



Existence results for Navier problems with degenerated (p,q) -Laplacian and (p,q) -Biharmonic operators

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Abstract

In this article, we prove the existence and uniqueness of solutions for the Navier problem

$$(P) \begin{cases} \Delta [\omega(x)(|\Delta u|^{p-2}\Delta u + |\Delta u|^{q-2}\Delta u)] \\ - \operatorname{div} [\omega(x)(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u)] \\ = f(x) - \operatorname{div}(G(x)), \quad \text{in } \Omega, \\ u(x) = \Delta u = 0, \quad \text{in } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of \mathbb{R}^N ($N \geq 2$), $f \in L^{p'}(\Omega, \omega)$ and $\frac{G}{\omega} \in [L^{q'}(\Omega, \omega)]^N$.

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1. Introduction and Preliminaries

The main purpose of this paper (see Theorem 3.3) is to establish the existence and uniqueness of solutions for the Navier problem

$$(P) \begin{cases} Lu(x) = f(x) - \operatorname{div}(G(x)), \quad \text{in } \Omega, \\ u(x) = \Delta u(x) = 0, \quad \text{in } \partial\Omega, \end{cases}$$

where

$$Lu(x) = \Delta [\omega(x)(|\Delta u|^{p-2}\Delta u + |\Delta u|^{q-2}\Delta u)] \\ - \operatorname{div} [\omega(x)(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u)],$$

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$\Omega \subset \mathbb{R}^N$ is a bounded open set, $\frac{f}{\omega} \in L^{p'}(\Omega, \omega)$, $\frac{G}{\omega} \in [L^{q'}(\Omega, \omega)]^N$, ω is a weight function (i.e., a locally integrable function on \mathbb{R}^N such that $0 < \omega(x) < \infty$ a.e. $x \in \mathbb{R}^N$), Δ is the Laplacian operator, $1 < q < p < \infty$, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1], [4], [5], [7], [8] and [11]). The type of a weight depends on the equation type.

A class of weights, which is particularly well understood, is the class of A_p weights that was introduced by B.Muckenhoupt in the early 1970's (see [8]). These classes have found many useful applications in harmonic analysis (see [9] and [10]). Another reason for studying A_p -weights is the fact that powers of the distance to submanifolds of \mathbb{R}^N often belong to A_p (see [3] and [11]). There are, in fact, many interesting examples of weights (see [7] for p -admissible weights).

In the non-degenerate case (i.e. with $\omega(x) \equiv 1$), for all $f \in L^p(\Omega)$ the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x), & \text{in } \Omega \\ u(x) = 0, & \text{in } \partial\Omega \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [6]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x), & \text{in } \Omega \\ u(x) = 0, & \text{in } \partial\Omega \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [2]), where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator. In the degenerate case, the degenerated p -Laplacian has been studied in [3].

The paper is organized as follow. In Section 2 we present the definitions and basic results. In Section 3 we prove our main result about existence and uniqueness of solutions for problem (P).

2. Definitions and Basic Results

By a weight we shall mean a locally integrable function ω on \mathbb{R}^N such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^N$. Every weight ω gives rise to a measure on the measurable subsets of \mathbb{R}^N through integration. This measure will be denoted by μ . Thus, $\mu(E) = \int_E \omega(x) dx$ for measurable sets $E \subset \mathbb{R}^N$.

Definition 2.1. Let $1 \leq p < \infty$. A weight ω is said to be an A_p -weight, if there is a positive constant C such that, for every ball $B \subset \mathbb{R}^N$

$$\begin{aligned} \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)}(x) dx \right)^{p-1} &\leq C, \text{ if } p > 1, \\ \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\operatorname{ess\,sup}_{x \in B} \frac{1}{\omega(x)} \right) &\leq C, \text{ if } p = 1, \end{aligned}$$

where $|\cdot|$ denotes the N -dimensional Lebesgue measure in \mathbb{R}^N . The infimum over all such constants C is called the A_p - constant of ω and is denoted by $C_{p,\omega}$.

If $1 < q \leq p$, then $A_q \subset A_p$ (see [5], [7] or [11] for more information about A_p - weights). As an example of an A_p -weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if $-N < \alpha < N(p - 1)$ (see [11], Chapter IX, Corollary 4.4). If $\varphi \in BMO(\mathbb{R}^N)$, then $\omega(x) = e^{\alpha \varphi(x)} \in A_2$ for some $\alpha > 0$ (see [9]).

Remark 2.2. If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|} \right)^p \leq C_{p,\omega} \frac{\mu(E)}{\mu(B)}$$

for all measurable subsets E of B (see 15.5 strong doubling property in [7]). Therefore, $\mu(E) = 0$ if and only if $|E| = 0$; so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e.. □

Definition 2.3. Let ω be a weight. We shall denote by $L^p(\Omega, \omega)$ ($1 \leq p < \infty$) the Banach space of all measurable functions f defined in Ω for which

$$\|f\|_{L^p(\Omega, \omega)} = \left(\int_{\Omega} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty.$$

We denote $[L^p(\Omega, \omega)]^N = L^p(\Omega, \omega) \times \dots \times L^p(\Omega, \omega)$.

Remark 2.4. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$ (see [11], Remark 1.2.4). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 2.5. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 < p < \infty$, k be a nonnegative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega, \omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega, \omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega, \omega)$, $1 \leq |\alpha| \leq k$. The norm in the space $W^{k,p}(\Omega, \omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega, \omega)} = \left(\int_{\Omega} |u(x)|^p \omega(x) dx + \sum_{1 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u(x)|^p \omega(x) dx \right)^{1/p}. \tag{2.1}$$

We also define the space $W_0^{k,p}(\Omega, \omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1). We have that the spaces $W^{k,p}(\Omega, \omega)$ and $W_0^{k,p}(\Omega, \omega)$ are Banach spaces (see Proposition 2.1.2 in [11]). The dual space of $W_0^{1,p}(\Omega, \omega)$ is the space $[W_0^{1,p}(\Omega, \omega)]^* = W^{-1,p'}(\Omega, \omega)$,

$$W^{-1,p'}(\Omega, \omega) = \{T = f - \text{div}(G) : G = (g_1, \dots, g_N), \frac{f}{\omega}, \frac{g_j}{\omega} \in L^{p'}(\Omega, \omega)\}.$$

It is evident that a weight function ω which satisfies $0 < C_1 \leq \omega(x) \leq C_2$, for a.e. $x \in \Omega$, gives nothing new (the space $W^{k,p}(\Omega, \omega)$ is then identical with the classical Sobolev space $W^{k,p}(\Omega)$). Consequently, we shall be interested in all above such weight functions ω which either vanish somewhere in $\Omega \cup \partial\Omega$ or increase to infinity (or both).

