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Bivariate Generalized Kantorovich Forms of Exponential Sampling Series: Some New Results

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Highlights

- The aforementioned operators are employed in the mathematical modeling of seismic waves.
- The proposed operators have applications in physics.
- Quantitative estimates of the proposed operators are addressed.

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Abstract

In this paper, we begin by establishing a rigorous upper bound for the difference between the operators $\left(\mathcal{K}_W^{\varrho,\mathfrak{G}}\right)$ and $\left(\mathcal{E}_W^{\varrho}\right)$, providing a precise measure of the approximation error inherent in the proposed operators. Building on this foundation, we proceed to derive a quantitative Voronovskaja-type formula, which offers a detailed characterization of the asymptotic behavior of the operator under consideration. Finally, to demonstrate the practical relevance and applicability of the theoretical results, we present several illustrative examples of kernels that are compatible with the proposed framework.

1. INTRODUCTION AND PRELIMINARIES

In this section, we begin by presenting a background on sampling-type operators, followed by certain preliminary constructions that will be employed in our subsequent results.

1.1. History of Sampling Type Operators

The classical sampling theorem was independently formulated in [1-3]. Sampling-type operators and their applications constitute a central and challenging topic in approximation theory, with significant relevance to signal and image processing (see, e.g., [4-7]). The family of operators is defined as

$$(\mathfrak{S}_W g)(\mathcal{X}) := \sum_{\Bbbk\in\mathbb{Z}} g\left(\frac{\Bbbk}{W}\right) sinc(W\mathcal{X}-\Bbbk),\ W>0, \mathcal{X}\in\mathbb{R}. \tag{1.1}$$

Here, sinc is the "normalized sine cardinal" function. We know that if g is a band-limited function with support in $[-\pi W, \pi W]$, then

$$(\mathfrak{S}_W g)(\mathcal{X}) = g(\mathcal{X}).$$

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Equivalently, g can be reconstructed without loss from its discrete samples taken at uniformly distributed points on $\mathbb{R}$, given by the set $\left\{\frac{\Bbbk}{W} : \Bbbk \in \mathbb{Z}\right\}$.

To treat non–band-limited functions, Butzer and coauthors [8-10] substituted the sinc kernel with generalized kernels meeting approximate-identity and Strang–Fix conditions, and introduced

$$\left(S_W^{\chi} g\right)(\mathcal{X}) := \sum_{\Bbbk\in\mathbb{Z}} g\left(\frac{\Bbbk}{W}\right)\chi(W\mathcal{X} - \Bbbk) \tag{1.2}$$

whenever the series converges absolutely. The remainder of classical sampling theory is then developed through Fourier transform techniques (see, e.g., [11, 12]). In recent years, extensive research has been devoted to the study of generalized sampling series across various function spaces. Linear combination forms have been investigated in [13], while Kantorovich-type and Durrmeyer-type forms have been explored in [14-17] and [18, 19], respectively. Extensions to bivariate and multivariate forms have also been considered [20-24], highlighting the versatility of these series in higher-dimensional settings. Saturation results, which describe the limits of approximation, were addressed in [25], and a novel type of sampling series was introduced in [26], further enriching the theory. More recently, attention has been given to the convergence properties of (1.2) and its variants for functions belonging to weighted spaces of continuous functions, emphasizing their practical applicability in approximation theory (see, e.g., [27-31]). Collectively, these studies illustrate both the breadth and depth of developments in the field, providing a solid foundation for ongoing research.

Motivated by practical challenges in light scattering and diffraction, researchers in optical physics and engineering during the 1980s developed a mathematical model that has since been applied in radio astronomy and related optical-physics problems (see, e.g., [32-35]). They formulated the inverse problem via

$$\mathcal{F}(x) = \int_0^{\infty} \Psi(xu) f(u)\frac{du}{u},$$

in which $\mathcal{F}, \Psi, f$ are the measured data, the kernel and the target function to be estimated respectively. An alternative perspective is that the Mellin transform, recognized as a highly effective tool for solving such problems, allows the solution to be represented as a series of samples. Within this framework, the family of exponential sampling operators is defined for any function $g: \mathbb{R}^+ \to \mathbb{C}$ by

$$(E_W g)(x) := \sum_{\Bbbk\in\mathbb{Z}} g\left(e^{\frac{\Bbbk}{W}}\right) \mathrm{lin}_{c/W}\left(e^{-\Bbbk} x^W\right),\ W > 0, c \in \mathbb{R}, x \in \mathbb{R}^+. \tag{1.3}$$

Here, lin_c is the "linearly compensated sinc" function. In this context, if g is a Mellin band-limited function, then

$$(E_W g)(x) = g(x)$$

for every $x \in \mathbb{R}^+$ (see [36]).

Bardaro et al. in [37] formulated a robust generalization of the exponential sampling series (Equation (1.3)) by substituting the classical lin_c kernel with a more general kernel φ defined on $\mathbb{R}^+$. The kernel φ is required to satisfy the standard approximate-identity hypotheses (formulated with respect to the multiplicative measure $\frac{du}{u}$ on $\mathbb{R}^+$), typically including normalization, integrability, and concentration under dilation, together with appropriate moment or vanishing-moment conditions when higher approximation orders are sought. Under these hypotheses—and given customary smoothness or decay assumptions on the underlying function—the generalized sampling operators converge to the original function in the relevant norms (e.g., uniform convergence on compact sets or convergence in L^p with respect to the multiplicative measure), and the approximation rates depend on the moment conditions of φ and the regularity of the target function. This framework therefore recovers the classical exponential sampling expansion as a special case (when $\varphi = \mathrm{lin}_c$) while admitting a wider variety of kernels tailored to different localization and approximation requirements. For $g: \mathbb{R}^+ \to \mathbb{C}$, this extension permits the reconstruction of functions that need not be Mellin band-limited. The generalized exponential sampling operator introduced in [37] is defined by

$$(E_W^{\varphi} g)(x) := \sum_{\Bbbk\in\mathbb{Z}} \varphi\left(e^{-\Bbbk}x^W\right) g\left(e^{\frac{\Bbbk}{W}}\right),\ W > 0, x \in \mathbb{R}^+, \tag{1.4}$$

whenever the series converges absolutely.

