A NEW FAMILY OF THE CONTINUOUS DISTRIBUTIONS: THE EXTENDED WEIBULL-G FAMILY

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ABSTRACT. In this study, we present a new family of continuous distributions via an extended form of the Weibull distribution. Some special members of the newly defined family are discussed and the new univariate continuous distributions are introduced. The mathematical properties are obtained for any members of the family such as expansions of the density, hazard rate function, quantile function, moments and order statistics. We obtain the distribution parameters by maximum likelihood method. The simulation study to evaluate the performance of the estimated parameters based on the selected member of the this new family is also given. The lifetime data example is discussed to illustrate the applicability of the distribution.

1. INTRODUCTION

The ordinary distributions can not always be sufficient to model the real data. Furthermore, in many applied areas such as lifetime analysis there is a strong need for extended forms of the ordinary distributions. There are several methods to obtain generalized or generated G families of the distributions by adding one or more flexibility parameters to baseline distribution in the literature. In this method, the baseline distributions have been developed in terms of either flexibility or the goodness-of-fit to real data. The well-known distribution families are the followings: the Marshall-Olkin-G (MO-G) by Marshall and Olkin [27], the beta-G by Eugene et al. [21], the gamma-G by Zografos and Balakrishanan [41], the Kumaraswamy-G (Kw-G) by Cordeiro and de Castro [16], the McDonald-G by Alexander et al. [2], the odd exponentiated generalized-G by Cordeiro et al. [17], the transformed-transformer (T-X) by Alzaatreh et al. [8], the exponentiated T-X by Alzaghal et al. [11], the Weibull-G by Bourguignon et al. [13], T-X{Y}-quantile based approach by Aljarrah et al. [6], T-normal{Y} by Alzaatreh et al. [10], Lomax-G by Cordeiro

et al. [18], the Kumaraswamy odd log-logistic-G by Alizadeh et al. [5], the two-sided generalized-G by Korkmaz and Genç [25], the Harris-G family by Batsidis and Lemonte[12], the beta-MO family by Alizadeh et al.[3], the Kw-MO-G by Alizadeh et al.[4], the logistic-G by Tahir et al. [37], the type 2 Weibull-G by Tahir et al. [39], generalized gamma-G by Alzaatreh et al. [9] and the beta odd log-logistic-G by Cordeiro et al. [15]. For more details on some well-established-G families, the one may see Tahir and Nadarajah [38].

Alzaatreh et al. [8] defined the T-X family of the cumulative distribution function (cdf) by

$$F\left(x\right) = \int_a^{W[G(x;\boldsymbol{\xi})]} r\left(t\right)dt, \tag{1.1}$$

where $r(t)$ the probability density function (pdf) of the random variable of $T \in [a,b]$ for $-\infty < a < b < \infty$, $G\left(x\right)$ is a cdf with parameter vector $\boldsymbol{\xi}$ of the baseline X random variable and $W\left[G\left(x\right)\right]$ is a function of the baseline cdf which satisfies the following conditions: i)$W\left[G\left(x\right)\right] \in [a,b]$, ii)$W\left[G\left(x\right)\right]$ is the differentiable and monotonically non-decreasing, iii)$\lim_{x \to -\infty} W\left[G\left(x\right)\right] = a$ and $\lim_{x \to \infty} W\left[G\left(x\right)\right] = b$. The equation (1.1) explains the above all distribution families with one formulation. The pdf of the $T - X$ family is given by

$$f\left(x\right) = \left\{\frac{\partial}{\partial x} W\left[G\left(x;\boldsymbol{\xi}\right)\right]\right\} r\left(W\left[G\left(x;\boldsymbol{\xi}\right)\right]\right). \tag{1.2}$$

We note that the support of the $T - X$ family is same with X random variable. On the other hand, the Weibull distribution is the most popular distribution in the statistics and reliability studies. The cdf and pdf of the Weibull distribution are given by

$$G_W\left(x;\alpha,\beta\right) = 1 - \mathrm{e}^{-\alpha x^\beta}$$

and

$$g_W\left(x;\alpha,\beta\right) = \alpha\beta x^{\beta-1}\mathrm{e}^{-\alpha x^\beta},$$

respectively, where $x > 0$, $\alpha > 0$ is scale parameter and $\beta > 0$ is shape parameter. Its pdf shapes have unimodal or reversed J-shaped. Also, its hazard rate function (hrf), defined by $h\left(x;\boldsymbol{\xi}\right) = g\left(x;\boldsymbol{\xi}\right)/(1 - G\left(x;\boldsymbol{\xi}\right))$, has the monotone shaped such as decreasing, increasing, and constant shaped. However, these properties can be inadequate for the Weibull distribution since the empirical approaches to real data are often non-monotone hrf shapes such as unimodal, bathtube, and various shaped specifically in the lifetime applications. Thus, in the literature there are many generalizations that makes it more flexible. For example, the extended Weibull distribution, denoted by $EW\left(\alpha,\beta,\lambda\right)$, have been introduced by Peng and Yang [32] with following cdf, pdf and hrf

$$G_{EW}(x; \alpha, \beta, \lambda) = 1 - \exp\left\{-\alpha x^\beta e^{-\lambda/x}\right\}, \tag{1.3}$$

$$g_{EW}(x; \alpha, \beta, \lambda) = \alpha(\lambda + \beta x) x^{\beta-2} \exp\left\{-\alpha x^\beta e^{-\lambda/x} - \lambda/x\right\} \tag{1.4}$$

and

$$h_{EW}(x; \alpha, \beta, \lambda) = \alpha(\lambda + \beta x) x^{\beta-2} \exp\left\{-\lambda/x\right\},$$

respectively, where $x > 0$, $\alpha > 0$ is scale parameter and $\beta, \lambda > 0$ are shape parameters. The authors have shown that the EW density has unimodal shape and its hrf is the unimodal or increasing shaped. When $\lambda = 0$, the EW distribution reduces the ordinary Weibull distribution. A good review on Weibull models, and its modifications are argued in Lai et al. [26] and Almalki and Nadarajah [7]. Also, there are some generated families based on ordinary Weibull distribution in the literature. They are the following: the Weibull-X family [8, 19], the Weibull-G family[13], the exponentiated Weibull-X family[11], the second type Weibull-G family[39] and the additive Weibull-G family[23].

The aim of the paper is to propose a new flexible family of distributions based on T-X family using the EW as the generator. In this way, we will utilize the flexibility of the baseline distribution for modelling the data. The paper is outlined as followings. In Section 2, we define the new Weibull-G family. The special cases and some members of the new family is given Section 3. In section 4 provides the expansions for the new density. The quantile function is given by Section 5. The moment and moment generating functions is obtained by Section 6. Section 7 gives the order statistics and its moments. In Section 8, the estimation of the model parameters is obtained by the method of maximum likelihood. In Section 9 includes a simulation study for the maximum likelihood estimations of the parameters and an application to real lifetime data. Finally, concluding remarks are presented in Section 10.

2. The New Family

Let $G(x; \boldsymbol{\xi})$ and $g(x; \boldsymbol{\xi})$ are the baseline cdf and pdf belong to continuous random variable respectively. Applying $W[G(x; \boldsymbol{\xi})] = \frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}$ from (1.1), (1.2), (1.3) and (1.4) we define the cdf and pdf of the new family by

$$
\begin{aligned}
F_{EW-G}(x; \alpha, \beta, \lambda, \boldsymbol{\xi}) &= \int_0^{\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}} f_{EW}(t; \alpha, \beta, \lambda)\, dt \\
&= 1 - \exp\left\{-\alpha\left(\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right)^\beta e^{-\lambda\left(\frac{1-G(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})}\right)}\right\} \tag{2.1}
\end{aligned}
$$

and

$$f_{EW-G}(x; \alpha, \beta, \lambda, \boldsymbol{\xi}) = \alpha g(x; \boldsymbol{\xi}) \left[\lambda + \beta \left(\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})} \right) \right] \frac{G(x;\boldsymbol{\xi})^{\beta-2}}{[1 - G(x;\boldsymbol{\xi})]^{\beta}}$$

$$\times \exp \left\{ -\alpha \left(\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})} \right)^{\beta} e^{-\lambda \left(\frac{1-G(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})} \right)} - \lambda \left(\frac{1-G(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})} \right) \right\} \quad (2.2)$$

respectively, where $\alpha > 0$ is scale parameter and $\beta, \lambda > 0$ are shape parameters. In this way, we extend the baseline distribution G with three extra parameters. Thus, the additional parameters will ensure to establish a more flexible distribution and we will obtain various shapes of the pdf and hrf. Further, the main motivations for using the new family are to make the kurtosis more flexible compared to the baseline, to produce a skewness for symmetrical distributions, to generate various shapes of the pdf and hrf and to provide better fits than other generated models. We can also say that if T has EW random variable, then $X = G^{-1}\left(\frac{T}{T+1}\right)$ random variable has the pdf in (2.2). We call new family the extended Weibull-G distribution and denote it by $X \sim EW - G(\alpha, \beta, \lambda, \boldsymbol{\xi})$.
The hrf of the $EW - G$ family is given by

$$h_{EW-G}(x; \alpha, \beta, \lambda; \boldsymbol{\xi}) = \alpha g(x; \boldsymbol{\xi}) \left[\lambda + \beta \left(\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})} \right) \right] \frac{G(x;\boldsymbol{\xi})^{\beta-2}}{[1-G(x;\boldsymbol{\xi})]^{\beta}}$$

$$\times \exp \left\{ -\lambda \left(\frac{1-G(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})} \right) \right\}.$$