We need the following basics results.

Theorem 2.6. (The weighted Sobolev inequality) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let ω be an A_p -weight, $1 < p < \infty$. Then there exists positive constants C_Ω and δ such that for all $f \in C_0^\infty(\Omega)$ and $1 \leq \eta \leq N/(N-1) + \delta$

$$\|f\|_{L^{\eta p}(\Omega, \omega)} \leq C_\Omega \|\nabla f\|_{L^p(\Omega, \omega)}, \tag{2.2}$$

where C_Ω may be taken to depend only on N , the A_p - constant of ω , p and the diameter of Ω .

Proof. See [4], Theorem 1.3. □

Lemma 2.7. (a) Let $1 < p < \infty$, then exists a constant C_p such that for all $\xi, \eta \in \mathbb{R}^N$

$$|\xi|^{p-2}\xi - |\eta|^{p-2}\eta \leq C_p |\xi - \eta| (|\xi| + |\eta|)^{p-2}.$$

(b) Let $1 < p < \infty$. There exist two positive constants α_p and β_p such that for every $\xi, \eta \in \mathbb{R}^N$ ($N \geq 1$)

$$\alpha_p (|\xi| + |\eta|)^{p-2} |\xi - \eta|^2 \leq \langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta \rangle \leq \beta_p (|\xi| + |\eta|)^{p-2} |\xi - \eta|^2,$$

where $\langle \cdot, \cdot \rangle$ denotes here the Euclidean scalar product in \mathbb{R}^N .

Proof. See Proposition 17.2 and Proposition 17.3 in [2]. □

3. Main Results

We denote by $X = W^{2,p}(\Omega, \omega) \cap W_0^{1,p}(\Omega, \omega)$ with the norm

$$\|u\|_X = \left(\int_{\Omega} |\nabla u|^p \omega \, dx + \int_{\Omega} |\Delta u|^p \omega \, dx \right)^{1/p}.$$

In this section we prove the existence and uniqueness of weak solutions $u \in X$ to the Navier problem

$$(P) \begin{cases} Lu(x) = f(x) - \operatorname{div}(G(x)), & \text{in } \Omega, \\ u(x) = \Delta u = 0, & \text{in } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of \mathbb{R}^N ($N \geq 2$), $\frac{f}{\omega} \in L^{p'}(\Omega, \omega)$ and $\frac{G}{\omega} \in [L^{q'}(\Omega, \omega)]^N$, $G = (g_1, \dots, g_N)$.

Definition 3.1. We say that $u \in X$ is a weak solution for problem (P) if

$$\begin{aligned} & \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \varphi \omega \, dx + \int_{\Omega} |\Delta u|^{q-2} \Delta u \Delta \varphi \omega \, dx \\ & + \int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \omega \, dx + \int_{\Omega} |\nabla u|^{q-2} \langle \nabla u, \nabla \varphi \rangle \omega \, dx \\ & = \int_{\Omega} f \varphi \, dx + \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx, \end{aligned} \tag{3.1}$$

for all $\varphi \in X$, with $f/\omega \in L^{p'}(\Omega, \omega)$ and $G/\omega \in [L^{q'}(\Omega, \omega)]^N$, where $\langle \cdot, \cdot \rangle$ denotes here the Euclidean scalar product in \mathbb{R}^N .

Remark 3.2. (i) Since $1 < q < p < \infty$, there exists a constant $C_{p,q} = [\mu(\Omega)]^{(p-q)/pq}$ such that

$$\|u\|_{L^q(\Omega, \omega)} \leq C_{p,q} \|u\|_{L^p(\Omega, \omega)}, \text{ where } \mu(\Omega) = \int_{\Omega} \omega(x) \, dx.$$

(ii) Since $1 < q < p < \infty$, then $1 < p' < q' < \infty$, and there exists a constant

$$\tilde{C}_{p,q} = [\mu(\Omega)]^{(q'-p')/q'p'}$$

such that $\|u\|_{L^{p'}(\Omega, \omega)} \leq \tilde{C}_{p,q} \|u\|_{L^{q'}(\Omega, \omega)}$.

(iii) By (ii), if $G/\omega \in [L^{q'}(\Omega, \omega)]^N$, then $G/\omega \in [L^{p'}(\Omega, \omega)]^N$.

Hence, $T = f - \operatorname{div}(G) \in [W_0^{1,p}(\Omega, \omega)]^*$.

Theorem 3.3. Let $\omega \in A_p$, $1 < q < p < \infty$, $\frac{f}{\omega} \in L^{p'}(\Omega, \omega)$ and $\frac{G}{\omega} \in [L^{q'}(\Omega, \omega)]^N$. Then the problem (P) has a unique solution $u \in X$ and

$$\|u\|_X \leq \left[C_{\Omega} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} + C_{p,q} \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \right]^{1/(p-1)},$$

where C_{Ω} is the constant in Theorem 2.6 and $C_{p,q}$ is the constant in Remark 3.2 (i).

