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# A STUDY ON A PARTIALLY NULL CURVE IN $\mathbb{E}_2^4$

ESEN İYİGÜN

Abstract. In this paper, we study with Frenet equations which are given in [3] of a partially null curve in Semi- Euclidean 4-space $\mathbb{E}_2^4$ with index 2. By using the Frenet equations, we give some theorem and corollary. A characterization of a hyperbolic partially null curve in $\mathbb{E}_2^4$ is given. Additionally, we examine harmonic curvatures and curvatures of this curve in $\mathbb{E}_2^4$.

## 1. Introduction

Semi-Euclidean $4-$space $\mathbb{E}_2^4$ with index 2 is the Euclidean $4-$space $\mathbb{E}^4$ equipped with an indefinite flat metric $g$ given by

$$g = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2,$$

where $(x_1, x_2, x_3, x_4)$ is a rectangular coordinate system of $\mathbb{E}_2^4$. A vector $v = (v_1, v_2, v_3, v_4)$ in $\mathbb{E}_2^4$ is a called a spacelike, a timelike or a null (lightlike), if respectively holds $g(v,v) > 0$, $g(v,v) < 0$ or $g(v,v) = 0$ and $v \neq 0 = (0,0,0,0)$. The norm of a vector $v$ is given by $\|v\| = \sqrt{|g(v,v)|}$. Two vectors $v$ and $w$ in $\mathbb{E}_2^4$ are said to be orthogonal, if $g(v,w) = 0$.

An arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_2^4$ can locally be spacelike, timelike or null, if respectively all of its velocity $\alpha'(s)$ are spacelike, timelike or null. Spacelike or timelike curve $\alpha(s)$ is said to be parametrized by arclength functions $s$, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. Let $a, b$ be two spacelike vectors in $\mathbb{E}_2^4$, then, there is unique real number $0 \leq \delta \leq \pi$, called angle between $a$ and $b$, such that $g(a,b) = \|a\| \, \|b\| \cos \delta$.

We also recall that the pseudosphere $S_2^3$ and the pseudohyperbolic space $H_1^3$ are the hyperquadrics in $\mathbb{E}_2^4$, defined respectively by:

$$\begin{aligned} S_2^3(c,r) &= \left\{ \alpha \epsilon \mathbb{E}_2^4 : g(\alpha - c, \alpha - c) = r^2 \right\}, \\ H_1^3(c,-r) &= \left\{ \alpha \epsilon \mathbb{E}_2^4 : g(\alpha - c, \alpha - c) = -r^2 \right\}, \end{aligned}$$

where center $c$ and radius $r \epsilon \mathbb{R}^+$ [1].

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Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors in $\mathbb{E}^4_2$. The vector product in $\mathbb{E}^4_2$ is defined with the determinant

$$a\Lambda b\Lambda c = -\begin{vmatrix} -e_1 & -e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix},$$

where $e_1, e_2, e_3$, and $e_4$ are coordinate direction vectors. Also, Frenet apparatus of a partially null curve in $\mathbb{E}^4_2$ are

$$T = \alpha'(s),\ N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|},\ B_2 = \frac{T\Lambda N\Lambda\alpha'''}{\|T\Lambda N\Lambda\alpha'''\|},\ B_1 = N\Lambda T\Lambda B_2$$

[4]

## 2. A Partially Null Curve in $\mathbb{E}^4_2$

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $\alpha = \alpha(s)$ in $\mathbb{E}^4_2$. Then $T, N, B_1, B_2$ are, respectively, the tangent, the principal normal ,the first binormal and the second binormal vector fields. Recall that spacelike curve with timelike principal normal and a null first and second binormal is called a partially null curve in $\mathbb{E}^4_2$. Then for a partially null curve $\alpha$ in $\mathbb{E}^4_2$, the following Frenet equations are given in [3]

$$\left\{\begin{array}{c} T'(s) = k_1(s)N(s), \\ N'(s) = k_1(s)T(s) + k_2(s)B_1(s), \\ B_1'(s) = k_3(s)B_1(s), \\ B_2'(s) = -\varepsilon_2 k_2(s)N(s) - k_3(s)B_2(s). \end{array}\right\} \tag{2.1}$$

where $T, N, B_1$ and $B_2$ are mutually orthogonal vectors satisfying equations

$$\left\{\begin{array}{c} g(T,T) = \varepsilon_1 = \pm 1,\ g(N,N) = \varepsilon_2 = \pm 1,\ \text{whereby } \varepsilon_1\varepsilon_2 = -1, \\ g(B_1, B_2) = 1,\ g(B_1, B_1) = g(B_2, B_2) = 0, \\ g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(N,B_2) = 0. \end{array}\right\} \tag{2.2}$$