3. Some special $EW - G$ distributions

The $EW - G$ family presents alternative generalized distributions, since (2.1) and (2.2) generate the more flexible distribution than the baseline distribution. The members of the $EW - G$ family of distributions are specialized by taking G as the well-known distributions in the literature. Some of them are $EW - uniform$, $EW - Weibull$ and $EW - normal$ distributions (please see Table 1 for the others). Also, we note that when $\lambda = 0$, the $EW - G$ family is the Weibull-G family [13] and it is reduced to a new family, which is named by extended exponential-G family of the distributions, for the $\beta = 1$. The details are given in the following subsections.

3.1. **EW-Uniform Distribution.** As in the first example, suppose that the baseline distribution has an uniform distribution in the interval $(0, \theta)$, where $\boldsymbol{\xi} = (\theta)$. Then $G(x; \theta) = x/\theta$ and $g(x; \theta) = 1/\theta$. The EW-Uniform (EW-U) cdf is given by

$$F_{EW-U}(x; \alpha, \beta, \lambda, \theta) = 1 - \exp \left\{ -\alpha \left(\frac{x}{\theta-x} \right)^{\beta} e^{-\lambda \left(\frac{\theta-x}{x} \right)} \right\}, \quad (3.1)$$

Table 1. Some members of the EW-G family

Distribution	ξ	$\dfrac{G(x;\xi)}{1-G(x;\xi)}$	CDF
Exponential $(x>0)$	θ	$e^{\theta x}-1$	$1-\exp\left\{-\alpha\left(e^{\theta x}-1\right)^{\beta}e^{-\lambda/\left(e^{\theta x}-1\right)}\right\}$
Fréchet $(x>0)$	(θ,γ)	$\left(e^{\theta x^{-\gamma}}-1\right)^{-1}$	$1-\exp\left\{-\alpha\left(e^{\theta x^{-\gamma}}-1\right)^{-\beta}e^{-\lambda\left(e^{\theta x^{-\gamma}}-1\right)}\right\}$
Pareto $(x>\theta)$	(θ,k)	$(x/\theta)^{k}-1$	$1-\exp\left\{-\alpha\left((x/\theta)^{k}-1\right)^{-\beta}e^{-\lambda/\left((x/\theta)^{k}-1\right)}\right\}$
Gumbel $(-\infty<x<\infty)$	(μ,σ)	$\left(e^{e^{-(x-\mu)/\sigma}}-1\right)^{-1}$	$1-\exp\left\{-\alpha\left(e^{e^{-(x-\mu)/\sigma}}-1\right)^{-\beta}e^{-\lambda\left(e^{e^{-(x-\mu)/\sigma}}-1\right)}\right\}$
Kumaraswamy $(0<x<1)$	(a,b)	$\left(1-x^{a}\right)^{-b}-1$	$1-\exp\left\{-\alpha\left[\left(1-x^{a}\right)^{-b}-1\right]^{-\beta}e^{-\lambda/\left[\left(1-x^{a}\right)^{-b}-1\right]}\right\}$
Two-sided power [40] $(0<x<1)$	(a,b)	$\begin{cases}\left(\frac{1}{b}\left(\frac{b}{x}\right)^{a}-1\right)^{-1}, & 0<x<b \\ \frac{1}{1-b}\left(\frac{1-b}{1-x}\right)^{a}-1, & b<x<1\end{cases}$	$1-\begin{cases}e^{-\alpha\left[\frac{1}{b}\left(\frac{b}{x}\right)^{a}-1\right]^{-\beta}-\lambda\left[\frac{1}{b}\left(\frac{b}{x}\right)^{a}-1\right]}, & 0<x<b \\ e^{-\alpha\left[\frac{1}{1-b}\left(\frac{1-b}{1-x}\right)^{a}-1\right]^{-\beta}/e^{-\lambda/\left[\frac{1}{1-b}\left(\frac{1-b}{1-x}\right)^{a}-1\right]}}, & b<x<1\end{cases}$
Lindley $(x>0)$	θ	$\frac{\theta+1}{\theta+1+\theta x}e^{\theta x}-1$	$1-\exp\left\{-\alpha\left(\frac{\theta+1}{\theta+1+\theta x}e^{\theta x}-1\right)^{\beta}e^{-\lambda/\left(\frac{\theta+1}{\theta+1+\theta x}e^{\theta x}-1\right)}\right\}$

where $0 < x < \theta$; $\alpha, \beta, \lambda > 0$. For $\lambda = 0$, we obtain the Weibull-uniform distribution [13] or Phani distribution [33]. Furthermore, the (3.1) can be named as the extended Phani distribution. The pdf of (3.1) is the following

$$f_{EW-U}\left(x; \alpha, \beta, \lambda, \theta\right) = \alpha\theta \left[\lambda + \beta\left(\frac{x}{\theta-x}\right)\right] \frac{x^{\beta-2}}{(\theta-x)^{\beta}}$$
$$\times \exp\left\{-\alpha\left(\frac{x}{\theta-x}\right)^{\beta} e^{-\lambda\left(\frac{\theta-x}{x}\right)} - \lambda\left(\frac{\theta-x}{x}\right)\right\}.$$

3.2. EW-Weibull Distribution.

We now consider the Weibull distribution as a baseline distribution with pdf $g\left(x; \gamma, \theta\right) = \gamma\theta x^{\gamma-1} e^{-\theta x^{\gamma}}$ and cdf $G\left(x; \theta, \gamma\right) = 1 - e^{-\theta x^{\gamma}}$, where $\boldsymbol{\xi} = (\gamma, \theta)$. Then we have the cdf of the EW-Weibull distribution (EW-W) by the following

$$F_{EW-W}\left(x; \alpha, \beta, \lambda, \gamma, \theta\right) = 1 - \exp\left\{-\alpha\left(e^{\theta x^{\gamma}} - 1\right)^{\beta} e^{-\lambda/\left(e^{\theta x^{\gamma}}-1\right)}\right\}, \qquad (3.2)$$

where where $x > 0$ and $\alpha, \beta, \lambda, \gamma, \theta > 0$. The (3.2) contains some important submodels which some of them are newly defined. These models are the followings:

- When $\lambda = 0$, we obtained Weibull-Weibull distribution [13].
- The ordinary exponential power distribution [36] is obtained $\alpha = \beta = 1$ and $\lambda = 0$.
- The Chen distribution [14] is obtained by $\theta = \beta = 1$ and $\lambda = 0$.
- For $\gamma = 1$, the EW-exponential distribution, denoted by $EW-E\left(\alpha, \beta, \lambda, \theta\right)$, is obtained (new).
- The extended exponential-Weibull distribution is obtained for $\beta = 1$ (new).
- The model reduces to the extended exponential power distribution for $\alpha = \beta = 1$ (new).
- The extended Chen distribution is obtained by $\theta = \beta = 1$ (new).
- For $\beta = \gamma = 1$ and $\alpha = \delta/\theta$, $(\delta > 0)$, the extended Gompertz distribution is obtained (new). The ordinary Gompertz distribution is obtained for the same case with $\lambda = 0$.