Proof. (I) *Existence.* By Theorem 2.6 (with $\eta = 1$), we have that

$$\begin{aligned} \left| \int_{\Omega} f \varphi \, dx \right| & \leq \left(\int_{\Omega} \left| \frac{f}{\omega} \right|^{p'} \omega \, dx \right)^{1/p'} \left(\int_{\Omega} |\varphi|^p \omega \, dx \right)^{1/p} \\ & \leq C_{\Omega} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} \| |\nabla \varphi| \|_{L^p(\Omega, \omega)} \\ & \leq C_{\Omega} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} \|\varphi\|_X, \end{aligned} \tag{3.2}$$

and by Remark 3.2(i)

$$\begin{aligned}
 \left| \int_{\Omega} \langle G, \nabla \varphi \rangle dx \right| &\leq \int_{\Omega} |\langle G, \nabla \varphi \rangle| dx \\
 &\leq \int_{\Omega} |G| |\nabla \varphi| dx \\
 &= \int_{\Omega} \frac{|G|}{\omega} |\nabla \varphi| \omega dx \\
 &\leq \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \|\nabla \varphi\|_{L^q(\Omega, \omega)} \\
 &\leq C_{p,q} \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \|\nabla \varphi\|_{L^p(\Omega, \omega)} \\
 &\leq C_{p,q} \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \|\varphi\|_X.
 \end{aligned} \tag{3.3}$$

Define the functional $J : X \rightarrow \mathbb{R}$ by

$$\begin{aligned}
 J(\varphi) &= \frac{1}{p} \int_{\Omega} |\Delta \varphi|^p \omega dx + \frac{1}{q} \int_{\Omega} |\Delta \varphi|^q \omega dx \\
 &\quad + \frac{1}{p} \int_{\Omega} |\nabla \varphi|^p \omega dx + \frac{1}{q} \int_{\Omega} |\nabla \varphi|^q \omega dx - \int_{\Omega} f \varphi dx - \int_{\Omega} \langle G, \nabla \varphi \rangle dx.
 \end{aligned}$$

Using (3.2), (3.3), Remark 3.2(i) and Young’s inequality, we have that

$$\begin{aligned}
 J(\varphi) &\geq \frac{1}{p} \int_{\Omega} |\Delta \varphi|^p \omega dx + \frac{1}{q} \int_{\Omega} |\Delta \varphi|^q \omega dx \\
 &\quad + \frac{1}{p} \int_{\Omega} |\nabla \varphi|^p \omega dx + \frac{1}{q} \int_{\Omega} |\nabla \varphi|^q \omega dx \\
 &\quad - \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} \|\varphi\|_{L^p(\Omega, \omega)} - \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \|\nabla \varphi\|_{L^q(\Omega, \omega)} \\
 &\geq \frac{1}{p} \int_{\Omega} |\nabla \varphi|^p \omega dx + \frac{1}{q} \int_{\Omega} |\nabla \varphi|^q \omega dx \\
 &\quad - C_{\Omega} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} \|\nabla \varphi\|_{L^p(\Omega, \omega)} \\
 &\quad - \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \|\nabla \varphi\|_{L^q(\Omega, \omega)} \\
 &\geq \frac{1}{p} \int_{\Omega} |\nabla \varphi|^p \omega dx + \frac{1}{q} \int_{\Omega} |\nabla \varphi|^q \omega dx \\
 &\quad - \frac{C_{\Omega}^{p'}}{p'} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)}^{p'} - \frac{1}{p} \|\nabla \varphi\|_{L^p(\Omega, \omega)}^p - \frac{1}{q'} \left\| \frac{|G|}{\omega} \right\|_{L^{p'}(\Omega, \omega)}^{q'} \\
 &\quad - \frac{1}{q} \|\nabla \varphi\|_{L^q(\Omega, \omega)}^q \\
 &= - \frac{C_{\Omega}^{p'}}{p'} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)}^{p'} - \frac{1}{q'} \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)}^{q'}
 \end{aligned}$$

that is, J is bounded from below. Let $\{u_n\}$ be a minimizing sequence, that is, a sequence such that

$$J(u_n) \rightarrow \inf_{\varphi \in X} J(\varphi).$$

Then for n large enough, we obtain

$$\begin{aligned} 0 \geq J(u_n) &= \frac{1}{p} \int_{\Omega} |\Delta u_n|^p \omega \, dx + \frac{1}{q} \int_{\Omega} |\Delta u_n|^q \omega \, dx \\ &+ \frac{1}{p} \int_{\Omega} |\nabla u_n|^p \omega \, dx + \frac{1}{q} \int_{\Omega} |\nabla u_n|^q \omega \, dx \\ &- \int_{\Omega} f u_n \, dx - \int_{\Omega} \langle G, \nabla u_n \rangle \, dx, \end{aligned}$$

and we have

$$\begin{aligned} &\frac{1}{p} \int_{\Omega} |\Delta u_n|^p \omega \, dx + \frac{1}{p} \int_{\Omega} |\nabla u_n|^p \omega \, dx \\ &\leq \frac{1}{p} \int_{\Omega} |\Delta u_n|^p \omega \, dx + \frac{1}{q} \int_{\Omega} |\Delta u_n|^q \omega \, dx + \frac{1}{p} \int_{\Omega} |\nabla u_n|^p \omega \, dx \\ &+ \frac{1}{q} \int_{\Omega} |\nabla u_n|^q \omega \, dx \\ &\leq \int_{\Omega} f u_n \, dx + \int_{\Omega} \langle G, u_n \rangle \, dx. \end{aligned}$$

Hence, by Theorem 2.6 (with $\eta = 1$) and Remark ??(i), we obtain

$$\begin{aligned} \|u_n\|_X^p &= \int_{\Omega} |\Delta u_n|^p \omega \, dx + \int_{\Omega} |\nabla u_n|^p \omega \, dx \\ &\leq p \left(\int_{\Omega} f u_n \, dx + \int_{\Omega} \langle G, \nabla u_n \rangle \, dx \right) \\ &\leq p \left(\left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} \|u_n\|_{L^p(\Omega, \omega)} + \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \|\nabla u_n\|_{L^q(\Omega, \omega)} \right) \\ &\leq p \left(C_{\Omega} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} \|\nabla u_n\|_{L^p(\Omega, \omega)} + C_{p,q} \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \|\nabla u_n\|_{L^p(\Omega, \omega)} \right) \\ &\leq p \left(C_{\Omega} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} + C_{p,q} \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \right) \|u_n\|_X. \end{aligned}$$