Subsequently, to facilitate applications in the analysis of seismic waves, Bardaro et al. [38] (see [39] for another study in which seismic wave modeling is carried out using exponential-type sampling operators) proposed a bivariate extension of (1.4), given by

$$(\mathcal{E}_W^{\varrho} f)(x_1, x_2) = \sum_{(k,j)\in\mathbb{Z}^2} f\left(e^{\frac{k}{W}}, e^{\frac{j}{W}}\right) \varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right),\ W > 0, (x_1, x_2) \in \mathbb{R}^2_+, \tag{1.5}$$

where $f: \mathbb{R}^2_+ \to \mathbb{R}$ is assumed such that the series converges for every $(x_1, x_2) \in \mathbb{R}^2_+$ and $\varrho: \mathbb{R}^2_+ \to \mathbb{R}$ is a continuous function (kernel) which satisfies the following assumptions:

- for every $(x_1, x_2) \in \mathbb{R}^2_+$,

  $$\sum_{(k,j)\in\mathbb{Z}^2} \varrho\left(e^{-k}x_1, e^{-j}x_2\right) = 1,$$

- it holds

  $$M_0(\varrho) := \sup_{(x_1,x_2)\in\mathbb{R}^2_+} \sum_{(k,j)\in\mathbb{Z}^2} \left|\varrho\left(e^{-k}x_2, e^{-j}x_2\right)\right| < \infty,$$

- $\lim\limits_{\gamma\to\infty} \sum_{(k,j)\notin B_\gamma(\log x_1, \log x_2)} \left|\varrho\left(e^{-k}x_1, e^{-j}x_2\right)\right| = 0$

  uniformly for $(x_1, x_2) \in \mathbb{R}^2_+$. Here, for any $\gamma > 0$ and $(x_1, x_2) \in \mathbb{R}^2_+$, we denote by $B_\gamma(x_1, x_2)$ the open ball of radius γ centered at (x_1, x_2), defined as

$$B_\gamma(x_1, x_2) := \{(u, v) \in \mathbb{R}^2_+ \mid (x_1 - u)^2 + (x_2 - v)^2 < \gamma^2\}.$$

In recent years, a considerable body of work has been devoted to exploring various forms and function spaces of the series (1.4). Notable contributions include investigations in Mellin–Lebesgue spaces [40, 41], studies on linear combinations [42], analyses of Kantorovich forms [43, 44] and Durrmeyer forms [45], as well as developments of generalized Kantorovich forms [46]. Furthermore, extensions to bivariate forms [38, 39], multivariate forms [47, 48], and logarithmically weighted spaces of continuous functions [49-51] have also been presented, reflecting the growing interest and applicability of these series in diverse mathematical contexts.

Another significant operator in Mellin analysis is the convolution operator introduced by Butzer and Jansche [52] (see also [53, 54]). Furthermore, in [55], Bardaro and Mantellini defined a Mellin-type convolution operator of the form

$$(T_W f)(x_1, x_2) := \int_{\mathbb{R}^2_+} K_W\left(t x_1^{-1}, v x_2^{-1}\right) f(t, v) \frac{dt}{t}\frac{dv}{v}, \tag{1.6}$$

where $\mathscr{f}$ belongs to L^p-space, $1 \leq p < \infty$, and $K_W : \mathbb{R}_+^2 \to \mathbb{R}$ is a kernel satisfying the suitable assumptions.

In [56], Acar et al. (2024) constructed bivariate generalized Kantorovich variants of the exponential sampling series by using the bivariate Mellin–Gauss–Weierstrass convolution integral operator. For a function $\mathscr{f}$ on $\mathbb{R}_+^2$ the operator is defined by

$$(\mathcal{K}_W \mathscr{f})(x_1, x_2) := \int_{\mathbb{R}_+^2} \mathfrak{G}_W\,(t, v)\mathscr{f}(tx_1, vx_2)\frac{dt}{t}\frac{dv}{v}, \tag{1.7}$$

where the integrating measure $\frac{dt}{t}\frac{dv}{v}$ is the multiplicative (Mellin) Haar measure. The kernel $\mathfrak{G}_W$ appearing in (1.7) is the Mellin–Gauss–Weierstrass kernel given by

$$\mathfrak{G}_W(x_1, x_2) = \frac{W^2}{4\pi} e^{\left(-\frac{W^2}{4}(\log^2 x_1 + \log^2 x_2)\right)},\ (x_1, x_2) \in \mathbb{R}_+^2, W > 1. \tag{1.8}$$

The bivariate generalized Kantorovich forms of exponential sampling series is defined by

$$\left(\mathcal{K}_W^{\varrho,\mathfrak{G}} \mathscr{f}\right)(x_1, x_2)$$

$$:= \sum_{(k,j)\in\mathbb{Z}^2} \varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right) \frac{\int_{\mathbb{R}_+^2} \mathfrak{G}_W(z_1, z_2)\mathscr{f}\left(z_1 e^{\frac{k}{W}}, z_2 e^{\frac{j}{W}}\right)\frac{dz_1}{z_1}\frac{dz_2}{z_2}}{\int_{\mathbb{R}_+^2} \mathfrak{G}_{\varpi}(z_1, z_2)\frac{dz_1}{z_1}\frac{dz_2}{z_2}}$$

$$= \sum_{(k,j)\in\mathbb{Z}^2} \varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right) \int_{\mathbb{R}_+^2} \mathfrak{G}_W\left(z_1 e^{-\frac{k}{W}}, z_2 e^{-\frac{j}{W}}\right)\mathscr{f}(z_1, z_2)\frac{dz_1}{z_1}\frac{dz_2}{z_2}, \tag{1.9}$$

where $\mathscr{f} : \mathbb{R}_+^2 \to \mathbb{R}$ is an integrable function for which the above series converges for every $(x_1, x_2) \in \mathbb{R}_+^2$ and $\varrho : \mathbb{R}_+^2 \to \mathbb{R}$ is a kernel which satisfies the above assumptions.

1.2. Basic Notations

Let $\mathbb{N}^2, \mathbb{N}_0^2$ and $\mathbb{Z}^2$ denote the sets of vectors $\boldsymbol{k} = (k_1, k_2)$ whose components k_1, k_2 are, respectively, positive integers, nonnegative integers and integers. For any $\boldsymbol{k}$, we write $|\boldsymbol{k}| = k_1 + k_2$. Moreover $\mathbb{R}^2$ denotes the two-dimensional Euclidean space consisting of all vectors (x_1, x_2) with $x_1, x_2 \in \mathbb{R}$.

Let $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2) \in \mathbb{R}^2$, and $\alpha \in \mathbb{R}$. We write $\boldsymbol{x} > \boldsymbol{y}$ if and only if $x_j > y_j$, $j = 1,2$ and we denote $\mathbf{1} := (1,1)$ and $\mathbf{0} := (0,0)$. The space of all vectors $\boldsymbol{x} > \mathbf{0}$ is denoted by $\mathbb{R}_+^2$. As usual, we define:

$$\boldsymbol{x} + \boldsymbol{y} := (x_1 + y_1, x_2 + y_2),\ \alpha\boldsymbol{x} := (\alpha x_1, \alpha x_2).$$

Further, for vectors in $\mathbb{R}_+^2$, we define component-wise multiplication and division as

$$\boldsymbol{x}\boldsymbol{y} := (x_1 y_1, x_2 y_2),$$
$$\frac{\boldsymbol{x}}{\boldsymbol{y}} := \left(\frac{x_1}{y_1}, \frac{x_2}{y_2}\right).$$

We also adopt the following notation:

$$\log\boldsymbol{x} := (\log x_1, \log x_2) \text{ with } \boldsymbol{x} > \mathbf{0},$$
$$\alpha^{\boldsymbol{x}} := (\alpha^{x_1}, \alpha^{x_2}) \text{ with } \alpha > 0,$$
$$\boldsymbol{x}^{\boldsymbol{y}} := \left(x_1^{y_1}, x_2^{y_2}\right).$$

For $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^2$,

$\|\boldsymbol{x}\|:=\sqrt{x_1^2+x_2^2}$ and $d(\boldsymbol{x},\boldsymbol{y}):=\|\boldsymbol{x}-\boldsymbol{y}\|$.