And here,

$$\left\{\begin{array}{c} k_1(s) = g(T'(s), N(s))\varepsilon_2, \\ k_2(s) = g(N'(s), B_2(s)), \\ k_3(s) = g(B_1'(s), B_2(s)) \end{array}\right.$$

are first, second and third curvature of the curve $\alpha$, respectively. In the sequel, in [3] prove that $k_3(s) = 0$ for each $s$. Consequently, there are only two curvatures $k_1(s)$ and $k_2(s)$ in this case. Thus, Frenet equations are as follows:

$$\left\{\begin{array}{c} T'(s) = k_1(s)N(s), \\ N'(s) = k_1(s)T(s) + k_2(s)B_1(s), \\ B_1'(s) = 0, \\ B_2'(s) = -\varepsilon_2 k_2(s)N(s). \end{array}\right\} \tag{2.3}$$

**Theorem 1.** [3]*Let $\alpha$ be a partially null curve in $\mathbb{E}_2^4$. $\{T, N, B_1, B_2\}$ is the Frenet frame of $\alpha$. $T, N, B_1, B_2$ are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Then*

$$\left\{\begin{array}{c} g(T',T) = g(N',N) = g(B_1',B_1) = g(B_2',B_2) = 0, \\ g(T',N) = -g(N',T), \\ g(T',B_1) = -g(T,B_1'), \\ g(N',B_1) = -g(N,B_1'), \\ g(B_1',B_2) = -g(B_1,B_2'), \\ g(T,B_2') = -g(T',B_2), \\ g(N',B_2) = -g(N,B_2'). \end{array}\right\} \tag{2.4}$$

**Theorem 2.** *Let $\alpha$ be a partially null curve in $\mathbb{E}_2^4$ with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$ and $k_3(s) = 0$ for each $s$. $\{T, N, B_1, B_2\}$ is the Frenet frame of $\alpha$. $T, N, B_1, B_2$ are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Then*

$$\left\{\begin{array}{c} g(T',T) = g(T',B_1) = g(T',B_2) = g(N',B_1) = g(N',N) = 0, \\ g(B_1',T) = g(B_1',N) = g(B_1',B_1) = g(B_1',B_2) = 0, \\ g(B_2',T) = g(B_2',B_1) = g(B_2',B_2) = 0, \\ g(T',N) = \varepsilon_2 k_1, \\ g(N',T) = \varepsilon_1 k_1, \\ g(N',B_2) = -g(B_2',N). \end{array}\right\} \tag{2.5}$$

**Corollary 1.** *There is only one curvature $k_1$ in a previous theorem.*

**Corollary 2.** *i) If $\varepsilon_2 = 1$, then $g(T',N) = k_1$.*
*ii) If $\varepsilon_2 = -1$, then $g(T',N) = -k_1$.*
*iii) If $\varepsilon_1 = 1$, then $g(N',T) = k_1$.*
*iv) If $\varepsilon_1 = -1$, then $g(N',T) = -k_1$.*

**Theorem 3.** *Let $\alpha$ be a partially null curve in $\mathbb{E}_2^4$ with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$ and $k_3(s) = 0$ for each $s$. $\{T, N, B_1, B_2\}$ is the Frenet frame of $\alpha$. $T, N, B_1, B_2$ are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Then*

$$\left\{\begin{array}{c} g(T'',N) = g(T'',B_1) = g(N'',T) = g(N'',B_1) = g(N'',B_2) = 0, \\ g(B_1'',T) = g(B_1'',N) = g(B_1'',B_1) = g(B_1'',B_2) = g(B_2'',N) = g(B_2'',B_1) = 0, \\ g(T'',T) = -\varepsilon_2 k_1^2, \\ g(T'',B_2) = g(B_2'',T) = k_1 k_2, \\ g(N'',N) = -\varepsilon_1 k_1^2, \\ g(B_2'',B_2) = -\varepsilon_2 k_2^2. \end{array}\right.$$