Hence,

$$\|u_n\|_X \leq \left[p \left(C_{\Omega} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} + C_{p,q} \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \right) \right]^{1/(p-1)}.$$

Therefore $\{u_n\}$ is bounded in X . Since X is reflexive, there exists a subsequence, still denoted by $\{u_n\}$, and a function $u \in X$ such that $u_n \rightharpoonup u$ in X . Since,

$$X \ni \varphi \mapsto \int_{\Omega} f \varphi \, dx + \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx,$$

and

$$X \ni \varphi \mapsto \|\Delta \varphi\|_{L^p(\Omega, \omega)} + \|\Delta \varphi\|_{L^q(\Omega, \omega)} + \|\nabla \varphi\|_{L^p(\Omega, \omega)} + \|\nabla \varphi\|_{L^q(\Omega, \omega)},$$

are continuous then J is continuous. Moreover since $1 < q < p < \infty$ we have that J is convex and thus lower semi-continuous for the weak convergence. It follows that

$$J(u) \leq \liminf_n J(u_n) = \inf_{\varphi \in X} J(\varphi),$$

and thus u is a minimizer of J on X (see Theorem 25.C and Corollary 25.15 in [12]). For any $\varphi \in X$ the function

$$\begin{aligned} \lambda \mapsto & \frac{1}{p} \int_{\Omega} |\Delta(u + \lambda\varphi)|^p \omega \, dx + \frac{1}{q} \int_{\Omega} |\Delta(u + \lambda\varphi)|^q \omega \, dx \\ & + \frac{1}{p} \int_{\Omega} |\nabla(u + \lambda\varphi)|^p \omega \, dx + \frac{1}{q} \int_{\Omega} |\nabla(u + \lambda\varphi)|^q \omega \, dx \\ & - \int_{\Omega} (u + \lambda\varphi) f \, dx - \int_{\Omega} \langle G, \nabla(u + \lambda\varphi) \rangle \, dx \end{aligned}$$

has a minimum at $\lambda = 0$. Hence,

$$\left. \frac{d}{d\lambda} \left(J(u + \lambda\varphi) \right) \right|_{\lambda=0} = 0, \quad \forall \varphi \in X.$$

We have

$$\frac{d}{d\lambda} \left(|\nabla(u + \lambda\varphi)|^p \omega \right) = p \{ |\nabla(u + \lambda\varphi)|^{p-2} (\langle \nabla u, \nabla \varphi \rangle + \lambda |\nabla \varphi|^2) \} \omega,$$

and

$$\frac{d}{d\lambda} \left(|\Delta(u + \lambda\varphi)|^p \omega \right) = p |\Delta u + \lambda \Delta \varphi|^{p-2} (\Delta u + \lambda \Delta \varphi) \Delta \varphi \omega.$$

Then we obtain

$$\begin{aligned} 0 &= \left. \frac{d}{d\lambda} \left(J(u + \lambda\varphi) \right) \right|_{\lambda=0} \\ &= \left[\frac{1}{p} \left(p \int_{\Omega} |\nabla(u + \lambda\varphi)|^{p-2} (\langle \nabla u, \nabla \varphi \rangle + \lambda |\nabla \varphi|^2) \omega \, dx \right. \right. \\ &+ p \int_{\Omega} |\Delta u + \lambda \Delta \varphi|^{p-2} (\Delta u + \lambda \Delta \varphi) \Delta \varphi \omega \, dx \\ &+ \frac{1}{q} \left(q \int_{\Omega} |\nabla(u + \lambda\varphi)|^{q-2} (\langle \nabla u, \nabla \varphi \rangle + \lambda |\nabla \varphi|^2) \omega \, dx \right. \\ &+ q \int_{\Omega} |\Delta u + \lambda \Delta \varphi|^{q-2} (\Delta u + \lambda \Delta \varphi) \Delta \varphi \omega \, dx \\ &\left. \left. - \int_{\Omega} \varphi f \, dx - \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx \right) \right]_{\lambda=0} \\ &= \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \varphi \omega \, dx + \int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \omega \, dx \\ &+ \int_{\Omega} |\Delta u|^{q-2} \Delta u \Delta \varphi \omega \, dx + \int_{\Omega} |\nabla u|^{q-2} \langle \nabla u, \nabla \varphi \rangle \omega \, dx \\ &- \int_{\Omega} f \varphi \, dx - \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx. \end{aligned}$$

Therefore

$$\begin{aligned} & \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \varphi \omega \, dx + \int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \omega \, dx \\ &+ \int_{\Omega} |\Delta u|^{q-2} \Delta u \Delta \varphi \omega \, dx + \int_{\Omega} |\nabla u|^{q-2} \langle \nabla u, \nabla \varphi \rangle \omega \, dx \\ &= \int_{\Omega} f \varphi \, dx + \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx, \end{aligned}$$

for all $\varphi \in X$, that is, $u \in X$ is a solution of problem (P).