The pdf of the EW-W distribution is given by

$$f_{EW-W}\left(x; \alpha, \beta, \lambda, \gamma, \theta\right) = \alpha\gamma\theta x^{\gamma-1} e^{(\beta-1)\theta x^{\gamma}} \left[\lambda + \beta\left(e^{\theta x^{\gamma}} - 1\right)\right]$$
$$\times \left(1 - e^{-\theta x^{\gamma}}\right)^{\beta-2} e^{-\alpha\left(e^{\theta x^{\gamma}}-1\right)^{\beta} e^{-\lambda/\left(e^{\theta x^{\gamma}}-1\right)} - \lambda/\left(e^{\theta x^{\gamma}}-1\right)}.$$

3.3. EW-normal Distribution.

We define the EW-normal (EW-N) distribution from (2.1) by taking $G\left(x; \mu, \sigma\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$ and $g\left(x; \mu, \sigma\right) = \sigma^{-1}\phi\left(\frac{x-\mu}{\sigma}\right)$ to be the cdf and pdf of the normal distribution with $\boldsymbol{\xi} = (\mu, \sigma)$, respectively and where

$\phi\left(\cdot\right)$ and $\Phi\left(\cdot\right)$ are the pdf and cdf of the standard normal distribution, respectively. Then, the EW-N cdf is given by

$$F_{EW-N}\left(x\right) = 1 - \exp\left\{-\alpha\left(\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right)^{\beta} e^{-\lambda\left(\Phi\left(\frac{x-\mu}{\sigma}\right)^{-1}-1\right)}\right\}, \qquad (3.3)$$

where $-\infty < x, \mu < \infty$ and $\alpha, \beta, \lambda, \sigma > 0$. For $\mu = 0$ and $\sigma = 1$, we obtain the standard EW-N distribution. Also for $\lambda = 0$, the Weibull-normal distribution [13] is obtained. The corresponding pdf of the (3.3) is

$$f_{EW-N}\left(x\right) = \frac{\alpha}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\left[\lambda + \beta\left(\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right)\right]\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)^{\beta-2}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta}}$$

$$\times \exp\left\{-\alpha\left(\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right)^{\beta} e^{-\lambda\left(\Phi\left(\frac{x-\mu}{\sigma}\right)^{-1}-1\right)} - \lambda\left(\Phi\left(\frac{x-\mu}{\sigma}\right)^{-1}-1\right)\right\}.$$

In Figures 1 and 2, we draw some plots of the pdf and hrf of the EW-U, EW-W, EW-N distributions for selected parameter values. Figure 1 shows that the EW-G family ensures rich shaped distributions with various shapes for modelling. For example, the EW-G family brings not only bi-modal shape and peak point properties to ordinary Weibull and normal distributions but also brings peak point property to ordinary uniform distribution. It also brings uni-modal, bathtub and increasing shape properties to ordinary uniform distribution. Figure 2 reveals that this family can produce flexible hrf shapes such as decreasing, increasing, bathtub, upside-down bathtub, firstly unimodal then increasing. Other shapes can be obtained using another distribution. These shape properties show that the EW-G family can be very useful to fit different data sets with various shapes.

4. The Expansions for the EW-G density

In this Section, using the exponential power series expansions and generalized binom expansions we will obtain expansions of the EW-G density. These expansions are $\exp\left(z\right) = \sum\limits_{i=0}^{\infty}\frac{z^{i}}{i!}$ and $\left[1 - G\left(x;\boldsymbol{\xi}\right)\right]^{-a} = \sum_{j=0}^{\infty}\frac{\Gamma\left(a+j\right)}{j!\Gamma\left(a\right)}G\left(x;\boldsymbol{\xi}\right)^{j}$ respectively, where $\Gamma\left(a\right) = \int_{0}^{\infty}t^{a-1}e^{-t}dt$ is the gamma function.

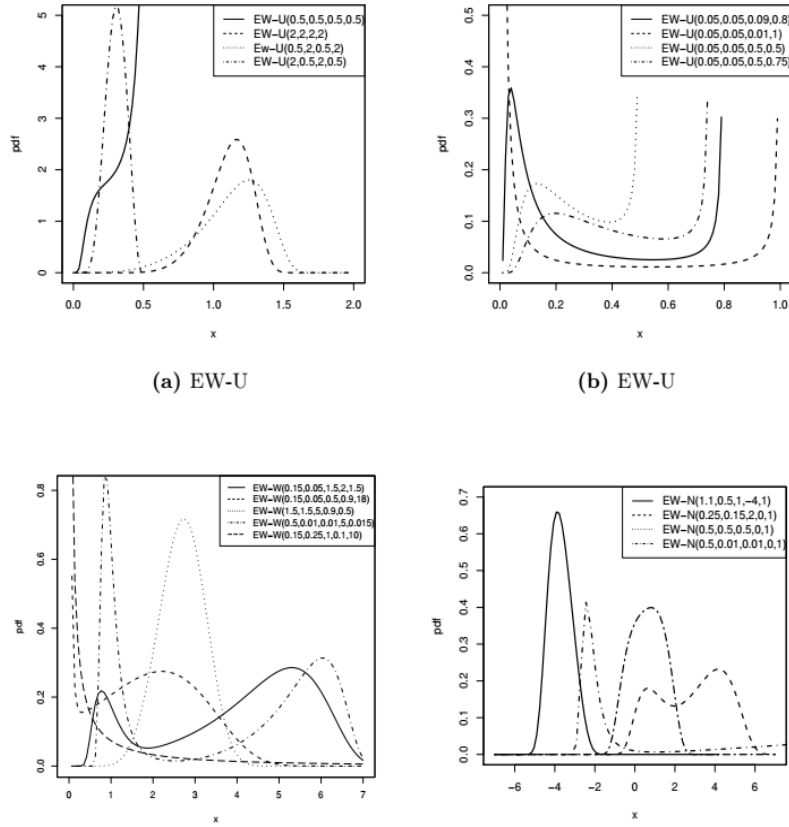When we use the exponential power series for twice, we have the following expansion

FIGURE 1. The plots of some member densities of the EW-G family

equation for the pdf

$$f_{EW-G}\left(x;\alpha,\beta,\lambda,\boldsymbol{\xi}\right) = \alpha\left[\lambda + \beta\left(\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right)\right]\frac{g\left(x;\boldsymbol{\xi}\right)}{\left[1-G\left(x;\boldsymbol{\xi}\right)\right]^{2}}$$

$$\times \sum_{k,i=0}^{\infty} \frac{(-1)^{i+k}\alpha^{k}[\lambda(k+1)]^{i}}{k!\,i!}\left(\frac{G\left(x;\boldsymbol{\xi}\right)}{1-G\left(x;\boldsymbol{\xi}\right)}\right)^{\beta k+\beta-i-2}.$$

Now using the generalized binom expansion for the negative power terms of the $[1-G\left(x;\boldsymbol{\xi}\right)]$, we can write following equation for the EW-G density

$$f_{EW-G}\left(x\right) = \sum_{k,i,j=0}^{\infty} w_{1}^{(k,i,j)}\tau_{\beta k+\beta+j-i-1}\left(x\right) + \sum_{k,i,j=0}^{\infty} w_{2}^{(k,i,j)}\tau_{\beta k+\beta+j-i}\left(x\right),$$

(a) EW-U

(b) EW-W

(c) EW-N
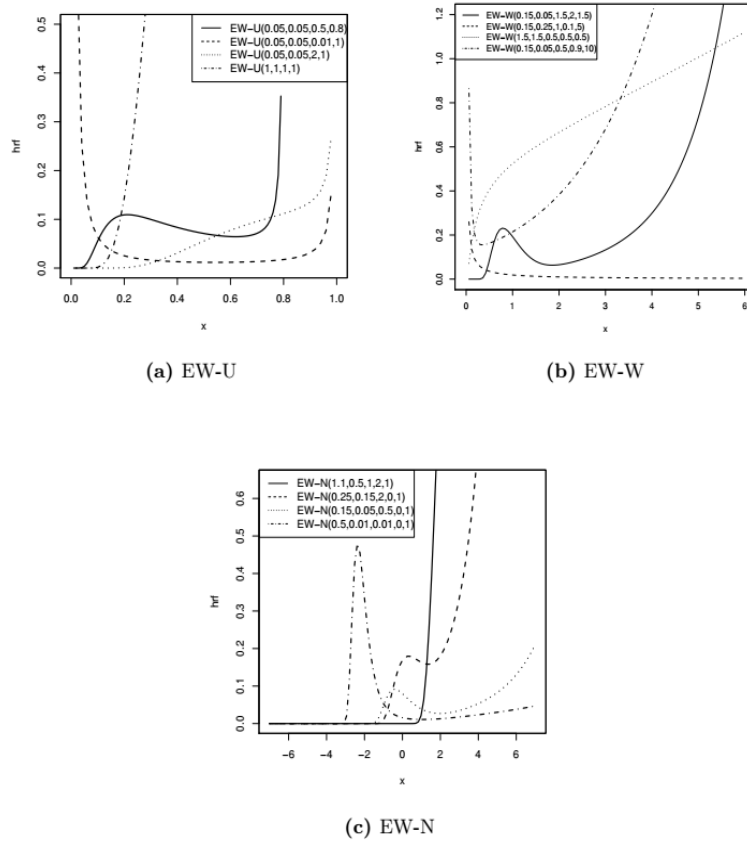
FIGURE 2. The hrf plots of the EW-U, EW-W and EW-N