Let $I\subset\mathbb{R}_+^2$ and denote by $C(I)$ the space of continuous functions on I. For $m\in\mathbb{N}$, let $C^m(I)$ be the subspace of $C(I)$ consisting of all functions f whose partial derivatives up to m exist and belong to $C(I)$.

Denote by $M(\mathbb{R}_+^2)$ the measurable functions and by $L^\infty(\mathbb{R}_+^2)$ the bounded functions. Set

$$L^1(\mathbb{R}_+^2):=\left\{f:\mathbb{R}_+^2\to\mathbb{C}: f\in M(\mathbb{R}_+^2),\int_{\mathbb{R}_+^2}|f(x_1,x_2)|\frac{dx_1}{x_1}\frac{dx_2}{x_2}<\infty\right\}.$$

Let $\boldsymbol{v}=(v_1,v_2)\in\mathbb{N}_0^2$. For $\boldsymbol{u}=(u_1,u_2)\in\mathbb{R}_+^2$, the algebraic moments of order $\boldsymbol{v}$ of ϱ are defined by

$$\mathrm{m}_{\boldsymbol{v}}^{|\boldsymbol{v}|}(\varrho,\boldsymbol{u}):=\mathrm{m}_{(v_1,v_2)}^{|\boldsymbol{v}|}(\varrho,\boldsymbol{u}):=\sum_{(k,j)\in\mathbb{Z}^2}\varrho\left(e^{-k}u_1,e^{-j}u_2\right)(k-\log u_1)^{v_1}(j-\log u_2)^{v_2}.$$

The absolute moments of order $\boldsymbol{v}$ of ϱ are defined by

$$\mathcal{M}_{\boldsymbol{v}}^{|\boldsymbol{v}|}(\varrho,\boldsymbol{u}):=\mathcal{M}_{(v_1,v_2)}^{|\boldsymbol{v}|}(\varrho,\boldsymbol{u}):=\sum_{(k,j)\in\mathbb{Z}^2}\left|\varrho\left(e^{-k}u_1,e^{-j}u_2\right)\right||k-\log u_1|^{v_1}|j-\log u_2|^{v_2}.$$

Finally, we set $\mathcal{M}_{\boldsymbol{v}}^{|\boldsymbol{v}|}(\varrho):=\sup_{\boldsymbol{u}\in\mathbb{R}_+^2}\mathcal{M}_{\boldsymbol{v}}^{|\boldsymbol{v}|}(\varrho,\boldsymbol{u})$.

Remark 1. $\mathcal{M}_{\boldsymbol{v}_1}^{|\boldsymbol{v}_1|}(\varrho)<\infty$ implies $\mathcal{M}_{\boldsymbol{v}_2}^{|\boldsymbol{v}_2|}(\varrho)<+\infty$ for any $\boldsymbol{v}_2$ such that $|\boldsymbol{v}_2|<|\boldsymbol{v}_1|$.

We recall the bivariate Mellin–Taylor formula.

Proposition 1. Let $m\in\mathbb{N}$ and $f\in C^m(\mathbb{R}_+^2)$. Fix $\boldsymbol{x}=(x_1,x_2)\in\mathbb{R}_+^2$ and let $\boldsymbol{t}=(t_1,t_2)\in\mathbb{R}_+^2$. Then

$$f(t_1x_1,t_2x_2) \tag{1.10}$$

$$=f(x_1,x_2)+\left(\Theta_{x_1}\log t_1+\Theta_{x_2}\log t_2\right)f(x_1,x_2)+\frac{1}{2!}\left(\Theta_{x_1}\log t_1+\Theta_{x_2}\log t_2\right)^2f(x_1,x_2)+\cdots$$

$$+\frac{1}{(m-1)!}\left(\Theta_{x_1}\log t_1+\Theta_{x_2}\log t_2\right)^{m-1}f(x_1,x_2)+\mathcal{R}_m(t_1,t_2),$$

where $\Theta_{x_i}f(\boldsymbol{x}):=x_i\frac{\partial f(\boldsymbol{x})}{\partial x_i}$ and the remainder is given by the Lagrange form

$$\mathcal{R}_m(t_1,t_2)=\frac{1}{m!}\left(\Theta_{x_1}\log t_1+\Theta_{x_2}\log t_2\right)^m f(r,s)$$

for some (r,s) on the straight-line segment

$$L_{(t_1,t_2)}=\{(1-\theta)(x_1,x_2)+\theta(t_1x_1,t_2x_2):0<\theta<1\}.$$

Remark 2. Logarithmic weighted spaces for bivariate functions extend the admissible target class. A function $\bar{\omega}$ is called a weight function if it is a continuous and positive function on the entire domain $\mathbb{R}_+^2$.

an important example is $\bar{\omega}(x_1,x_2)=\frac{1}{1+\log^2 x_1+\log^2 x_2}$, $x_1,x_2\in\mathbb{R}^+$. The associated logarithmic weighted space of continuous functions and its natural subspaces are defined as follows:

$$\begin{aligned}
\mathcal{B}_{\bar{\omega}}(\mathbb{R}_+^2)\quad &:=\left\{f:\mathbb{R}_+^2\to\mathbb{R}:\exists M>0 \text{ such that } \frac{|f(x_1,x_2)|}{1+\log^2 x_1+\log^2 x_2}\leq M \text{ for every } x_1,x_2\in\mathbb{R}^+\right\}\\
\mathcal{C}_{\bar{\omega}}(\mathbb{R}_+^2)\quad &:=C(\mathbb{R}_+^2)\cap\mathcal{B}_{\bar{\omega}}(\mathbb{R}_+^2)\\
\mathcal{C}_{\bar{\omega}}^*(\mathbb{R}_+^2)\quad &:=\left\{f\in\mathcal{C}_{\bar{\omega}}(\mathbb{R}_+^2):\exists \lim_{\|(x_1,x_2)\|\to\infty}\frac{|f(x_1,x_2)|}{1+\log^2 x_1+\log^2 x_2}\in\mathbb{R}\right\}.
\end{aligned}$$