(II) *Uniqueness.* If $u_1, u_2 \in X$ are two weak solutions of problem (P), we have for all $\varphi \in X$,

$$\begin{aligned} & \int_{\Omega} |\Delta u_1|^{p-2} \Delta u_1 \Delta \varphi \omega \, dx + \int_{\Omega} |\Delta u_1|^{q-2} \Delta u_1 \Delta \varphi \omega \, dx \\ & + \int_{\Omega} |\nabla u_1|^{p-2} \langle \nabla u_1, \nabla \varphi \rangle \omega \, dx + \int_{\Omega} |\nabla u_1|^{q-2} \langle \nabla u_1, \nabla \varphi \rangle \omega \, dx \\ & = \int_{\Omega} f \varphi \, dx + \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx, \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} |\Delta u_2|^{p-2} \Delta u_2 \Delta \varphi \omega \, dx + \int_{\Omega} |\Delta u_2|^{q-2} \Delta u_2 \Delta \varphi \omega \, dx \\ & + \int_{\Omega} |\nabla u_2|^{p-2} \langle \nabla u_2, \nabla \varphi \rangle \omega \, dx + \int_{\Omega} |\nabla u_2|^{q-2} \langle \nabla u_2, \nabla \varphi \rangle \omega \, dx \\ & = \int_{\Omega} f \varphi \, dx + \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx. \end{aligned}$$

Hence

$$\begin{aligned} & \int_{\Omega} \left(|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2 \right) \Delta \varphi \omega \, dx \\ & + \int_{\Omega} \left(|\Delta u_1|^{q-2} \Delta u_1 - |\Delta u_2|^{q-2} \Delta u_2 \right) \Delta \varphi \omega \, dx \\ & + \int_{\Omega} \left(|\nabla u_1|^{p-2} \langle \nabla u_1, \nabla \varphi \rangle - |\nabla u_2|^{p-2} \langle \nabla u_2, \nabla \varphi \rangle \right) \omega \, dx \\ & + \int_{\Omega} \left(|\nabla u_1|^{q-2} \langle \nabla u_1, \nabla \varphi \rangle - |\nabla u_2|^{q-2} \langle \nabla u_2, \nabla \varphi \rangle \right) \omega \, dx = 0. \end{aligned}$$

Taking $\varphi = u_1 - u_2$, and using Lemma 2.7(b) there exist positive constants $\alpha_p, \tilde{\alpha}_p, \alpha_q, \tilde{\alpha}_q$ such that

$$\begin{aligned} 0 & = \int_{\Omega} \left(|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2 \right) (\Delta u_1 - \Delta u_2) \omega \, dx \\ & + \int_{\Omega} \left(|\Delta u_1|^{q-2} \Delta u_1 - |\Delta u_2|^{q-2} \Delta u_2 \right) (\Delta u_1 - \Delta u_2) \omega \, dx \\ & + \int_{\Omega} \left(|\nabla u_1|^{p-2} \langle \nabla u_1, \nabla u_1 - \nabla u_2 \rangle - |\nabla u_2|^{p-2} \langle \nabla u_2, \nabla u_1 - \nabla u_2 \rangle \right) \omega \, dx \\ & + \int_{\Omega} \left(|\nabla u_1|^{q-2} \langle \nabla u_1, \nabla u_1 - \nabla u_2 \rangle - |\nabla u_2|^{q-2} \langle \nabla u_2, \nabla u_1 - \nabla u_2 \rangle \right) \omega \, dx \\ & = \int_{\Omega} \left(|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2 \right) (\Delta u_1 - \Delta u_2) \omega \, dx \\ & + \int_{\Omega} \left(|\Delta u_1|^{q-2} \Delta u_1 - |\Delta u_2|^{q-2} \Delta u_2 \right) (\Delta u_1 - \Delta u_2) \omega \, dx \\ & + \int_{\Omega} \langle |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2, \nabla u_1 - \nabla u_2 \rangle \omega \, dx \\ & + \int_{\Omega} \langle |\nabla u_1|^{q-2} \nabla u_1 - |\nabla u_2|^{q-2} \nabla u_2, \nabla u_1 - \nabla u_2 \rangle \omega \, dx \\ & \geq \alpha_p \int_{\Omega} \left(|\Delta u_1| + |\Delta u_2| \right)^{p-2} |\Delta u_1 - \Delta u_2|^2 \omega \, dx \\ & + \tilde{\alpha}_p \int_{\Omega} \left(|\nabla u_1| + |\nabla u_2| \right)^{p-2} |\nabla u_1 - \nabla u_2|^2 \omega \, dx \\ & + \alpha_q \int_{\Omega} \left(|\Delta u_1| + |\Delta u_2| \right)^{q-2} |\Delta u_1 - \Delta u_2|^2 \omega \, dx \\ & + \tilde{\alpha}_q \int_{\Omega} \left(|\nabla u_1| + |\nabla u_2| \right)^{q-2} |\nabla u_1 - \nabla u_2|^2 \omega \, dx. \end{aligned}$$

Therefore $\Delta u_1 = \Delta u_2$ and $\nabla u_1 = \nabla u_2$ a.e. and since $u_1, u_2 \in X$, then $u_1 = u_2$ a.e. (by Remark 2.2).

(III) *Estimate for $\|u\|_X$.*

In particular, for $\varphi = u \in X$ in Definition 3.1 we have

$$\begin{aligned} & \int_{\Omega} |\Delta u|^p \omega \, dx + \int_{\Omega} |\Delta u|^q \omega \, dx + \int_{\Omega} |\nabla u|^p \omega \, dx + \int_{\Omega} |\nabla u|^q \omega \, dx \\ &= \int_{\Omega} f u \, dx + \int_{\Omega} \langle G, \nabla u \rangle \, dx. \end{aligned}$$

Then, by Theorem 2.6 and Remark 3.2(i), we obtain

$$\begin{aligned} \|u\|_X^p &= \int_{\Omega} |\Delta u|^p \omega \, dx + \int_{\Omega} |\nabla u|^p \omega \, dx \\ &\leq \int_{\Omega} |\Delta u|^p \omega \, dx + \int_{\Omega} |\Delta u|^q \omega \, dx + \int_{\Omega} |\nabla u|^p \omega \, dx + \int_{\Omega} |\nabla u|^q \omega \, dx \\ &= \int_{\Omega} f u \, dx + \int_{\Omega} \langle G, \nabla u \rangle \, dx \\ &\leq \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} \|u\|_{L^p(\Omega, \omega)} + \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \|\nabla u\|_{L^q(\Omega, \omega)} \\ &\leq C_{\Omega} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} \|\nabla u\|_{L^p(\Omega, \omega)} + C_{p,q} \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \|\nabla u\|_{L^p(\Omega, \omega)} \\ &\leq \left(C_{\Omega} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} + C_{p,q} \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \right) \|u\|_X. \end{aligned}$$