where $\tau_a(x) = a g(x;\boldsymbol{\xi}) G(x;\boldsymbol{\xi})^{a-1}$ is the exponentiated-G (exp-G) pdf with the a power parameter,

$$w_1^{(k,i,j)} = \frac{(-1)^{i+k} \alpha^{k+1} \lambda^{i+1} (k+1)^i \Gamma(\beta k + \beta + j - 1)}{k!\, i!\, j!\, \Gamma(\beta k + \beta - i)(\beta k + \beta + j - i + 1)}$$

and

$$w_2^{(k,i,j)} = \frac{(-1)^{i+k} \beta \alpha^{k+1} [\lambda(k+1)]^i \Gamma(\beta k + \beta + j - i + 1)}{k!\, i!\, j!\, \Gamma(\beta k + \beta - i + 1)(\beta k + \beta + j - i)}.$$

We note that the $w_1^{(k,i,j)}$ and $w_2^{(k,i,j)}$ coefficients come from exponential power series and generalized binom expansions which are defined above. Finally by using exponentiated-G pdf, the pdf of the EW-G is given by

$$
\begin{aligned}
f_{EW-G}(x) \quad = \quad & \sum_{k,i,j=0}^{\infty} w_1^{(k,i,j)} \left(\beta k + \beta + j - i - 1 \right) g(x) G(x)^{(\beta k + \beta + j - i - 2)} \\
& + \sum_{k,i,j=0}^{\infty} w_2^{(k,i,j)} \left(\beta k + \beta + j - i \right) g(x) G(x)^{(\beta k + \beta + j - i - 1)}. \quad (4.1)
\end{aligned}
$$

Hence, from (4.1) we can say that the EW-G density can be explained as sum of the two density which are infinite combination of the exponentiated-G (exp-G) density functions. Thus, some mathematical properties of the EW-G model can be obtained directly from those properties of the exp-G distribution. For example, the ordinary moment and moment generating function of the EW-G distribution can be obtained immediately from those quantities of the exp-G distribution. The properties of exp-G distributions have been studied by many authors in recent years, see [30, 31].

5. Quantile Functions and Random Number Generation

The quantile functions (qf) are used in widespread in general statistics and often to obtain percentiles. The uth quantile, denoted by $x_u = Q(u)$, of the $EW - G$ distribution can be obtained by inversion of the cdf which is given by

$$
\alpha^{-1} \log(1 - u) + \left(\frac{G(x_u; \boldsymbol{\xi})}{1 - G(x_u; \boldsymbol{\xi})} \right)^{\beta} \exp \left\{ -\lambda \left(\frac{1 - G(x_u; \boldsymbol{\xi})}{G(x_u; \boldsymbol{\xi})} \right) \right\} = 0 \qquad (5.1)
$$

where $u \in (0, 1)$. Hence, if U is uniform random variable on (0,1), then X_U follows the $EW - G$ random variable.

The random number generation from any member of this family can be obtained with followings.

- The random number generation is same with solution of the (5.1) depend on baseline distribution G for $u \in (0, 1)$. This is directly solution of (2.1)
- if t is a random number on $EW(\alpha, \beta, \lambda)$ then $G^{-1}\left(\frac{t}{t+1} \right)$ is the random number on $EW - G(\alpha, \beta, \lambda, \boldsymbol{\xi})$ distribution, where t is solution of the (1.3) and $G^{-1}(\cdot)$ is the inverse of the baseline cdf or its qf.

These above expressions can be easily calculated with some packet programmes such as `R`, `Maple` and `Mathematica`.

The skewness and kurtosis of the $EW - G$ distribution can be derived by quantiles. By using (5.1) the Bowleys skewness [24] and the Moors kurtosis [28] are given by

$$
B = \frac{Q(3/4) - 2Q(2/4) + Q(1/4)}{Q(3/4) - Q(1/4)}
$$
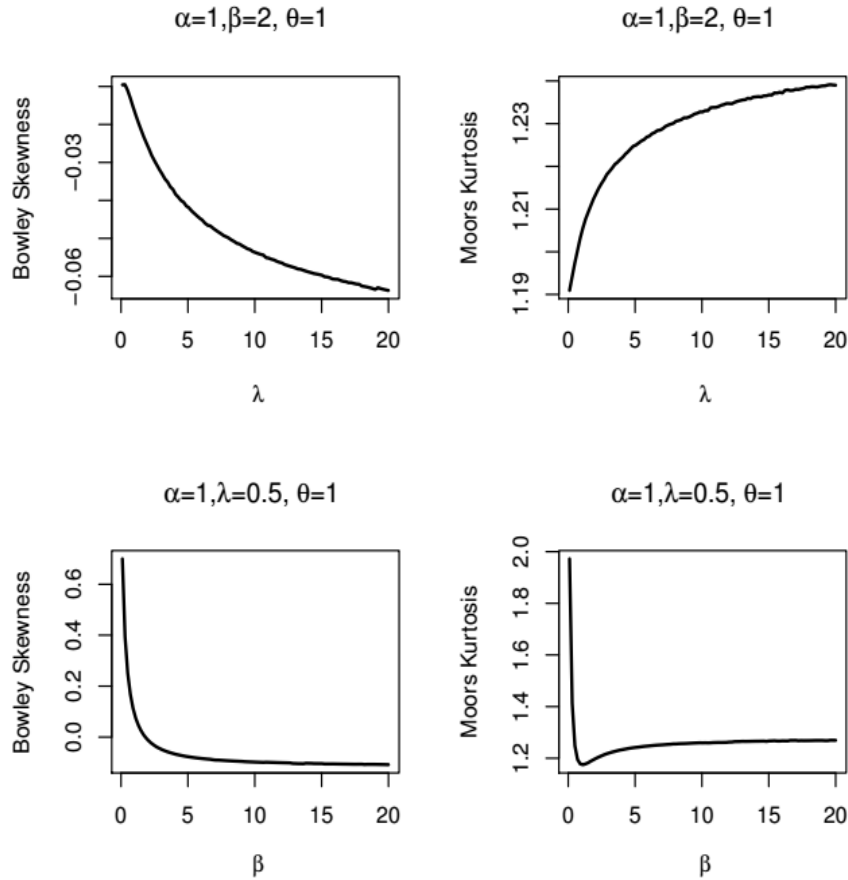
FIGURE 3. Graphics of the skewness and kurtosis measures for the EW-E distribution.

$$M = \frac{Q\left(7/8\right) - Q\left(5/8\right) + Q\left(3/8\right) - Q\left(1/8\right)}{Q\left(6/8\right) - Q\left(2/8\right)}$$

respectively. These measures are less sensitive to outliers and they exist even for distributions without moments. We draw B and M based on the EW-E distribution, introduced by Section 3, as a function of β and λ for fixed values other parameters in Figure 3. shows that the plots of the measures B and M for the EW-E distribution introduced in Section 3. These plots indicate that the member of the this family can model various data types in terms of skewness and kurtosis.

6. The Moments and Moment Generating Functions

In this section we obtain the expressions for the non-central moments and moment generating function of the EW-G family. Let Z_a be a random variable having the exp-G pdf τ_a with power parameter a and $\mu_r \equiv E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$, then the first formula for the rth moment of X follows from (4) as

$$E(X^r) = \sum_{k,i,j=0}^{\infty} w_1^{(k,i,j)} E\left(Z_{\beta k+\beta+j-i-1}^r\right) + \sum_{k,i,j=0}^{\infty} w_2^{(k,i,j)} E\left(Z_{\beta k+\beta+j-i}^r\right). \tag{6.1}$$

A second formula for $E(X^r)$ follows from (6.1) as

$$E(X^r) = \sum_{k,i,j=0}^{\infty} (\beta k + \beta + j - i - 1)\, w_1^{(k,i,j)} I(r, \beta k + \beta + j - i - 1)$$
$$+ \sum_{k,i,j=0}^{\infty} (\beta k + \beta + j - i) w_2^{(k,i,j)} I(r, \beta k + \beta + j - i),$$

where $I(a,b) = \int_0^1 [Q_G(u)]^a u^b du$.