Let $\mathcal{B}_{\bar{\omega}}(\mathbb{R}_+^2)$ denote the linear space introduced above and let the same symbol refer to its relevant subspaces. These spaces are equipped with the norm

$$\|f\|_{\bar{\omega}}\quad:=\sup_{(x_1,x_2)\in\mathbb{R}_+^2}\frac{|f(x_1,x_2)|}{1+\log^2 x_1+\log^2 x_2}.$$

In the aforementioned paper, a new modulus of continuity, referred to as the weighted logarithmic modulus of continuity, was also introduced in order to determine the rate of approximation of the operators in weighted spaces, that is, for $f\in\mathcal{C}_{\bar{\omega}}(\mathbb{R}_+^2)$, weighted logarithmic modulus of continuity is given by

$$\Omega(f;\varsigma_1,\varsigma_2):=\sup_{\substack{(x_1,x_2)\in\mathbb{R}_+^2\\ |\log t_1|\leq\varsigma_1,\ |\log t_2|\leq\varsigma_2}}\frac{|f(t_1x_1,t_2x_2)-f(x_1,x_2)|}{(1+\log^2 t_1+\log^2 t_2)(1+\log^2 x_1+\log^2 x_2)}. \tag{1.11}$$

We know that the modulus of continuity defined in (1.11) satisfies the following properties:

Lemma 1. Let $\varsigma_1,\varsigma_2>0$ and $n_1,n_2\in\mathbb{N}$. Then the following assertions hold:

a) for $f\in\mathcal{C}_{\bar{\omega}}(\mathbb{R}_+^2)$, $\Omega(f;\varsigma_1,\varsigma_2)<\infty$,

b) for all $f\in\mathcal{C}_{\bar{\omega}}(\mathbb{R}_+^2)$ $\Omega(f;n_1\varsigma_1,n_2\varsigma_2)\leq 2n_1^3n_2^3(1+\varsigma_1^2)(1+\varsigma_2^2)\Omega(f;\varsigma_1,\varsigma_2)$ and for any $\alpha_1,\alpha_2\in\mathbb{R}^+$,

$$\Omega(f;\alpha_1\varsigma_1,\alpha_2\varsigma_2)\leq 2(1+\alpha_1)^3(1+\alpha_2)^3(1+\varsigma_1^2)(1+\varsigma_2^2)\Omega(f;\varsigma_1,\varsigma_2),$$

c) for $f\in\mathcal{C}_{\bar{\omega}}^*(\mathbb{R}_+^2)$, $\lim_{(\varsigma_1,\varsigma_2)\to 0}\Omega(f;\varsigma_1,\varsigma_2)=0$,

d) for all $f\in\mathcal{C}_{\bar{\omega}}(\mathbb{R}_+^2)$ and $(h_1,h_2),(x_1,x_2)\in\mathbb{R}_+^2$,

$$\begin{aligned}
&|f(h_1,h_2)-f(x_1,x_2)|\\
&\leq\frac{128}{\bar{\omega}(x_1,x_2)}(1+\varsigma_1^2)^2(1+\varsigma_2^2)^2\Omega(f;\varsigma_1,\varsigma_2)\\
&\times\left[1+\frac{|\log h_1-\log x_1|^5}{\varsigma_1^5}+\frac{|\log h_2-\log x_2|^5}{\varsigma_2^5}+\frac{|\log h_1-\log x_1|^5}{\varsigma_1^5}\frac{|\log h_2-\log x_2|^5}{\varsigma_2^5}\right].
\end{aligned}$$

In the same paper again, in order to provide a quantitative form of the Voronovskaja-type formula, the following inequality was also established for $(\Theta_{x_1}f),(\Theta_{x_2}f)\in\mathcal{C}_{\bar{\omega}}^*(\mathbb{R}_+^2)$:

$$\begin{aligned}
&\left|\mathcal{R}_1\left(\frac{u_1}{x_1},\frac{u_2}{x_2}\right)\right|\\
&\leq\frac{2^{12}}{\bar{\omega}(x_1,x_2)}\Omega\left((\Theta_{x_1}f);\varsigma_1,\varsigma_2\right)\left(1+L_2^5+L_1^6+L_1^6L_2^5\right)
\end{aligned} \tag{1.12}$$

$$+\frac{2^{12}}{\bar{\omega}(x_1,x_2)}\Omega\left(\left(\Theta_{x_2}\mathscr{f}\right);\varsigma_1,\varsigma_2\right)\left(1+L_2^6+L_1^5+L_1^5L_2^6\right),$$

where $L_1:=\frac{|\log u_1-\log x_1|}{\varsigma_1}, L_2:=\frac{|\log u_2-\log x_2|}{\varsigma_2}, \varsigma_1,\varsigma_2\leq 1 \textit{ and } t_1x_1=u_1, t_2x_2=u_2.$

We are now ready to present our main results. In the present paper, we first determine an upper bound for difference of the operators $\left(\mathcal{K}_W^{\varrho,\mathfrak{G}}\right)$ and $\left(\mathcal{E}_W^{\varrho}\right)$. Subsequently, we construct a Voronovskaja-type formula in a quantitative sense using the inequality (1.12). Finally, we discuss several examples of kernels supporting the present theory.

2. MAIN RESULTS

In this section, we present and prove two main results that extend the present theory.

In the following theorem, we establish an upper bound for difference of the operators $\left(\mathcal{K}_W^{\varrho,\mathfrak{G}}\right)$ and $\left(\mathcal{E}_W^{\varrho}\right)$.

Theorem 1. Let ϱ be a kernel such that $\mathcal{M}_{(a,b)}^2(\varrho)<\infty$. If $\mathscr{f}\in\mathcal{C}_{\bar{\omega}}^*(\mathbb{R}_+^2)$, then the following inequality holds:

$$\left\|\left(\mathcal{K}_W^{\varrho,\mathfrak{G}}\mathscr{f}\right)-\left(\mathcal{E}_W^{\varrho}\mathscr{f}\right)\right\|_{\bar{\omega}}$$

$$\leq\left(2^{11}+\frac{2^{18}}{\sqrt{\pi}}+\frac{2^{23}}{\pi}\right)\Omega\left(\mathscr{f};\frac{1}{W},\frac{1}{W}\right)\mathcal{M}_{\mathbf{0}}(\varrho)$$

$$+\left(\frac{2^{11}}{W^2}+\frac{2^{18}}{\sqrt{\pi}W^2}+\frac{2^{23}}{\pi W^2}\right)\Omega\left(\mathscr{f};\frac{1}{W},\frac{1}{W}\right)\left(M_{(2,0)}^2(\varrho)+2W\mathcal{M}_{(1,0)}^1(\varrho)+W^2\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\left(\frac{2^{11}}{W^2}+\frac{2^{18}}{\sqrt{\pi}W^2}+\frac{2^{23}}{\pi W^2}\right)\Omega\left(\mathscr{f};\frac{1}{W},\frac{1}{W}\right)\left(\mathcal{M}_{(0,2)}^2(\varrho)+2W\mathcal{M}_{(0,1)}^1(\varrho)+W^2\mathcal{M}_{\mathbf{0}}(\varrho)\right).$$