Therefore,

$$\|u\|_X \leq \left(C_{\Omega} \left\| \frac{f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} + C_{p,q} \left\| \frac{|G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \right)^{1/(p-1)}.$$

□

Corollary 3.4. *Under the assumptions of Theorem 3.3 with $2 \leq q < p < \infty$. If $u_1, u_2 \in X$ are solutions of*

$$(P_1) \begin{cases} Lu_1(x) = f(x) - \operatorname{div}(G(x)), & \text{in } \Omega, \\ u_1(x) = \Delta u_1(x) = 0, & \text{in } \partial\Omega, \end{cases}$$

and

$$(P_2) \begin{cases} Lu_2(x) = \tilde{f}(x) - \operatorname{div}(\tilde{G}(x)), & \text{in } \Omega, \\ u_2(x) = \Delta u_2(x) = 0, & \text{in } \partial\Omega, \end{cases}$$

then

$$\|u_1 - u_2\|_X \leq \frac{1}{\gamma^{1/(p-1)}} \left(C_{\Omega} \left\| \frac{f - \tilde{f}}{\omega} \right\|_{L^{p'}(\Omega, \omega)} + C_{p,q} \left\| \frac{|G - \tilde{G}|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \right)^{1/(p-1)},$$

where γ is a positive constant, C_{Ω} and $C_{p,q}$ are the same constants of Theorem 3.3.

Proof. If u_1 and u_2 are solutions of (P1) and (P2) then for all $\varphi \in X$ we have

$$\begin{aligned}
 & \int_{\Omega} |\Delta u_1|^{p-2} \Delta u_1 \Delta \varphi \omega \, dx + \int_{\Omega} |\Delta u_1|^{q-2} \Delta u_1 \Delta \varphi \omega \, dx \\
 & + \int_{\Omega} |\nabla u_1|^{p-2} \langle \nabla u_1, \nabla \varphi \rangle \omega \, dx + \int_{\Omega} |\nabla u_1|^{q-2} \langle \nabla u_1, \nabla \varphi \rangle \omega \, dx \\
 & - \left(\int_{\Omega} |\Delta u_2|^{p-2} \Delta u_2 \Delta \varphi \omega \, dx + \int_{\Omega} |\Delta u_2|^{q-2} \Delta u_2 \Delta \varphi \omega \, dx \right. \\
 & \left. + \int_{\Omega} |\nabla u_2|^{p-2} \langle \nabla u_2, \nabla \varphi \rangle \omega \, dx + \int_{\Omega} |\nabla u_2|^{q-2} \langle \nabla u_2, \nabla \varphi \rangle \omega \, dx \right) \\
 & = \int_{\Omega} (f - \tilde{f}) \varphi \, dx + \int_{\Omega} \langle G - \tilde{G}, \nabla \varphi \rangle \, dx.
 \end{aligned} \tag{3.4}$$

In particular, for $\varphi = u_1 - u_2$, we obtain

(i) Since $2 \leq q < p < \infty$ and by Lemma 2.7(b), there exist two positive constants α_p and α_q such that

$$\begin{aligned}
 & \int_{\Omega} \left(|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2 \right) \Delta(u_1 - u_2) \omega \, dx \\
 & \geq \alpha_p \int_{\Omega} \left(|\Delta u_1| + |\Delta u_2| \right)^{p-2} |\Delta u_1 - \Delta u_2|^2 \omega \, dx \\
 & \geq \alpha_p \int_{\Omega} |\Delta u_1 - \Delta u_2|^{p-2} |\Delta u_1 - \Delta u_2|^2 \omega \, dx = \alpha_p \int_{\Omega} |\Delta(u_1 - u_2)|^p \omega \, dx,
 \end{aligned}$$

and analogously

$$\int_{\Omega} \left(|\Delta u_1|^{q-2} \Delta u_1 - |\Delta u_2|^{q-2} \Delta u_2 \right) \Delta(u_1 - u_2) \omega \, dx \geq \alpha_q \int_{\Omega} |\Delta(u_1 - u_2)|^q \omega \, dx \geq 0.$$

(ii) Since $2 \leq q < p < \infty$ and by Lemma 2.7(b), there exist two positive constants $\tilde{\alpha}_p$ and $\tilde{\alpha}_q$ such that

$$\begin{aligned}
 & \int_{\Omega} \left(|\nabla u_1|^{p-2} \langle \nabla u_1, \nabla(u_1 - u_2) \rangle - |\nabla u_2|^{p-2} \langle \nabla u_2, \nabla(u_1 - u_2) \rangle \right) \omega \, dx \\
 & = \int_{\Omega} \langle |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2, \nabla(u_1 - u_2) \rangle \omega \, dx \\
 & \geq \tilde{\alpha}_p \int_{\Omega} (|\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla u_1 - \nabla u_2|^2 \omega \, dx \\
 & \geq \tilde{\alpha}_p \int_{\Omega} |\nabla u_1 - \nabla u_2|^{p-2} |\nabla u_1 - \nabla u_2|^2 \omega \, dx = \tilde{\alpha}_p \int_{\Omega} |\nabla(u_1 - u_2)|^p \omega \, dx,
 \end{aligned}$$

and analogously,

$$\begin{aligned}
 & \int_{\Omega} \left(|\nabla u_1|^{q-2} \langle \nabla u_1, \nabla(u_1 - u_2) \rangle - |\nabla u_2|^{q-2} \langle \nabla u_2, \nabla(u_1 - u_2) \rangle \right) \omega \, dx \\
 & \geq \tilde{\alpha}_q \int_{\Omega} |\nabla(u_1 - u_2)|^q \omega \, dx \geq 0.
 \end{aligned}$$