The moment generating function (mgf), defined by $M(t) = E\left(e^{tX}\right)$, of the EW-G random variable can be obtained with (4) by

$$M(t) = \sum_{k,i,j=0}^{\infty} w_1^{(k,i,j)} M_{\beta k+\beta+j-i-1}(t) + \sum_{k,i,j=0}^{\infty} w_2^{(k,i,j)} M_{\beta k+\beta+j-i}(t), \tag{6.2}$$

where $M_a(t)$ denotes the mgf function of the Z_a random variable. A second formula for $M(t)$ follows from (6.2) as

$$M(t) = \sum_{k,i,j=0}^{\infty} (\beta k + \beta + j - i - 1)\, w_1^{(k,i,j)} \rho(t, \beta k + \beta + j - i - 1)$$
$$+ \sum_{k,i,j=0}^{\infty} (\beta k + \beta + j - i)\, w_2^{(k,i,j)} \rho(t, \beta k + \beta + j - i),$$

where $\rho(t,b) = \int_0^1 \exp\{tQ_G(u)\} u^b du$. The moments and mgf of some exp-G distributions are given by [31], which can be used to obtain $E(X^r)$ and $M(t)$.

7. Order Statistics

The pdf function $f_{i:n}(x)$ of the ith order statistics for $i = 1, \ldots, n$ from a random sample $X_1, X_2, \cdots, X_n$

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(x)\, F(x)^{i-1} [1 - F(x)]^{n-i}.$$

For the Ew-G family, the pdf of the ith order statistics is given by

$$f_{i:n}(x) = \frac{\alpha g(x;\boldsymbol{\xi})\, n!}{(i-1)!\,(n-i)!} \left[\lambda + \beta\left(\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right)\right] \frac{G(x;\boldsymbol{\xi})^{\beta-2}}{[1-G(x;\boldsymbol{\xi})]^\beta} e^{-\lambda\left(\frac{1-G(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})}\right)} \tag{7.1}$$
$$\times \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} \exp\left\{-\alpha(n+k-i+1)\left(\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right)^\beta e^{-\lambda\left(\frac{1-G(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})}\right)}\right\}.$$

From (7.1) especially the pdf of the maximum order statistic is

$$f_{n:n}(x) = \frac{n\alpha g(x;\boldsymbol{\xi})\,G(x;\boldsymbol{\xi})^{\beta-2}}{[1-G(x;\boldsymbol{\xi})]^{\beta}}\left[\lambda + \beta\left(\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right)\right]e^{-\lambda\left(\frac{1-G(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})}\right)}$$

$$\times \sum_{k=0}^{n-1}(-1)^k\binom{i-1}{k}\exp\left\{-\alpha(k+1)\left(\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right)^{\beta}e^{-\lambda\left(\frac{1-G(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})}\right)}\right\},$$

and the pdf of the minimum order statistic is

$$f_{1:n}(x) = \frac{n\alpha g(x;\boldsymbol{\xi})\,G(x;\boldsymbol{\xi})^{\beta-2}}{[1-G(x;\boldsymbol{\xi})]^{\beta}}\left[\lambda + \beta\left(\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right)\right]$$

$$\times \exp\left\{-\alpha n\left(\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right)^{\beta}e^{-\lambda\left(\frac{1-G(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})}\right)}-\lambda\left(\frac{1-G(x;\boldsymbol{\xi})}{G(x;\boldsymbol{\xi})}\right)\right\}.$$

Using series expansions in Section 4, we have the following pdf expansions of the ith order statistic $f_{i:n}(x)$

$$f_{1:n}(x) = \frac{n!}{(i-1)!(n-i)!}\left[\sum_{j,m,p=0}^{\infty}\sum_{k=0}^{i-1}w_1^{(k,j,m,p)}\tau_{\beta_j+\beta+p-m-1}(x;\boldsymbol{\xi})\right]$$

$$+ \left[\sum_{j,m,p=0}^{\infty}\sum_{k=0}^{i-1}w_2^{(k,j,m,p)}\tau_{\beta_j+\beta+p-m-1}(x;\boldsymbol{\xi})\right]$$

where $\tau_a(x;\boldsymbol{\xi})$ is exp-G distribution defined as before,

$$w_1^{(k,j,m,p)} = \binom{i-1}{k}\frac{(-1)^{i+k+m}(n+k+1-i)^j\alpha^{j+1}\lambda^{m+1}(j+1)^m\Gamma(\beta j+\beta-m-p-2)}{m!\,p!\,j!\,\Gamma(\beta j+\beta-m-2)(\beta j+\beta-m+p-1)}$$

and

$$w_2^{(k,j,m,p)} = \binom{i-1}{k}\frac{(-1)^{i+k+m}(n+k+1-i)^j\beta\alpha^{j+1}\lambda^m(j+1)^m\Gamma(\beta j+\beta-m-p+1)}{m!\,p!\,j!\,\Gamma(\beta j+\beta-m-1)(\beta j+\beta-m+p)}.$$

Hence, the rth, $r=1,2,...$, moments of the ith order statistic can be obtained as follows:

$$E(X_{i:n}^r) = \frac{n!}{(i-1)!(n-i)!}\left[\sum_{j,m,p=0}^{\infty}\sum_{k=0}^{i-1}w_1^{(k,j,m,p)}E(Z_1^r) + \sum_{j,m,p=0}^{\infty}\sum_{k=0}^{i-1}w_2^{(k,j,m,p)}E(Z_2^r)\right],$$

where Z_1 and Z_2 denote the exp-G distribution with power parameter $\beta j+\beta+ -m+p-1$ and $\beta j+\beta-m+p$ respectively.

8. Maximum Likelihood Estimation

We consider the estimation of the unknown parameters of the model parameters of the new family from complete samples only by maximum likelihood. Let

$x_1, \ldots, x_n$ be a random sample of the EW-G distribution with parameter vector $\boldsymbol{\Theta} = (\alpha, \beta, \lambda, \boldsymbol{\xi})^{\mathsf{T}}$. The log-likelihood function for Θ, say $\ell = \ell(\boldsymbol{\Theta})$, is given by

$$\ell \;=\; \ell(\boldsymbol{\Theta}) = n \log \alpha + \sum_{i=1}^{n} \log\left(g\left(x_i; \boldsymbol{\xi}\right)\right) + \sum_{i=1}^{n} \log\left[\lambda + \beta W\left(x_i; \boldsymbol{\xi}\right)\right] \quad (8.1)$$

$$+ \left(\beta - 2\right) \sum_{i=1}^{n} \log G\left(x_i; \boldsymbol{\xi}\right) - \beta \sum_{i=1}^{n} \log\left[1 - G\left(x_i; \boldsymbol{\xi}\right)\right]$$

$$- \alpha \sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{\beta} e^{-\lambda W(x_i; \boldsymbol{\xi})^{-1}} - \lambda \sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{-1},$$

where $W\left(x_i; \boldsymbol{\xi}\right) = \frac{G(x_i; \boldsymbol{\xi})}{1 - G(x_i; \boldsymbol{\xi})}$. The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (8.1). The last equation can be maximized either by using the different programs like `R` [34] (`optim`, `AdequecyModel`, `maxLik` functions) or by solving the nonlinear likelihood equations obtained by differentiating (8.1). The associated gradients or components of the score function, $(\ell_\alpha = \partial\ell/\partial\alpha, \ell_\beta = \partial\ell/\partial\beta, \ell_\lambda = \partial\ell/\partial\lambda, \ell_{\boldsymbol{\xi}} = \partial\ell/\partial\boldsymbol{\xi},)$, are given by

$$\ell_\alpha = \frac{n}{\alpha} - \sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{\beta} e^{-\lambda W(x_i; \boldsymbol{\xi})^{-1}},$$

$$\ell_\beta = \sum_{i=1}^{n} \frac{W\left(x_i; \boldsymbol{\xi}\right)}{\lambda + \beta W\left(x_i; \boldsymbol{\xi}\right)} + \sum_{i=1}^{n} \log W\left(x_i; \boldsymbol{\xi}\right) - \alpha \sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{\beta} e^{-\lambda W(x_i; \boldsymbol{\xi})^{-1}} \log W\left(x_i; \boldsymbol{\xi}\right),$$

$$\ell_\lambda = \sum_{i=1}^{n} \frac{1}{\lambda + \beta W\left(x_i; \boldsymbol{\xi}\right)} + \alpha \sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{\beta-1} e^{-\lambda W(x_i; \boldsymbol{\xi})^{-1}} - \sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{-1}.$$