Proof. By using definitions of the operators and the idendity

$$\int_{\mathbb{R}_+^2}\mathfrak{G}_W\left(z_1e^{-\frac{k}{W}},z_1e^{-\frac{j}{W}}\right)\frac{dz_1}{z_1}\frac{dz_2}{z_2}=\int_{\mathbb{R}_+^2}\mathfrak{G}_W\left(z_1,z_2\right)\frac{dz_1}{z_1}\frac{dz_2}{z_2}, \tag{2.1}$$

we can write

$$\left(\mathcal{K}_W^{\varrho,\mathfrak{G}}\mathscr{f}\right)(x_1,x_2)-\left(\mathcal{E}_W^{\varrho}\mathscr{f}\right)(x_1,x_2)$$

$$\leq\sum_{(k,j)\in\mathbb{Z}^2}\left|\varrho\left(e^{-k}x_1^W,e^{-j}x_2^W\right)\right|\int_{\mathbb{R}_+^2}\mathfrak{G}_W\left(z_1e^{-\frac{k}{W}},z_2e^{-\frac{j}{W}}\right)\left|\mathscr{f}(z_1,z_2)-\mathscr{f}\left(e^{\frac{k}{W}},e^{\frac{j}{W}}\right)\right|\frac{dz_1}{z_1}\frac{dz_2}{z_2}=:I.$$

Now, considering the property of Lemma 1 d) for $\varsigma_1,\varsigma_2\leq 1$, we obtain

$$I$$

$$\leq 2^{11}\Omega(\mathscr{f};\varsigma_1,\varsigma_2)\sum_{(k,j)\in\mathbb{Z}^2}\left|\varrho\left(e^{-k}x_1^W,e^{-j}x_2^W\right)\right|\left(1+\frac{k^2}{W^2}+\frac{j^2}{W^2}\right)\int_{\mathbb{R}_+^2}\mathfrak{G}_W\left(z_1e^{-\frac{k}{W}},z_2e^{-\frac{j}{W}}\right)$$
$$\times\left(1+\frac{\left|\log z_1-\frac{k}{W}\right|^5}{\varsigma_1^5}+\frac{\left|\log z_2-\frac{j}{W}\right|^5}{\varsigma_2^5}+\frac{\left|\log z_1-\frac{k}{W}\right|^5}{\varsigma_1^5}\frac{\left|\log z_2-\frac{j}{W}\right|^5}{\varsigma_2^5}\right)\frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

$$= 2^{11}\Omega(\not{f};\varsigma_1,\varsigma_2)\sum_{(k,j)\in\mathbb{Z}^2}\left|\varrho\left(e^{-k}x_1^W,e^{-j}x_2^W\right)\right|\left(1+\frac{k^2}{W^2}+\frac{j^2}{W^2}\right)\int_{\mathbb{R}_+^2}\mathfrak{G}_W(z_1,z_2)$$

$$\times\left(1+\frac{|\mathrm{log}z_1|^5}{\varsigma_1^5}+\frac{|\mathrm{log}z_2|^5}{\varsigma_2^5}+\frac{|\mathrm{log}z_1|^5}{\varsigma_1^5}\frac{|\mathrm{log}z_2|^5}{\varsigma_2^5}\right)\frac{dz_1}{z_1}\frac{dz_2}{z_2}=:I_1.$$

Since

$$k^2=(k-W\mathrm{log}x_1)^2+2W\mathrm{log}x_1(k-W\mathrm{log}x_1)+W^2\log^2 x_1,$$

$$j^2=(j-W\mathrm{log}x_2)^2+2W\mathrm{log}x_2(j-W\mathrm{log}x_2)+W^2\log^2 x_2,$$

we get

$$I_1$$

$$\leq 2^{11}\Omega(\not{f};\varsigma_1,\varsigma_2)\mathcal{M}_{\mathbf{0}}(\varrho)+\frac{2^{17}\Omega(\not{f};\varsigma_1,\varsigma_2)}{\sqrt{\pi}\varsigma_1^5W^5}\mathcal{M}_{\mathbf{0}}(\varrho)+\frac{2^{17}\Omega(\not{f};\varsigma_1,\varsigma_2)}{\sqrt{\pi}\varsigma_2^5W^5}\mathcal{M}_{\mathbf{0}}(\varrho)+\frac{2^{23}\Omega(\not{f};\varsigma_1,\varsigma_2)}{\pi\varsigma_1^5\varsigma_2^5W^{10}}\mathcal{M}_{\mathbf{0}}(\varrho)$$

$$+\frac{2^{11}}{W^2}\Omega(\not{f};\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(2,0)}^2(\varrho)+2W|\mathrm{log}x_1|\mathcal{M}_{(1,0)}^1(\varrho)+W^2\mathrm{log}^2x_1\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{17}}{\sqrt{\pi}\varsigma_1^5W^7}\Omega(\not{f};\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(2,0)}^2(\varrho)+2W|\mathrm{log}x_1|\mathcal{M}_{(1,0)}^1(\varrho)+W^2\mathrm{log}^2x_1\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{17}}{\sqrt{\pi}\varsigma_2^5W^7}\Omega(\not{f};\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(2,0)}^2(\varrho)+2W|\mathrm{log}x_1|\mathcal{M}_{(1,0)}^1(\varrho)+W^2\mathrm{log}^2x_1\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{23}}{\pi\varsigma_1^5\varsigma_2^5W^{12}}\Omega(\not{f};\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(2,0)}^2(\varrho)+2W|\mathrm{log}x_1|\mathcal{M}_{(1,0)}^1(\varrho)+W^2\mathrm{log}^2x_1\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{11}}{W^2}\Omega(\not{f};\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(0,2)}^2(\varrho)+2W|\mathrm{log}x_2|\mathcal{M}_{(0,1)}^1(\varrho)+W^2\mathrm{log}^2x_2\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{17}}{\sqrt{\pi}\varsigma_1^5W^7}\Omega(\not{f};\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(0,2)}^2(\varrho)+2W|\mathrm{log}x_2|\mathcal{M}_{(0,1)}^1(\varrho)+W^2\mathrm{log}^2x_2\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{17}}{\sqrt{\pi}\varsigma_2^5W^7}\Omega(\not{f};\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(0,2)}^2(\varrho)+2W|\mathrm{log}x_2|\mathcal{M}_{(0,1)}^1(\varrho)+W^2\mathrm{log}^2x_2\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{23}}{\pi\varsigma_1^5\varsigma_2^5W^{12}}\Omega(\not{f};\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(0,2)}^2(\varrho)+2W|\mathrm{log}x_2|\mathcal{M}_{(0,1)}^1(\varrho)+W^2\mathrm{log}^2x_2\mathcal{M}_{\mathbf{0}}(\varrho)\right).$$