(iii) By Remark 3.2(i) we have

$$\begin{aligned}
 & \left| \int_{\Omega} (f - \tilde{f})(u_1 - u_2) \, dx + \int_{\Omega} \langle G - \tilde{G}, \nabla(u_1 - u_2) \rangle \, dx \right| \\
 & \leq \left(C_{\Omega} \left\| \frac{f - \tilde{f}}{\omega} \right\|_{L^{p'}(\Omega, \omega)} + C_{p,q} \left\| \frac{|G - \tilde{G}|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \right) \|u_1 - u_2\|_X.
 \end{aligned}$$

Hence, with $\gamma = \min\{\alpha_p, \tilde{\alpha}_p\}$, we obtain in (3.4)

$$\begin{aligned} \gamma \|u_1 - u_2\|_X^p &\leq \alpha_p \int_{\Omega} |\Delta(u_1 - u_2)|^p \omega \, dx + \tilde{\alpha}_p \int_{\Omega} |\nabla(u_1 - u_2)|^p \omega \, dx \\ &\leq \left(C_{\Omega} \left\| \frac{f - \tilde{f}}{\omega} \right\|_{L^{p'}(\Omega, \omega)} + C_{p,q} \left\| \frac{|G - \tilde{G}|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \right) \|u_1 - u_2\|_X. \end{aligned}$$

Therefore,

$$\|u_1 - u_2\|_X \leq \frac{1}{\gamma^{1/(p-1)}} \left(C_{\Omega} \left\| \frac{f - \tilde{f}}{\omega} \right\|_{L^{p'}(\Omega, \omega)} + C_{p,q} \left\| \frac{|G - \tilde{G}|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \right)^{1/(p-1)}.$$

□

Corollary 3.5. *Assume $2 \leq q < p < \infty$. Let the assumptions of Theorem 3.3 be fulfilled, and let $\{f_m\}$ and $\{G_m\}$ be sequences of functions satisfying $\frac{f_m}{\omega} \rightarrow \frac{f}{\omega}$ in $L^{p'}(\Omega, \omega)$ and $\left\| \frac{|G_m - G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \rightarrow 0$ as $m \rightarrow \infty$. If $u_m \in X$ is a solution of the problem*

$$(P_m) \begin{cases} Lu_m(x) = f_m(x) - \operatorname{div}(G_m(x)), & \text{in } \Omega, \\ u_m(x) = \Delta u_m(x) = 0, & \text{in } \partial\Omega, \end{cases}$$

then $u_m \rightarrow u$ in X and u is a solution of problem (P).

Proof. By Corollary 3.4 we have

$$\begin{aligned} &\|u_m - u_r\|_X \\ &\leq \frac{1}{\gamma^{1/(p-1)}} \left(C_{\Omega} \left\| \frac{f_m - f_r}{\omega} \right\|_{L^{p'}(\Omega, \omega)} + C_{p,q} \left\| \frac{|G_m - G_r|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \right)^{1/(p-1)}. \end{aligned}$$

Therefore $\{u_m\}$ is a Cauchy sequence in X . Hence, there is $u \in X$ such that $u_m \rightarrow u$ in X . We have that u is a solution of problem (P). In fact, since u_m is a solution of (P_m) , for all $\varphi \in X$ we have

$$\begin{aligned} &\int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \varphi \omega \, dx + \int_{\Omega} |\Delta u|^{q-2} \Delta u \Delta \varphi \omega \, dx \\ &+ \int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \omega \, dx + \int_{\Omega} |\nabla u|^{q-2} \langle \nabla u, \nabla \varphi \rangle \omega \, dx \\ &= \int_{\Omega} \left(|\Delta u|^{p-2} \Delta u - |\Delta_m|^{p-2} \Delta u_m \right) \Delta \varphi \omega \, dx \\ &+ \int_{\Omega} \left(|\Delta u|^{q-2} \Delta u - |\Delta u_m|^{q-2} \Delta u_m \right) \Delta \varphi \omega \, dx \\ &+ \int_{\Omega} \left(|\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle - |\nabla u_m|^{p-2} \langle \nabla u_m, \nabla \varphi \rangle \right) \omega \, dx \\ &+ \int_{\Omega} \left(|\nabla u|^{q-2} \langle \nabla u, \nabla \varphi \rangle - |\nabla u_m|^{q-2} \langle \nabla u_m, \nabla \varphi \rangle \right) \omega \, dx \\ &+ \int_{\Omega} |\Delta u_m|^{p-2} \Delta u_m \Delta \varphi \omega \, dx + \int_{\Omega} |\Delta u_m|^{q-2} \Delta u_m \Delta \varphi \omega \, dx \\ &+ \int_{\Omega} |\nabla u_m|^{p-2} \langle \nabla u_m, \nabla \varphi \rangle \omega \, dx + \int_{\Omega} |\nabla u_m|^{q-2} \langle \nabla u_m, \nabla \varphi \rangle \omega \, dx \\ &= I_1 + I_2 + I_3 + I_4 + \int_{\Omega} f_m \varphi \, dx + \int_{\Omega} \langle G_m, \nabla \varphi \rangle \, dx \\ &= I_1 + I_2 + I_3 + I_4 + \int_{\Omega} f \varphi \, dx + \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx \\ &+ \int_{\Omega} (f_m - f) \varphi \, dx + \int_{\Omega} \langle G_m - G, \nabla \varphi \rangle \, dx, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
 I_1 &= \int_{\Omega} \left(|\Delta u|^{p-2} \Delta u - |\Delta u_m|^{p-2} \Delta u_m \right) \Delta \varphi \omega \, dx, \\
 I_2 &= \int_{\Omega} \left(|\Delta u|^{q-2} \Delta u - |\Delta u_m|^{q-2} \Delta u_m \right) \Delta \varphi \omega \, dx, \\
 I_3 &= \int_{\Omega} \left(|\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle - |\nabla u_m|^{p-2} \langle \nabla u_m, \nabla \varphi \rangle \right) \omega \, dx, \\
 I_4 &= \int_{\Omega} \left(|\nabla u|^{q-2} \langle \nabla u, \nabla \varphi \rangle - |\nabla u_m|^{q-2} \langle \nabla u_m, \nabla \varphi \rangle \right) \omega \, dx.
 \end{aligned}$$

