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CLASSICAL WAY OF LOOKING AT THE LANE-EMDEN EQUATION

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ABSTRACT. In this article, the well-known approximate and analytical solutions of the Lane-Emden equation applying Taylor series expansion are derived. To the best of author's knowledge nobody has overcome the singularity of the Lane-Emden equation at the origin as it is carried out here.

1. INTRODUCTION

The well known Lane-Emden (LE) equation [1, 2] reads

$$y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0 \tag{1.1}$$

subject to the initial conditions

$$y(0) = 1 \quad \text{and} \quad y'(0) = 0. \tag{1.2}$$

One of the main problems with the LE equation is the singularity at $x = 0$ which is a challenging task for many numerical methods to get its solution. Since the LE equation is important, for its importance see [1, 2, 3, 4, 5], it has been the focus of several studies. In general, the solution of the Lane-Emden equation can not be given analytically but only numerically. It is solved analytically only for $m = 0, 1$ and 5 in [3]. For the other values of m, to our best knowledge, analytic solutions are still missing and thus the equation must be dealt with other techniques to get at least either approximate solutions or exact solutions [6, 7, 8, 9].

2. MAIN RESULTS

Set

$$F(m, x, y, y') = xy'' + 2y' + xy^m = 0 \tag{2.1}$$

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where $y(x)$ is an analytic function. From here,

$$y'' + \frac{2}{x}y' + y^m = 0. \tag{2.2}$$

By using (1.2) and applying L' Hospital's rule to the second term in (2.2) one obtains

$$y''(0) + 2y''(0) + y^m(0) = 0.$$

That is,

$$y''(0) = -\frac{1}{3}. \tag{2.3}$$

Differentiating (2.1) with respect to x one obtains

$$xy''' + 3y'' + y^m + mxy^{m-1}y' = 0. \tag{2.4}$$

From here,

$$y''' + \frac{3y'' + y^m}{x} + my^{m-1}y' = 0. \tag{2.5}$$

By using (1.2), (2.3) and applying L' Hospital's rule to the second term in (2.5) one obtains

$$y'''(0) = 0. \tag{2.6}$$

Differentiating (2.4) with respect to x one obtains

$$xy^{(4)} + 4y''' + 2my^{m-1}y' + m(m-1)xy^{m-2}(y')^2 + mxy^{m-1}y'' = 0. \tag{2.7}$$

From here,

$$y^{(4)} + \frac{4y''' + 2my^{m-1}y'}{x} + m(m-1)y^{m-2}(y')^2 + my^{m-1}y'' = 0. \tag{2.8}$$

By using (1.2), (2.3), (2.6) and applying L' Hospital's rule to the second term in (2.8) one obtains

$$y^{(4)}(0) + 4y^{(4)}(0) + 2my^{m-1}(0)y''(0) + my^{m-1}(0)y''(0) = 0.$$

That is,

$$y^{(4)}(0) = \frac{m}{5}. \tag{2.9}$$

Differentiating (2.7) with respect to x one obtains

$$\begin{gathered} 5y^{(4)} + xy^{(5)} + 3m(m-1)y^{m-2}(y')^2 + 3my^{m-1}y'' \\ +m(m-1)(m-2)xy^{m-3}(y')^3 + 3m(m-1)xy^{m-2}y'y'' \\ + mxy^{m-1}y''' = 0. \end{gathered} \tag{2.10}$$