Now, if we divide both sides by $(1+\mathrm{log}^2x_1+\mathrm{log}^2x_2)$ and take the supremum over $(x_1,x_2)\in\mathbb{R}_+^2$, we have

$$\left\|\left(\mathcal{K}_W^{\varrho,\mathfrak{G}}\not{f}\right)-\left(\mathcal{E}_W^{\phi}\not{f}\right)\right\|_{\bar{\omega}}$$

$$\leq 2^{11}\Omega(\not{f};\varsigma_1,\varsigma_2)\mathcal{M}_{\mathbf{0}}(\varrho)+\frac{2^{17}\Omega(\not{f};\varsigma_1,\varsigma_2)}{\sqrt{\pi}\varsigma_1^5W^5}\mathcal{M}_{\mathbf{0}}(\varrho)+\frac{2^{17}\Omega(\not{f};\varsigma_1,\varsigma_2)}{\sqrt{\pi}\varsigma_2^5W^5}\mathcal{M}_{\mathbf{0}}(\varrho)+\frac{2^{23}\Omega(\not{f};\varsigma_1,\varsigma_2)}{\pi\varsigma_1^5\varsigma_2^5W^{10}}\mathcal{M}_{\mathbf{0}}(\varrho)$$

$$+\frac{2^{11}}{W^2}\Omega(\not{f};\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(2,0)}^2(\varrho)+2W\mathcal{M}_{(1,0)}^1(\varrho)+W^2\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{17}}{\sqrt{\pi}\varsigma_1^5W^7}\Omega(\not{f};\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(2,0)}^2(\varrho)+2W\mathcal{M}_{(1,0)}^1(\varrho)+W^2\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{17}}{\sqrt{\pi}\varsigma_2^5W^7}\Omega(f;\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(2,0)}^2(\varrho)+2W\mathcal{M}_{(1,0)}^1(\varrho)+W^2\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{23}}{\pi\varsigma_1^5\varsigma_2^5W^{12}}\Omega(f;\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(2,0)}^2(\varrho)+2W\mathcal{M}_{(1,0)}^1(\varrho)+W^2\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{11}}{W^2}\Omega(f;\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(0,2)}^2(\varrho)+2W\mathcal{M}_{(0,1)}^1(\varrho)+W^2\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{17}}{\sqrt{\pi}\varsigma_1^5W^7}\Omega(f;\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(0,2)}^2(\varrho)+2W\mathcal{M}_{(0,1)}^1(\varrho)+W^2\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{17}}{\sqrt{\pi}\varsigma_2^5W^7}\Omega(f;\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(0,2)}^2(\varrho)+2W\mathcal{M}_{(0,1)}^1(\varrho)+W^2\mathcal{M}_{\mathbf{0}}(\varrho)\right)$$

$$+\frac{2^{23}}{\pi\varsigma_1^5\varsigma_2^5W^{12}}\Omega(f;\varsigma_1,\varsigma_2)\left(\mathcal{M}_{(0,2)}^2(\varrho)+2W\mathcal{M}_{(0,1)}^1(\varrho)+W^2\mathcal{M}_{\mathbf{0}}(\varrho)\right).$$

Finally, if we set $\varsigma_1=\varsigma_2=W^{-1}$, the proof is completed. □

In the following theorem, we present a quantitative form of Voronovskaja-type theorem.

Theorem 2. Let ϱ be a kernel such that $M_{(a,b)}^{11}(\varrho)<\infty$. If $\left(\Theta_{x_1}f\right),\left(\Theta_{x_2}f\right)\in\mathcal{C}_{\bar{\omega}}^*(\mathbb{R}_+^2)$ and $m_{(a,b)}^1(\varrho,\boldsymbol{x}):=m_{(a,b)}^1(\varrho)\neq0$ is independent of $\boldsymbol{x}$, we obtain

$$\left|W\left[\left(\mathcal{K}_W^{\varrho,\mathfrak{G}}f\right)(x_1,x_2)-f(x_1,x_2)\right]-\left(\Theta_{x_1}f\right)(x_1,x_2)m_{(1,0)}^1(\varrho)+\left(\Theta_{x_2}f\right)(x_1,x_2)m_{(0,1)}^1(\varrho)\right|$$

$$\leq\frac{1}{\bar{\omega}(x_1,x_2)}\{\Omega\left(\Theta_{x_1}f;\frac{1}{W},\frac{1}{W}\right)\left[\left(121\cdot2^{12}+2^{22}/\sqrt{\pi}\right)\mathcal{M}_{\mathbf{0}}(\varrho)+2^{16}\mathcal{M}_{(0,5)}^5(\varrho)+2^{17}\mathcal{M}_{(6,0)}^6(\varrho)+64\cdot120\cdot2^{12}/\sqrt{\pi}\,\mathcal{M}_{(6,5)}^{11}(\varrho)\right]$$

$$+\Omega\left(\Theta_{x_2}f;\frac{1}{W},\frac{1}{W}\right)\left[\left(121\cdot2^{12}+2^{22}/\sqrt{\pi}\right)\mathcal{M}_{\mathbf{0}}(\varrho)+2^{17}\mathcal{M}_{(0,6)}^6(\varrho)+2^{17}\mathcal{M}_{(5,0)}^5(\varrho)+64\cdot120\cdot2^{12}/\sqrt{\pi}\,\mathcal{M}_{(5,6)}^{11}(\varrho)\right]\}.$$

Proof. By applying the Mellin–Taylor expansion (1.10) to the function f at the point $\boldsymbol{x}\in\mathbb{R}_+^2$ and using the identity given in (2.1), we obtain

$$\left(\mathcal{K}_W^{\varrho,\mathfrak{G}}f\right)(x_1,x_2)$$

$$=\sum_{(k,j)\in\mathbb{Z}^2}\varrho\left(e^{-k}x_1^W,e^{-j}x_2^W\right)\int_{\mathbb{R}_+^2}\mathfrak{G}_{\varpi}\left(z_1e^{-\frac{k}{W}},z_2e^{-\frac{j}{W}}\right)\left(f(x_1,x_2)+\left(\Theta_{x_1}f\right)(x_1,x_2)(\log z_1-\log x_1)\right.$$

$$\left.+\left(\Theta_{x_2}f\right)(x_1,x_2)(\log z_2-\log x_2)\right)\frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

