentions

Let $\prec_J$ be a quasi-order relation in BCC-algebra A generated by a BCC-ideal J as in Theorem 3.2. Thus the relation $\sim_J$, defined as in Corollary 3.1, is a BCC-congruence in A. Let $[a]_{\sim_J}$ be a class of the BCC-congruence $\sim_J$ generated by the element $a \in A$. It is clear the following

$$x \in [a]_{\sim_J} \iff x \sim_J a \iff (x \prec_J a \wedge a \prec_J x).$$

Let $L_{\prec_J}(a) = \{x \in A : ax \in J\}$ and $R_{\prec_J}(a) = \{y \in A : ya\}$ be denote the left and right class of the relation $\prec_J$ generated by the element a. Then $[a]_{\sim_J} = L_{\prec_J}(a) \cap R_{\prec_J}(a)$ holds. So, in accordance with Theorem 4 in [5], we have

$$[0]_{\sim_J} = L_{\prec_J}(0) \cap R_{\prec_J}(0) = A \cap J = J.$$

On the other hand, for the filter $F = \bigcup_{a\in A\backslash\{0\}} [a]_\sim$ generated as in Theorem 4.2, we have

$$\begin{aligned} F &= \cup_{a\in A\backslash\{0\}} [a]_{\sim_J} = \cup_{a\in A\backslash\{0\}} (L_{\prec_J}(a) \cap R_{\prec_J}(a)) \\ &= (\cup_{a\in A\backslash\{0\}} L_{\prec_J}(a)) \cap (\cup_{a\in A\backslash\{0\}} (R_{\prec_J}(a))). \end{aligned}$$

So, since the filter are shown as intersection of two sets from two classes of subsets in a BCC algebra it is reasonable to describe the elements of these two families in the BCC-algebra supplied with some quasi-order relation.

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