*Proof.* By using Equations (2.2), (2.3) and (2.5), we obtain the proof of the theorem. □

**Corollary 3.** *There are only two curvatures $k_1$ and $k_2$ in a previous theorem.*

**Corollary 4.** *i) If* $\varepsilon_2 = 1$ *then* $g(T'', T) = -k_1^2$ *and* $g(B_2'', B_2) = -k_2^2$.
*ii) If* $\varepsilon_2 = -1$ *then* $g(T'', T) = k_1^2$ *and* $g(B_2'', B_2) = k_2^2$.
*iii) If* $\varepsilon_1 = 1$, *then* $g(N'', N) = -k_1^2$.
*iv) If* $\varepsilon_1 = -1$, *then* $g(N'', N) = k_1^2$.

## 3. A Hyperbolic Partially Null Curve in $\mathbb{E}_2^4$

**Theorem 4.** [2]*A partially null unit speed curve* $\alpha(s)$ *in* $\mathbb{E}_2^4$ *with curvatures* $k_1 \neq 0$, $k_2 \neq 0$ *for each* $s \epsilon I \subset \mathbb{R}$ *has* $k_3 = 0$ *for each* $s$.

**Theorem 5.** [2]*Let* $\alpha = \alpha(s)$ *be a unit speed partially null curve in* $\mathbb{E}_2^4$ *with curvatures* $k_1 \neq 0$, $k_2 \neq 0$ *for each* $s$. *If* $\alpha$ *lies on* $\mathbb{S}_2^3$ *Lorentzian hypersphere, then*

$$\frac{1}{k_2}\frac{d}{ds}\left(\frac{1}{k_1}\right) = constant.$$

**Theorem 6.** *Let* $\alpha = \alpha(s)$ *be a partially null curve in* $\mathbb{E}_2^4$ *with curvatures* $k_1 \neq 0$, $k_2 \neq 0$ *and* $k_3 = 0$ *for each* $s \epsilon I \subset \mathbb{R}$. *If* $\alpha$ *lies on a pseudohyperbolic space* $\mathbb{H}_1^3(c, -r)$, *then*

$$k_1 = \frac{\varepsilon_2}{r} = constant; \ \text{if } \varepsilon_2 = 1,$$

*where radius* $r \epsilon \mathbb{R}^+$ *and with center* $c$ *in* $\mathbb{E}_2^4$.

*Proof.* Let us suppose that $\alpha = \alpha(s)$ lies on $\mathbb{H}_1^3$ with center $c$. By the definition, we have

$$g(\alpha - c, \alpha - c) = -r^2, \tag{3.1}$$

for every $s \epsilon I \subset \mathbb{R}$. Differentiating (3.1), four times with respect to $s$ and using Frenet equations, we have, respectively,

$$\begin{cases} g(T, \alpha - c) = 0, \\ g(N, \alpha - c) = -\dfrac{\varepsilon_1}{k_1}, \\ g(B_1, \alpha - c) = 0, \\ g(B_2, \alpha - c) = 0. \end{cases}$$

Let us decompose $\alpha - c$ by

$$\alpha - c = -\frac{\varepsilon_1}{k_1}N.$$

Finally, if we calculate

$$g(\alpha - c, \alpha - c) = -r^2,$$

we easily obtain

$$k_1 = \frac{\varepsilon_2}{r} = constant,$$

where is $\varepsilon_2 = 1$. □

**Corollary 5.** *If* $\alpha$ *is a spacelike curve, that is,* $\varepsilon_2 = 1$ *then* $k_1 = \dfrac{1}{r}$.

## 4. Harmonic Curvatures of a Partially Null Curve in $\mathbb{E}_2^4$

**Definition 1.** [6]*Let $\alpha$ be a partially null curve in $\mathbb{E}_2^4$. The harmonic functions*

$$H_j : I \longrightarrow \mathbb{R} \quad , \quad j = 0, 1$$

*defined by*

$$\begin{cases} H_0 = 0, \ H_1 = \dfrac{k_1}{k_2}, \ (k_2 \neq 0), \end{cases}$$

*are called the harmonic curvatures of $\alpha$. Here, $k_1$ and $k_2$ are Frenet curvatures of $\alpha$.*