$$+\sum_{(k,j)\in\mathbb{Z}^2}\varrho\left(e^{-k}x_1^W,e^{-j}x_2^W\right)\int_{\mathbb{R}_+^2}\mathfrak{G}_W(z_1,z_2)R_1\left(z_1e^{\frac{k}{W}},z_2e^{\frac{j}{W}}\right)\frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

$$=f(x_1,x_2)+\frac{\left(\Theta_{x_1}f\right)(x_1,x_2)}{W}m_{(1,0)}^1(\varrho)+\frac{\left(\Theta_{x_2}f\right)(x_1,x_2)}{W}m_{(0,1)}^1(\varrho)+J,$$

where $J =: \sum_{(k,j)\in\mathbb{Z}^2} \varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right) \int_{\mathbb{R}_+^2} \mathfrak{G}_W\,(z_1,z_2) R_1\left(z_1 e^{\frac{k}{W}}, z_2 e^{\frac{j}{W}}\right) \frac{dz_1}{z_1}\frac{dz_2}{z_2}$. Now, employing the inequality (1.12), we get

$$|J|$$

$$\leq \sum_{(k,j)\in\mathbb{Z}^2} \left|\varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right)\right| \int_{\mathbb{R}_+^2} \mathfrak{G}_W\,(z_1,z_2) \left|R_1\left(z_1 e^{\frac{k}{W}}, z_2 e^{\frac{j}{W}}\right)\right| \frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

$$\leq \frac{2^{12}}{\bar{\omega}(x_1,x_2)} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_1}\not{f}\right);\varsigma_1,\varsigma_2\right) \sum_{(k,j)\in\mathbb{Z}^2} \left|\varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right)\right|$$

$$+\frac{2^{12}}{\bar{\omega}(x_1,x_2)} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_1}\not{f}\right);\varsigma_1,\varsigma_2\right) \sum_{(k,j)\in\mathbb{Z}^2} \left|\varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right)\right| \int_{\mathbb{R}_+^2} \mathfrak{G}_W\,(z_1,z_2) \frac{\left|\log\left(z_2 e^{\frac{j}{W}}\right)-\log x_2\right|^5}{\varsigma_2^5} \frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

$$+\frac{2^{12}}{\bar{\omega}(x_1,x_2)} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_1}\not{f}\right);\varsigma_1,\varsigma_2\right) \sum_{(k,j)\in\mathbb{Z}^2} \left|\varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right)\right| \int_{\mathbb{R}_+^2} \mathfrak{G}_W\,(z_1,z_2) \frac{\left|\log\left(z_1 e^{\frac{k}{W}}\right)-\log x_1\right|^6}{\varsigma_1^6} \frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

$$+\frac{2^{12}}{\bar{\omega}(x_1,x_2)} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_1}\not{f}\right);\varsigma_1,\varsigma_2\right) \sum_{(k,j)\in\mathbb{Z}^2} \left|\varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right)\right| \int_{\mathbb{R}_+^2} \mathfrak{G}_W\,(z_1,z_2)$$

$$\times \frac{\left|\log\left(z_1 e^{\frac{k}{W}}\right)-\log x_1\right|^6}{\varsigma_1^6} \frac{\left|\log\left(z_2 e^{\frac{j}{W}}\right)-\log x_2\right|^5}{\varsigma_2^5} \frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

$$+\frac{2^{12}}{\bar{\omega}(x_1,x_2)} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_2}\not{f}\right);\varsigma_1,\varsigma_2\right) \sum_{(k,j)\in\mathbb{Z}^2} \left|\varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right)\right|$$

$$+\frac{2^{12}}{\bar{\omega}(x_1,x_2)} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_2}\not{f}\right);\varsigma_1,\varsigma_2\right) \sum_{(k,j)\in\mathbb{Z}^2} \left|\varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right)\right| \int_{\mathbb{R}_+^2} \mathfrak{G}_W\,(z_1,z_2) \frac{\left|\log\left(z_2 e^{\frac{j}{W}}\right)-\log x_2\right|^6}{\varsigma_2^6} \frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

$$+\frac{2^{12}}{\bar{\omega}(x_1,x_2)} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_2}\not{f}\right);\varsigma_1,\varsigma_2\right) \sum_{(k,j)\in\mathbb{Z}^2} \left|\varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right)\right| \int_{\mathbb{R}_+^2} \mathfrak{G}_W\,(z_1,z_2) \frac{\left|\log\left(z_1 e^{\frac{k}{W}}\right)-\log x_1\right|^5}{\varsigma_1^5} \frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

$$+\frac{2^{12}}{\bar{\omega}(x_1,x_2)} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_2}\not{f}\right);\varsigma_1,\varsigma_2\right) \sum_{(k,j)\in\mathbb{Z}^2} \left|\varrho\left(e^{-k}x_1^W, e^{-j}x_2^W\right)\right| \int_{\mathbb{R}_+^2} \mathfrak{G}_W\,(z_1,z_2)$$

$$\times \frac{\left|\log\left(z_1 e^{\frac{k}{W}}\right)-\log x_1\right|^5}{\varsigma_1^5} \frac{\left|\log\left(z_2 e^{\frac{j}{W}}\right)-\log x_2\right|^6}{\varsigma_2^6} \frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

$$\leq \frac{2^{12}}{\bar{\omega}(x_1,x_2)} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_1}\not{f}\right);\varsigma_1,\varsigma_2\right) \mathcal{M}_{\mathbf{0}}(\varrho) + \frac{2^{22}}{\bar{\omega}(x_1,x_2)\sqrt{\pi}\delta_2^5 W^5} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_1}\not{f}\right);\varsigma_1,\varsigma_2\right) \mathcal{M}_{\mathbf{0}}(\varrho)$$

$$+\frac{2^{16}}{\bar{\omega}(x_1,x_2)\varsigma_2^5 W^5} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_1}\not{f}\right);\varsigma_1,\varsigma_2\right) \mathcal{M}_{(0,5)}^5(\varrho) + \frac{120\times 2^{12}}{\bar{\omega}(x_1,x_2)\varsigma_1^6 W^6} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_1}\not{f}\right);\varsigma_1,\varsigma_2\right) \mathcal{M}_{\mathbf{0}}(\varrho)$$
$$+\frac{2^{17}}{\bar{\omega}(x_1,x_2)\varsigma_1^6 W^6} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_1}\not{f}\right);\varsigma_1,\varsigma_2\right) \mathcal{M}_{(6,0)}^6(\varrho) + \frac{64\times 120\times 2^{12}}{\bar{\omega}(x_1,x_2)\sqrt{\pi}\varsigma_1^6\varsigma_2^5 W^{11}} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_1}\not{f}\right);\varsigma_1,\varsigma_2\right) \mathcal{M}_{(6,5)}^{11}(\varrho)$$