For parameter vector from the baseline distribution, the associated gradients are found to

$$\ell_{\boldsymbol{\xi}} \;=\; \sum_{i=1}^{n} \frac{\partial g\left(x_i; \boldsymbol{\xi}\right)/\partial\boldsymbol{\xi}_k}{g\left(x_i; \boldsymbol{\xi}\right)} + \beta \sum_{i=1}^{n} \frac{\partial W\left(x_i; \boldsymbol{\xi}\right)/\partial\boldsymbol{\xi}_k}{\lambda + \beta W\left(x_i; \boldsymbol{\xi}\right)} + \beta \sum_{i=1}^{n} \frac{\partial W\left(x_i; \boldsymbol{\xi}\right)/\partial\boldsymbol{\xi}_k}{W\left(x_i; \boldsymbol{\xi}\right)}$$

$$- \alpha \sum_{i=1}^{n} \left(\beta W\left(x_i; \boldsymbol{\xi}\right) + \lambda\right) W\left(x_i; \boldsymbol{\xi}\right)^{\beta-2} \partial W\left(x_i; \boldsymbol{\xi}\right)/\partial\boldsymbol{\xi}_k e^{-\lambda W(x_i; \boldsymbol{\xi})^{-1}}$$

$$- 2 \sum_{i=1}^{n} \frac{\partial G\left(x_i; \boldsymbol{\xi}\right)/\partial\boldsymbol{\xi}_k}{G\left(x_i; \boldsymbol{\xi}\right)} + \lambda \sum_{i=1}^{n} \frac{\partial W\left(x_i; \boldsymbol{\xi}\right)/\partial\boldsymbol{\xi}_k}{W\left(x_i; \boldsymbol{\xi}\right)^2}.$$

We can obtain the estimates of the unknown parameters by setting the score vector to zero. These equations cannot be solved analytically, and a statistical software with an optimization routine implementation should be used to solve them numerically using an iterative method such as Newton-Raphson.

For interval estimation and hypothesis tests on the parameters in Θ, we need $(k+3) \times (k+3)$ dimensions the asymptotic variance-covariance matrix of the maximum likelihood estimators of the parameters. This matrix is obtained by the inverse of the Fisher information matrix, $J(\boldsymbol{\Theta}) = -E\left(\ell_{ij}\right), i, j = \alpha, \beta, \lambda, \boldsymbol{\xi}$, whose elements are negative of the expected values of the second partial derivatives of the log-likelihood function with respect to the parameters. The ℓ_{ij} are given in **Appendix**. The multivariate normal $N_{k+3}(0, J(\widehat{\boldsymbol{\Theta}})^{-1})$ distribution, can be used to provide approximate confidence intervals for the unknown parameters since the maximum likelihood estamations has asymptotic normal distribution under standard regularity conditions, where $J(\widehat{\boldsymbol{\Theta}})$ is the observed Fisher information matrix evaluated at $\widehat{\boldsymbol{\Theta}}$. Then, approximate $100(1-\delta)\%$ confidence intervals for α, β, λ and $\boldsymbol{\xi}$ can be determined by:

$$\widehat{\alpha} \pm z_{\delta/2}\sqrt{\widehat{J_{\alpha\alpha}}}, \quad \widehat{\beta} \pm z_{\delta/2}\sqrt{\widehat{J_{\beta\beta}}} \quad \widehat{\lambda} \pm z_{\delta/2}\sqrt{\widehat{J_{\lambda\lambda}}} \text{ and } \quad \widehat{\boldsymbol{\xi}} \pm z_{\delta/2}\sqrt{\widehat{J_{\boldsymbol{\xi}\boldsymbol{\xi}}}} \text{ , where}$$

$z_{\delta/2}$ is the upper δth percentile of the standard normal model.

9. Data Analysis

In this section, we give a simulation study and a real data application. The EW-E distribution, introduced by Section 2, is considered for both simulation study and real data application. The computations of the maximum likelihood estimates of all parameters for all the distributions are obtained by using the `maxLik` function in `R` program. This function also gives the numerically differentiated observed Fisher information matrix.

9.1. **A Simulation Study.** To evaluate the performance of the maximum likelihood estimates, we generate 1,000 samples of sizes 20 and 100 from the $EW - E(\alpha, \beta, \lambda, \theta)$ distribution. The random number generation is obtained by inverse of the $EW - E$ cdf by using uniroot routine in `R` programme. The results of the simulation are reported in Table 2. We observe that the estimates approach true values as the sample size increases implying the consistency of the estimates. Also, we observe from Table 2 that the estimates are quite stable and get closer to the true values as the sample sizes increase.

9.2. **A Real Data Application.** We analyze the data set studied by Abouammoh et al. [1], which represent the lifetime in days of 40 patients suffering from leukemia from one of the Ministry of Health Hospitals in Saudi Arabia. The data also have been analyzed by Sarhan et al. [35]. The data are: 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852. By using this data set we compare the EW-E distribution with Weibull-exponential distribution [13] (WE) , exponentiated exponential

TABLE 1. Emprical means and standard errors (given in parentheses) for different values of the EW-E distribution parameters

Parameters	$n = 20$				$n = 100$			
$(\alpha, \beta, \lambda, \theta)$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\theta}$
(0.5,1,1,1)	0.5561	1.0265	1.2604	1.2208	0.4852	0.9994	0.9874	1.0784
	(0.5242)	(0.3421)	(0.5397)	(0.3551)	(0.2201)	(0.1034)	(0.2395)	(0.1519)
(1,0.5,1,1)	0.8975	0.5479	1.2557	1.3434	1.0357	0.5078	1.1566	1.0729
	(0.5311)	(0.2685)	(0.7318)	(0.4081)	(0.3209)	(0.1271)	(0.4056)	(0.2575)
(1,1,0.5,1)	0.8329	0.9538	0.8950	1.4467	0.9733	0.9641	0.6335	1.1321
	(0.5806)	(0.3954)	(0.6962)	(0.4359)	(0.3413)	(0.2258)	(0.2913)	(0.2564)
(0.5,1,2,0.5)	0.5530	1.0473	2.1562	0.6002	0.5239	0.9858	2.0560	0.5423
	(0.6769)	(0.3385)	(0.4719)	(0.3517)	(0.4153)	(0.1490)	(0.1897)	(0.2255)
(1,1,1,1)	0.9858	1.0170	1.3429	1.2495	0.9953	0.9979	1.1047	1.0613
	(0.6712)	(0.4209)	(0.7423)	(0.2935)	(0.3777)	(0.2410)	(0.3757)	(0.1218)
(2,1,2,1)	1.9769	1.2256	2.0804	1.0141	1.9952	1.0526	2.0112	1.0015
	(0.5567)	(0.5345)	(0.8411)	(0.1427)	(0.3264)	(0.2553)	(0.4807)	(0.0826)
(2,2,2,1)	2.0181	2.1996	2.1338	1.0127	2.0046	2.0293	2.0244	1.0033
	(0.2763)	(0.5589)	(0.6094)	(0.0892)	(0.1371)	(0.2578)	(0.2880)	(0.0432)
(1,2,2,1)	1.0255	2.0668	2.1314	1.0507	1.0049	2.0364	2.0412	1.0086
	(0.4819)	(0.5077)	(0.5628)	(0.1202)	(0.2683)	(0.2274)	(0.2862)	(0.0459)
(2,1,1,1)	2.0798	1.1468	1.4166	1.1505	2.0222	1.0306	1.0948	1.0383
	(0.6870)	(0.6405)	(1.1427)	(0.3423)	(0.2873)	(0.3331)	(0.4896)	(0.1494)
(2,1,1,0.5)	2.0264	1.1673	1.1502	0.5250	2.0044	1.0526	1.0269	0.5050
	(0.3336)	(0.4857)	(0.5250)	(0.1053)	(0.1649)	(0.2586)	(0.3347)	(0.0526)
(2,1,0.5,2)	1.7869	1.0328	0.8045	2.6994	1.9062	0.9863	0.5815	2.1822
	(0.8221)	(0.5741)	(0.7987)	(0.7481)	(0.4192)	(0.2630)	(0.2692)	(0.3410)
(0.5,1,1,2)	0.5537	1.0099	1.2713	2.4654	0.5170	0.9928	1.0217	2.0125
	(0.5456)	(0.4595)	(0.8717)	(0.3566)	(0.2071)	(0.1431)	(0.2824)	(0.2463)
(2,2,2,2)	2.0414	2.2564	2.2273	2.0633	2.0399	2.0530	2.1254	2.0365
	(0.6845)	(0.8436)	(1.1878)	(0.3090)	(0.4394)	(0.4936)	(0.7837)	(0.1963)
(1,2,3,4)	1.2731	2.1368	3.3794	4.2206	1.0747	2.0284	3.0753	4.1244
	(1.0245)	(0.7431)	(1.1488)	(0.3575)	(0.6978)	(0.3931)	(0.7309)	(0.1965)
(4,3,2,1)	4.0096	3.1979	2.1511	1.0144	4.0021	3.0275	2.0254	1.0034
	(0.0964)	(0.6106)	(0.6073)	(0.0826)	(0.0440)	(0.2806)	(0.2691)	(0.0370)