From here,

$$y^{(5)} + \frac{5y^{(4)} + 3m(m-1)(m-2)y^{m-2}(y')^2 + 3my^{m-1}y''}{x}$$

$$+m(m-1)(m-2)y^{m-3}(y')^3 + 3m(m-1)y^{m-2}y'y''$$

$$+ my^{m-1}y''' = 0 \tag{2.11}$$

By using (1.2), (2.3), (2.6), (2.9) and applying L' Hospital's rule to the second term in (2.11) one obtains

$$y^{(5)}(0) = 0. \tag{2.12}$$

Differentiating (2.10) with respect to x one obtains

$$\begin{gathered}6y^{(5)} + xy^{(6)} + 4m(m-1)(m-2)y^{m-3}(y')^3 + 12m(m-1)y^{m-2}y'y'' \\ +4my^{m-1}y''' + m(m-1)(m-2)(m-3)xy^{m-4}(y')^4 \\ +6m(m-1)(m-2)xy^{m-3}(y')^2y'' + 3m(m-1)xy^{m-2}(y'')^2 \\ + 4m(m-1)xy^{m-2}y'y''' + mxy^{m-1}y^{(4)} = 0.\end{gathered} \tag{2.13}$$

From here,

$$\begin{gathered}y^{(6)} + \frac{6y^{(5)} + 4m(m-1)(m-2)y^{m-3}(y')^3}{x} \\ \frac{+12m(m-1)y^{m-2}y'y'' + 4my^{m-1}y'''}{x} \\ +m(m-1)(m-2)(m-3)y^{m-4}(y')^4 \\ +6m(m-1)(m-2)y^{m-3}(y')^2y'' + 3m(m-1)y^{m-2}(y'')^2 \\ + 4m(m-1)y^{m-2}y'y''' + my^{m-1}y^{(4)} = 0.\end{gathered} \tag{2.14}$$

By using (1.2), (2.3), (2.6), (2.9), (2.12) and applying L' Hospital's rule to the second and the third terms in (2.14) one obtains

$$\begin{gathered}7y^{(6)}(0) + 12m(m-1)y^{m-2}(0)(y''(0))^2 + 4my^{m-1}(0)y^{(4)}(0) \\ +3m(m-1)y^{m-2}(0)(y''(0))^2 + my^{m-1}(0)y^{(4)}(0) = 0.\end{gathered}$$

That is,

$$y^{(6)}(0) = \frac{-m(8m-5)}{3.7}. \tag{2.15}$$

Differentiating (2.13) with respect to x one obtains

$$\begin{gathered}7y^{(6)} + xy^{(7)} + 5m(m-1)(m-2)(m-3)y^{m-4}(y')^4 \\ +30m(m-1)(m-2)y^{m-3}(y')^2y'' + 15m(m-1)y^{m-2}(y'')^2 \\ +20m(m-1)y^{m-2}y'y''' + 5my^{m-1}y^{(4)} \\ +m(m-1)(m-2)(m-3)(m-4)xy^{m-5}(y')^5 \\ +10m(m-1)(m-2)(m-3)xy^{m-4}(y')^3y''\end{gathered}$$

$$+15m(m-1)(m-2)xy^{m-3}y'(y'')^2+10m(m-1)(m-2)xy^{m-3}(y')^2y'''$$
$$+10m(m-1)xy^{m-2}y''y'''+5m(m-1)xy^{m-2}y'y^{(4)}$$
$$+\,mxy^{m-1}y^{(5)}. \tag{2.16}$$

From here,

$$y^{(7)}+\frac{7y^{(6)}+5m(m-1)(m-2)(m-3)y^{m-4}(y')^4}{x}$$
$$\frac{+30m(m-1)(m-2)y^{m-3}(y')^2y''+15m(m-1)y^{m-2}(y'')^2}{x}$$
$$+\frac{20m(m-1)y^{m-2}y'y'''+5my^{m-1}y^{(4)}}{x}$$
$$+m(m-1)(m-2)(m-3)(m-4)y^{m-5}(y')^5$$
$$+10m(m-1)(m-2)(m-3)y^{m-4}(y')^3y''$$
$$+15m(m-1)(m-2)y^{m-3}y'(y'')^2$$
$$+10m(m-1)(m-2)y^{m-3}(y')^2y'''$$
$$+10m(m-1)y^{m-2}y''y'''+5m(m-1)y^{m-2}y'y^{(4)}$$
$$+\,my^{m-1}y^{(5)}. \tag{2.17}$$