Now, the Theorem 2 and the Theorem 3 can be given in terms of harmonic curvatures as follows:

**Theorem 7.** *Let $\alpha$ be a partially null curve in $\mathbb{E}_2^4$. $T, N, B_2$ are, respectively, the tangent, the principal normal and the second binormal vector fields. Then*

$$\begin{cases} g(T', N) = \varepsilon_2 k_2 H_1, \quad g(T'', T) = -\varepsilon_2 k_2^2 H_1^2, \qquad g(B_2'', B_2) = -\varepsilon_2 \dfrac{k_1^2}{H_1^2}, \\ g(N', T) = \varepsilon_1 k_2 H_1, \quad g(N'', N) = -\varepsilon_1 k_2^2 H_1^2, \quad g(T'', B_2) = g(B_2'', T) = k_2^2 H_1, \end{cases}$$

*where $k_1$, $k_2$ are curvatures and $H_1$ is harmonic curvature of the curve $\alpha$.*

*Proof.* By using the definition of the harmonic curvatures, we obtain the proof of the theorem. □

**Corollary 6.** *i) If $\varepsilon_1 = 1$ then $g(N', T) = k_2 H_1$ and $g(N'', N) = -k_2^2 H_1^2$.*
*ii) If $\varepsilon_1 = -1$ then $g(N', T) = -k_2 H_1$ and $g(N'', N) = k_2^2 H_1^2$.*
*iii) If $\varepsilon_2 = 1$, then*

$$g(T', N) = k_2 H_1, \quad g(T'', T) = -k_2^2 H_1^2, \quad g(B_2'', B_2) = -\frac{k_1^2}{H_1^2}.$$

*iv) If $\varepsilon_2 = -1$, then*

$$g(T', N) = -k_2 H_1, \quad g(T'', T) = k_2^2 H_1^2, \quad g(B_2'', B_2) = \frac{k_1^2}{H_1^2}.$$

**Theorem 8.** *Let $\alpha$ be a partially null curve in $\mathbb{E}_2^4$ where $\{T, N, B_1, B_2\}$ is the Frenet frame of $\alpha$ and $k_1$, $k_2$ , $k_3$ are curvatures of $\alpha$. If $k_1 \neq 0, k_2 \neq 0$ and $k_3 = 0$ then*

$$\nabla_T^4 T - k_1^4 T - \frac{k_1^4}{H_1} B_1 = 0,$$

*where $\nabla_T T = T'$ and $\nabla$ is the Levi-Civita connection of $\mathbb{E}_2^4$.*

*Proof.* Since $k_3 = 0$, from Equation (2.3), we have

$$\nabla_T T = k_1 N \Longrightarrow \nabla_T^2 T = k_1 \nabla_T N \Longrightarrow \nabla_T^3 T = k_1 \nabla_T^2 N \Longrightarrow \nabla_T^4 T = k_1 \nabla_T^3 N.$$

Since

$$\nabla_T N = k_1 T + k_2 B_1,$$

$$\nabla_T^2 N = k_1^2 N,$$

we have

$$\nabla_T^3 N = k_1^3 T + k_1^2 k_2 B_1.$$

In that case

$$\nabla_T^4 T = k_1 \nabla_T^3 N = k_1^4 T + k_1^3 k_2 B_1.$$

Thus we have

$$\nabla_T^4 T - k_1^4 T - \frac{k_1^4}{H_1} B_1 = 0,$$

where $k_2 = \dfrac{k_1}{H_1}$. □

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*Current address*: Esen İyigün: Uludag University, Faculty of Art and Science, Department of Mathematics, 16059, Görükle, Bursa, Turkey.

*E-mail address*: esen@uludag.edu.tr

ORCID Address: http://orcid.org/0000-0001-6821-0248