distribution [22] (EE), exponentiated Weibull distribution [29] (Exp-W), EW distribution and Kumaraswamy-exponential-Weibull distribution [20] (Kw-EW). The cdfs of these models are given below (for $x > 0$):

$$F_{EE}\left(x; \alpha, \theta\right) = \left(1 - e^{-\theta x}\right)^{\alpha}, \; \alpha, \theta > 0.$$

$$F_{Exp-W}(x;\alpha,\theta) = \left(1 - e^{-(\theta x)^{\beta}}\right)^{\alpha}, \ \alpha,\beta,\theta > 0,$$

$$F_{WE}(x;\alpha,\beta,\theta) = 1 - \exp\left\{-\alpha\left(e^{\theta x} - 1\right)^{\beta}\right\}, \ \alpha,\beta,\theta > 0,$$

$$F_{Kw-EW}(x,\alpha,\beta,\lambda,\gamma,\theta) = 1 - (1 - (1 - \exp\{-\theta x - \lambda x^{\gamma}\})^{\alpha})^{\beta}, \ \alpha,\beta,\lambda,\gamma,\theta > 0,$$

The model selection is applied using the Kolmogorov-Smirnov (K-S) statistics, Akaike information criterion (AIC) and estimated log-likelihood ($\hat{\ell}$) values. The AIC value is given by $AIC = -2\hat{\ell} + 2p$ where p is the number of the estimated model parameters and n is sample size. When searching the best fit among others to data, the distribution with the smallest AIC and K-S values and the biggest $\hat{\ell}$ and p values of the K-S statistics is chosen.

By using this data, for exponentiated generalized linear failure rate distribution, Sarhan et al. [35] have obtained K-S statistics , goodness-of-fit statistics, and its p-value as 0.0917 and 0.8591 respectively. For EW distribution, we obtain these values as 0.1330 (K-S) and 0.4797 (p-value). Table 3 lists the maximum likelihood estimations (and the corresponding standard errors in parentheses) of the parameters, $\hat{\ell}$, AIC value and K-S statistic for above fitted models. On the Table 3, we observe that the EW-E distribution has the smallest AIC and K-S values and has the biggest $\hat{\ell}$ and p values. For this reason, it could be chosen as the best model among the other models under these criteria.

The inverse of the numerically evaluated Hessian (or observed Fisher information) matrix at the solution found is given by

$$J^{-1}\left(\widehat{\boldsymbol{\Theta}}\right) = \begin{pmatrix} 0.00031440 & -0.0001693 & -0.0002109 & 0.0000093 \\ -0.0001693 & 0.0006020 & -0.0129784 & -0.000145 \\ -0.0002109 & -0.0129784 & 17.9347331 & 0.0039429 \\ -0.0000093 & -0.000145 & 0.0039429 & 0.0000455 \end{pmatrix}.$$

The 90% confidence intervals for the EW-E distribution parameters α, β, λ and θ are then computed as [0.0051, 0.0633], [0.0506, 0.1312], [30.9130, 44.8458], and [0.0163, 0.0383] respectively.

We sketched the fitted densities of the application models for this data in Figure 4. The Figure 4 shows that the EW-E model fits to lukemia data as bi-modal shaped and others fit to this data as uni-modal shaped. Fig. 5(a) gives the fitted hrf of the EW-E model. This figure shows firstly unimodal shaped and then an increasing shaped hrf for the data set. The emprical cdf and fitted EW-E cdf are given by Fig 5(b). Hence, the EW-E the distribution could be appropriate to fit such data.
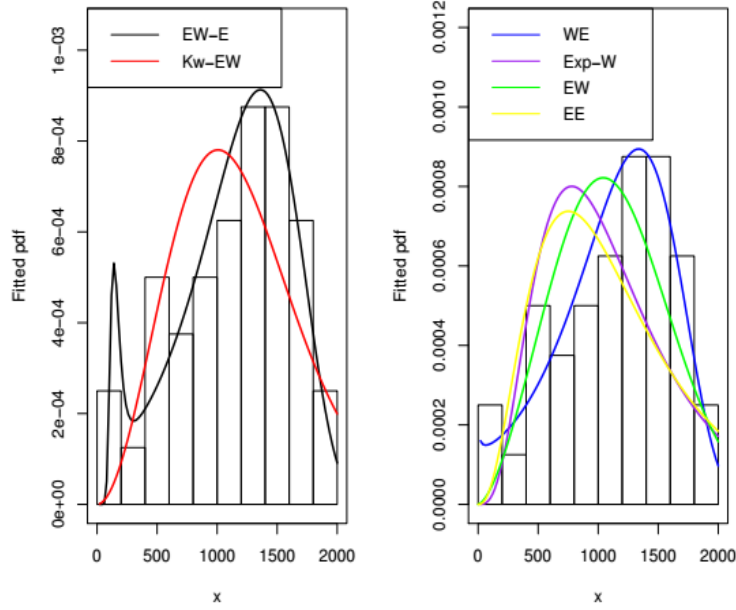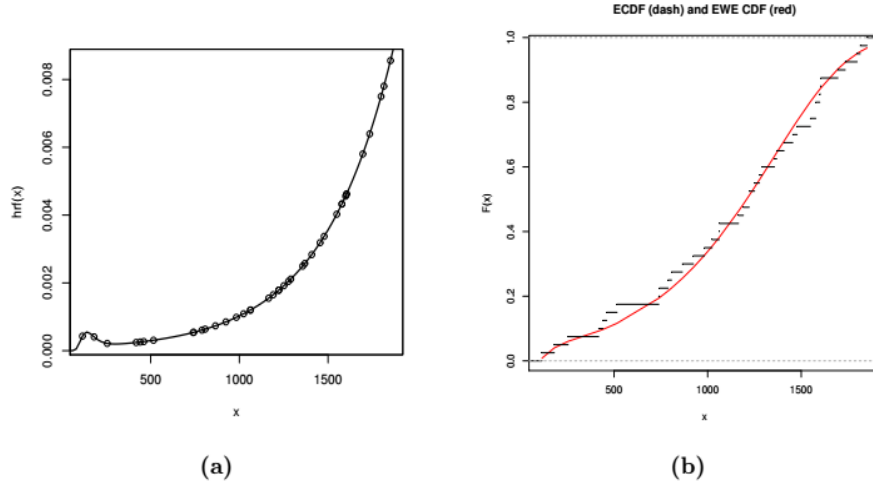
FIGURE 4. Fitted pdfs for the lukemia data.

FIGURE 5. (a) The fitted hrf of the EW-E, (b) the fitted EW-E cdf
and emprical cdf for lukemia data

TABLE 2. The maximum likelihood estimates of parameters, standard erros of the estimates (in parentheses), $\widehat{\ell}$, AIC and K-S values of the application models (p-values are given in [·])

Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{\theta}$	$\widehat{\gamma}$	$\widehat{\ell}$	AIC	K-S
EW-E	0.0342	0.0909	37.8794	0.0273	–	-298.8885	605.7770	0.0765
	(0.0177)	(0.0245)	(4.2349)	(0.0067)				[0.9735]
Kw-EW	3.0392	17.1361	0.00014	0.00027	0.9831	-306.1389	622.2778	0.1113
	(0.3265)	(5.5975)	(0.0013)	(0.0011)	(0.0363)			[0.7047]
EE	3.6502	–	–	0.0017	–	-310.1557	624.3114	0.1655
	(0.9319)			(0.0002)				[0.2232]
Exp-W	4.7443	0.9797	–	0.0020	–	-310.7864	627.5728	0.1911
	(0.8542)	(0.0790)		(0.0001)				[0.1076]
WE	0.0414	0.8480	–	0.0028	–	-300.1153	606.2306	0.0783
	(0.0286)	(0.4979)		(0.0019)				[0.9668]

10. Concluding Remarks

We defined a new Weibull family of the distributions in order to provide great flexibility to any continuous distribution by adding three extra parameters. We called it extended Weibull-G family. The ordinary Weibull-G family is [13] particular case of the proposed distribution family. Some special models of this new family are discussed. The shapes of the density and hazard rate function were investigated for EW-U, EW-W and EW-N distributions. We studied its general structural properties such as expansions for density, quantile functions, moments, moment generating functions and order statistics under general settings. The model parameters were estimated using maximum likelihood method. A simulation study was performed to see performance maximum likelihood estimation, and we observed that the estimations are quite stable and get closer to the true values as the sample sizes increase. Its usefulness was illustrated by an application to real lifetime data. Further, we saw that this family could generate a bi-modal shaped distribution such as bi-modal Weibull and bi-modal normal distributions (see, Section 3). The ordinary exponential distribution in simple structure, which has only reversed-J shaped and constant hrf shaped, have been become as a bi-modal shaped and a firstly uni-modal then increasing hrf shaped (see, Figures 4 and 5). Finally we can say that adding parameters to any continuous distribution via EW-G family construction increases its flexibility. We hope that the EW-G family may be extensively used in statistics.