$$+\frac{2^{12}}{\bar{\omega}(x_1,x_2)} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_2}\not{f}\right);\varsigma_1,\varsigma_2\right) \mathcal{M}_{\mathbf{0}}(\varrho) + \frac{120\times 2^{12}}{\bar{\omega}(x_1,x_2)\varsigma_2^6 W^6} \mathit{\Omega}\left(\left(\mathit{\Theta}_{x_2}\not{f}\right);\varsigma_1,\varsigma_2\right) \mathcal{M}_{\mathbf{0}}(\varrho)$$

$$+\frac{2^{17}}{\bar{\omega}(x_1,x_2)\varsigma_2^6 W^6}\Omega\left((\Theta_{x_2}f);\varsigma_1,\varsigma_2\right)\mathcal{M}^6_{(0,6)}(\varrho)+\frac{2^{22}}{\bar{\omega}(x_1,x_2)\sqrt{\pi}\varsigma_1^5 W^5}\Omega\left((\Theta_{x_2}f);\varsigma_1,\varsigma_2\right)\mathcal{M}_{\mathbf{0}}(\varrho)$$

$$+\frac{2^{17}}{\bar{\omega}(x_1,x_2)\varsigma_1^5 W^5}\Omega\left((\Theta_{x_2}f);\varsigma_1,\varsigma_2\right)\mathcal{M}^5_{(5,0)}(\varrho)+\frac{64\times 120\times 2^{12}}{\bar{\omega}(x_1,x_2)\sqrt{\pi}\varsigma_1^5\varsigma_2^6 W^{11}}\Omega\left((\Theta_{x_2}f);\varsigma_1,\varsigma_2\right)\mathcal{M}^{11}_{(5,6)}(\varrho).$$

Finally, if we set $\varsigma_1=\varsigma_2=W^{-1}$, the proof is completed. □

3. SEVERAL EXAMPLES OF KERNELS

In this section, we present specific kernel functions that satisfy the required assumptions. To begin, we recall the constructions of univariate kernel functions.

3.1. Mellin-Spline Kernel

For $n\in\mathbb{N}$, the one-dimensional Mellin-Spline of order n is defined as

$$\mathfrak{B}_n(x):=\frac{1}{(n-1)!}\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{\ell}\left(\frac{n}{2}+\log x-\ell\right)_+^{n-1},\ x\in\mathbb{R}_+.$$

By taking the tensor (product) of the one-dimensional Mellin–Spline kernels, one obtains a separable, box-type Mellin–Spline kernel. For $n,m\in\mathbb{N}$, define the kernel $\mathfrak{B}_{n,m}:\mathbb{R}_+^2\to\mathbb{R}$ by

$$\mathfrak{B}_{n,m}(x_1,x_2):=\mathfrak{B}_n(x_1)\mathfrak{B}_m(x_2),\ (x_1,x_2)\in\mathbb{R}_+^2,$$

where $\mathfrak{B}_n$ and $\mathfrak{B}_m$ are the one-dimensional Mellin–Spline kernels of orders n and m, respectively. This kernel is separable (box-type) and inherits regularity and moment properties from the one-dimensional factors $\mathfrak{B}_n$ and $\mathfrak{B}_m$.

Figure 1 shows graph of the bivariate Mellin-Spline kernel for $n=m=3$.

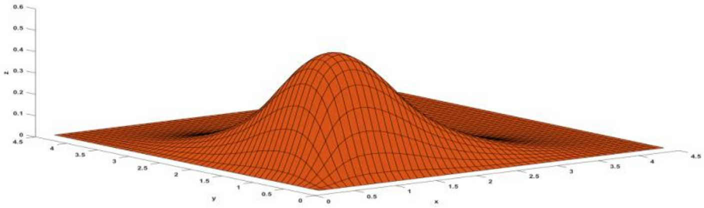

Figure 1. *Graph of* $\mathfrak{B}_{n,m}$ *for* $n=m=3$.

3.2. Mellin-Jackson Kernel

For $c\in\mathbb{R}$, the one-dimensional generalized Mellin–Jackson kernel is defined by

$$\mathfrak{J}_{\gamma,\beta}(x):=d_{\gamma,\beta}t^{-c}\operatorname{sinc}^{2\beta}\left(\frac{\log x}{2\gamma\beta\pi}\right),$$

where $x\in\mathbb{R}_+,\beta\in\mathbb{N},\gamma\geq 1$, and $d_{\gamma,\beta}$ is the normalization constant determined by $d_{\gamma,\beta}^{-1}:=\int_{\mathbb{R}_+}\operatorname{sinc}^{2\beta}\left(\frac{\log y}{2\gamma\beta\pi}\right)\frac{dy}{y}$.

With the choice $c=0,\gamma=1$ and $\beta=2$, the corresponding box-type (separable) Mellin–Jackson kernel on $\mathbb{R}_+^2$ is given by the tensor product

$$\widetilde{\mathfrak{J}}_{1,2}(x_1,x_2) := \mathfrak{J}_{1,2}(x_1)\mathfrak{J}_{1,2}(x_2),\ (x_1,x_2) \in \mathbb{R}^2_+.$$

Figure 2 shows graph of the bivariate Mellin-Jackson kernel for $c = 0, \gamma = 1$ and $\beta = 2$.

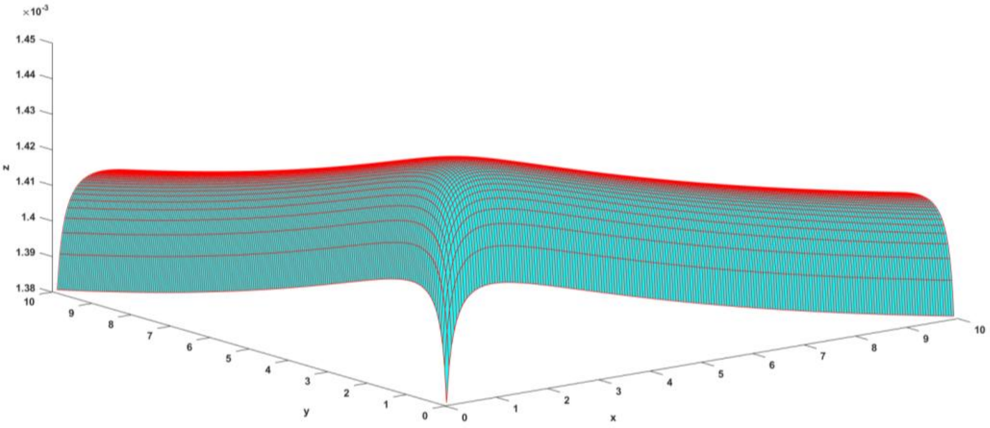

Figure 2. *Graph of* $\widetilde{\mathfrak{J}}_{1,2}$.

DECLARATION OF ARTIFICIAL INTELLIGENCE (AI) USE

Generative AI agents or large language models (LLMs) were used only to improve clarity and quality of the English language presentation of the manuscript.

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

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