By using (1.2), (2.3), (2.6), (2.9), (2.12), (2.15) and applying L' Hospital's rule to the second, the third and the fourth terms in (2.17) one obtains

$$y^{(7)}(0)=0.$$

Similarly, keep applying the same argument as above we get

$$y^{(8)}(0)=\frac{m(122m^2-183m+70)}{9.9},$$

$$y^{(9)}(0)=0 \quad \text{and}$$

$$y^{(10)}(0)=-\frac{m(5032m^3-12642m^2+10805m-3150)}{11.45}.$$

So for any m one formally has the following Taylor series expansion at $x=0$.

$$y(x)=\sum_{n=0}^{\infty}\frac{y^{(n)}(0)}{n!}x^n$$

$$=1-\frac{1}{3!}x^2+\frac{m}{5!}x^4-\frac{m(8m-5)}{3.7!}x^6+\frac{m(122m^2-183m+70)}{9.9!}x^8$$

$$-\frac{m(5032m^3 - 12642m^2 + 10805m - 3150)}{45.11!}x^{10} + \cdots . \tag{2.18}$$

Theorem 1. *Let* (1.2) *hold. Then the serial solution of the* (1.1) *at* $x = 0$ *has the form* (2.18).

It is also well known that

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \cdots \quad \text{where} \quad |x| < 1. \tag{2.19}$$

So one formally has the following well-known results.

Corollary 1. *If* $m = 0$, *then the series* (2.18) *converges to well-known solution*

$$y(x) = 1 - \frac{x^2}{6}.$$

Corollary 2. *If* $n \to \infty$ *and* $m = 1$, *then the series* (2.18) *which is in the form*

$$y(x) = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \frac{1}{9!}x^8 - \frac{1}{11!}x^{10} + \cdots$$

converges to well-known solution

$$y(x) = \frac{\sin x}{x}.$$

Corollary 3. *If* $n \to \infty$, $m = 5$ *and by using* (2.19), *then the series* (2.18) *which is in the form*

$$y(x) = 1 - \frac{1}{6}x^2 + \frac{1}{24}x^4 - \frac{5}{432}x^6 + \cdots$$

converges to well-known solution

$$y(x) = (1 + \frac{x^2}{3})^{-\frac{1}{2}}.$$

This elegant analytic result is also rediscovered analytically and numerically recently [10, 11].

Corollary 4. *If* $n \to \infty$ *and* $m = 2$, *then the series* (2.18) *has the form*

$$y(x) = 1 - \frac{1}{6}x^2 + \frac{1}{60}x^4 - \frac{11}{7560}x^6 + \cdots .$$

Corollary 5. *If* $n \to \infty$ *and* $m = 3$, *then the series* (2.18) *has the form*

$$y(x) = 1 - \frac{1}{6}x^2 + \frac{1}{40}x^4 - \frac{19}{5040}x^6 + \cdots .$$

Corollary 6. *If* $n \to \infty$ *and* $m = 4$, *then the series* (2.18) *has the form*

$$y(x) = 1 - \frac{1}{6}x^2 + \frac{1}{30}x^4 - \frac{1}{140}x^6 + \cdots .$$

3. Conclusion

Series solutions to the LE equation for any real m up to tenth terms have been accurately obtained by working directly to the original differential equation. The singularity of the Lane-Emden equation at the origin as it is carried out here has been also evaluated differently from the previous published papers in the literature. It is confirmed that the power series solutions obtained here satisfying (1.1) and (1.2) guarantees that numerical results obtained in [6, 7, 8, 9] are also correct. In [6, 7, 8, 9, 10, 11] and other numerical techniques which are not mentioned here considered $m \geq 0$ but this study considers the LE equation for any real m. Power series solutions are useful since they give a good approximation to the solution on a small domain and it is comparatively easy to analyze the behavior of the equation.

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