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Appendix

In here, we give second order derivative of ℓ respect to parameters.

$$\ell_{\alpha\alpha} = -\frac{n}{\alpha^2},$$

$$\ell_{\alpha\beta} = -\sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{\beta} e^{-\lambda W(x_i;\boldsymbol{\xi})^{-1}} \log W\left(x_i; \boldsymbol{\xi}\right),$$

$$\ell_{\alpha\lambda} = \sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{\beta-1} e^{-\lambda W(x_i;\boldsymbol{\xi})^{-1}},$$

$$\ell_{\alpha\boldsymbol{\xi}_k} = -\sum_{i=1}^{n} \left(\beta W\left(x_i; \boldsymbol{\xi}\right) + \lambda\right) W\left(x_i; \boldsymbol{\xi}\right)^{\beta-2} \partial W\left(x_i; \boldsymbol{\xi}\right)/\partial\boldsymbol{\xi}_k e^{-\lambda W(x_i;\boldsymbol{\xi})^{-1}},$$

$$\ell_{\beta\beta} = -\sum_{i=1}^{n} \frac{W^2\left(x_i; \boldsymbol{\xi}\right)}{\left[\lambda + \beta W\left(x_i; \boldsymbol{\xi}\right)\right]^2} - \alpha\sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{\beta} e^{-\lambda W(x_i;\boldsymbol{\xi})^{-1}} \log^2 W\left(x_i; \boldsymbol{\xi}\right),$$

$$\ell_{\beta\lambda} = -\sum_{i=1}^{n} \frac{1}{\left[\lambda + \beta W\left(x_i; \boldsymbol{\xi}\right)\right]^2} - \alpha\sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{\beta-2} e^{-\lambda W(x_i;\boldsymbol{\xi})^{-1}},$$

$$\ell_{\beta\boldsymbol{\xi}_k} = -\alpha\sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{\beta-2} \left[\beta + \left(\beta W\left(x_i; \boldsymbol{\xi}\right) + \lambda\right) \log W\left(x_i; \boldsymbol{\xi}\right)\right] \frac{\partial W(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k} e^{-\lambda W(x_i;\boldsymbol{\xi})^{-1}}$$
$$+ \sum_{i=1}^{n} \frac{\partial W(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k} W^{-1}\left(x_i; \boldsymbol{\xi}\right) + \sum_{i=1}^{n} \frac{\partial W(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k} \left[\lambda + \beta W\left(x_i; \boldsymbol{\xi}\right)\right]^{-1}$$
$$- \beta\sum_{i=1}^{n} \frac{\partial W(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k} \frac{W\left(x_i; \boldsymbol{\xi}\right)}{\left[\lambda + \beta W\left(x_i; \boldsymbol{\xi}\right)\right]^2},$$

$$\ell_{\lambda\lambda} = -\sum_{i=1}^{n} \frac{1}{\left[\lambda + \beta W\left(x_i; \boldsymbol{\xi}\right)\right]^2} - \alpha\sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{\beta-2} e^{-\lambda W(x_i;\boldsymbol{\xi})^{-1}},$$

$$\ell_{\lambda\boldsymbol{\xi}} = -\beta \sum_{i=1}^{n} \left[\lambda + \beta W\left(x_i; \boldsymbol{\xi}\right)\right]^{-2} \frac{\partial W(x_i;\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_k} + \lambda \sum_{i=1}^{n} W\left(x_i; \boldsymbol{\xi}\right)^{-2} \frac{\partial W(x_i;\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_k}$$

$$- \alpha \sum_{i=1}^{n} \left(1 - \beta - \lambda W\left(x_i; \boldsymbol{\xi}\right)^{-1}\right) W\left(x_i; \boldsymbol{\xi}\right)^{\beta-2} \frac{\partial W(x_i;\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_k} e^{-\lambda W(x_i;\boldsymbol{\xi})^{-1}},$$

$$\ell_{\boldsymbol{\xi}_k \boldsymbol{\xi}_l} = \sum_{i=1}^{n} \frac{g''_{kl}(x_i;\boldsymbol{\xi})}{g(x_i;\boldsymbol{\xi})} - \sum_{i=1}^{n} \frac{g'_l(x_i;\boldsymbol{\xi})g'_k(x_i;\boldsymbol{\xi})}{g^2(x_i;\boldsymbol{\xi})} + \beta \sum_{i=1}^{n} \frac{(\lambda+\beta W(x_i;\boldsymbol{\xi}))W''_{kl}(x_i;\boldsymbol{\xi})-\beta W'_l(x_i;\boldsymbol{\xi})W'_k(x_i;\boldsymbol{\xi})}{[\lambda+\beta W(x_i;\boldsymbol{\xi})]^2}$$

$$+ \beta \sum_{i=1}^{n} \frac{W''_{kl}(x_i;\boldsymbol{\xi})W(x_i;\boldsymbol{\xi})-W'_l(x_i;\boldsymbol{\xi})W'_k(x_i;\boldsymbol{\xi})}{W(x_i;\boldsymbol{\xi})^2} + 2 \sum_{i=1}^{n} \frac{G'_k(x_i;\boldsymbol{\xi})G'_l(x_i;\boldsymbol{\xi})}{G(x_i;\boldsymbol{\xi})^2} - 2 \sum_{i=1}^{n} \frac{G''_{kl}(x_i;\boldsymbol{\xi})}{G(x_i;\boldsymbol{\xi})}$$

$$- \alpha \sum_{i=1}^{n} \left(\beta W\left(x_i; \boldsymbol{\xi}\right) + \lambda\right) W\left(x_i; \boldsymbol{\xi}\right)^{\beta-2} \left(W''_{kl}\left(x_i; \boldsymbol{\xi}\right) - \frac{W'_l(x_i;\boldsymbol{\xi})}{W(x_i;\boldsymbol{\xi})^2}\right) e^{-\lambda W(x_i;\boldsymbol{\xi})^{-1}}$$

$$- \alpha \sum_{i=1}^{n} \frac{W'_l(x_i;\boldsymbol{\xi})W'_k(x_i;\boldsymbol{\xi})W(x_i;\boldsymbol{\xi})^{\beta-3}[\beta W(x_i;\boldsymbol{\xi})+(\beta-2)(\beta W(x_i;\boldsymbol{\xi})+\lambda)]}{e^{\lambda W(x_i;\boldsymbol{\xi})^{-1}}}$$

$$+ \lambda \sum_{i=1}^{n} \frac{W''_{kl}\left(x_i; \boldsymbol{\xi}\right)}{W\left(x_i; \boldsymbol{\xi}\right)^2} - 2\lambda \sum_{i=1}^{n} \frac{W'_l\left(x_i; \boldsymbol{\xi}\right)W'_k\left(x_i; \boldsymbol{\xi}\right)}{W\left(x_i; \boldsymbol{\xi}\right)^3},$$

where $z'_k\left(\cdot; \boldsymbol{\xi}\right) = \frac{\partial}{\partial \boldsymbol{\xi}_k} z\left(\cdot; \boldsymbol{\xi}\right)$ and $z''_{kl}\left(\cdot; \boldsymbol{\xi}\right) = \frac{\partial^2}{\partial \boldsymbol{\xi}_k \partial \boldsymbol{\xi}_l} z\left(\cdot; \boldsymbol{\xi}\right).$

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