We have that:

(i) By Lemma 2.7(a) there exists $C_p > 0$ such that

$$\begin{aligned}
 |I_1| &\leq \int_{\Omega} \left| |\Delta u|^{p-2} \Delta u - |\Delta u_m|^{p-2} \Delta u_m \right| |\Delta \varphi| \omega \, dx \\
 &\leq C_p \int_{\Omega} |\Delta u - \Delta u_m| (|\Delta u| + |\Delta u_m|)^{p-2} |\Delta \varphi| \omega \, dx.
 \end{aligned}$$

Let $r = p/(p - 2)$. Since $\frac{1}{p} + \frac{1}{p} + \frac{1}{r} = 1$, by the Generalized Hölder inequality we obtain

$$\begin{aligned}
 |I_1| &\leq C_p \left(\int_{\Omega} |\Delta u - \Delta u_m|^p \omega \, dx \right)^{1/p} \left(\int_{\Omega} |\Delta \varphi|^p \omega \, dx \right)^{1/p} \left(\int_{\Omega} (|\Delta u| + |\Delta u_m|)^{(p-2)r} \omega \, dx \right)^{1/r} \\
 &\leq C_p \|u - u_m\|_X \|\varphi\|_X \| |\Delta u| + |\Delta u_m| \|_{L^p(\Omega, \omega)}^{(p-2)}.
 \end{aligned}$$

Now, since $u_m \rightarrow u$ in X , then exists a constant $M > 0$ such that $\|u_m\|_X \leq M$. Hence,

$$\| |\Delta u| + |\Delta u_m| \|_{L^p(\Omega, \omega)} \leq \|u\|_X + \|u_m\|_X \leq 2M. \tag{3.6}$$

Therefore,

$$\begin{aligned}
 |I_1| &\leq C_p (2M)^{p-2} \|u - u_m\|_X \|\varphi\|_X \\
 &= C_1 \|u - u_m\|_X \|\varphi\|_X.
 \end{aligned}$$

Analogously, there exists a constant C_3 such that

$$|I_3| \leq C_3 \|u - u_m\|_X \|\varphi\|_X.$$

(ii) By Lemma 2.7(a) there exists a positive constant C_q such that

$$\begin{aligned}
 |I_2| &\leq \int_{\Omega} \left| |\Delta u|^{q-2} \Delta u - |\Delta u_m|^{q-2} \Delta u_m \right| |\Delta \varphi| \omega \, dx \\
 &\leq C_q \int_{\Omega} |\Delta u - \Delta u_m| (|\Delta u| + |\Delta u_m|)^{q-2} |\Delta \varphi| \omega \, dx.
 \end{aligned}$$

Let $s = q/(q - 2)$ (if $2 < q < p < \infty$). Since $\frac{1}{q} + \frac{1}{q} + \frac{1}{s} = 1$, by the Generalized Hölder inequality we obtain

$$\begin{aligned}
 |I_2| &\leq C_q \left(\int_{\Omega} |\Delta u - \Delta u_m|^q \omega \, dx \right)^{1/q} \left(\int_{\Omega} |\Delta \varphi|^q \omega \, dx \right)^{1/q} \left(\int_{\Omega} (|\Delta u| + |\Delta u_m|)^{(q-2)s} \omega \, dx \right)^{1/s} \\
 &= C_q \|\Delta u - \Delta u_m\|_{L^q(\Omega, \omega)} \|\Delta \varphi\|_{L^q(\Omega, \omega)} \| |\Delta u| + |\Delta u_m| \|_{L^q(\Omega, \omega)}^{q-2}.
 \end{aligned}$$

Now, by Remark 3.2(i) and (3.6) we have

$$\begin{aligned} |I_2| &\leq C_q C_{p,q} \|\Delta u - \Delta u_m\|_{L^p(\Omega, \omega)} C_{p,q} \|\Delta \varphi\|_{L^p(\Omega, \omega)} C_{p,q}^{q-2} \|\Delta u\| + \|\Delta u_m\|_{L^p(\Omega, \omega)}^{q-2} \\ &\leq C_q C_{p,q}^q \|u - u_m\|_X \|\varphi\|_X (2M)^{q-2} \\ &= C_2 \|u - u_m\|_X \|\varphi\|_X. \end{aligned}$$

Analogously, there exists a positive constant C_4 such that

$$|I_4| \leq C_4 \|u - u_m\|_X \|\varphi\|_X.$$

In case $q = 2$, we have $|I_2|, |I_4| \leq C_{p,2}^2 \|u - u_m\|_X \|\varphi\|_X$.

Therefore, we have $I_1, I_2, I_3, I_4 \rightarrow 0$ when $m \rightarrow \infty$.

(iii) We also have

$$\begin{aligned} &\left| \int_{\Omega} (f_m - f) \varphi \, dx + \int_{\Omega} \langle G_m - G, \nabla \varphi \rangle \, dx \right| \\ &\left(C_{\Omega} \left\| \frac{f_m - f}{\omega} \right\|_{L^{p'}(\Omega, \omega)} + C_{p,q} \left\| \frac{|G_m - G|}{\omega} \right\|_{L^{q'}(\Omega, \omega)} \right) \|\varphi\|_X \rightarrow 0, \end{aligned}$$

when $m \rightarrow \infty$. Therefore, in (3.5), we obtain when $m \rightarrow \infty$ that

$$\begin{aligned} &\int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \varphi \, \omega \, dx + \int_{\Omega} |\Delta u|^{q-2} \Delta u \Delta \varphi \, \omega \, dx \\ &+ \int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \, \omega \, dx + \int_{\Omega} |\nabla u|^{q-2} \langle \nabla u, \nabla \varphi \rangle \, \omega \, dx \\ &= \int_{\Omega} f \varphi \, dx + \int_{\Omega} \langle G, \nabla \varphi \rangle \, dx, \end{aligned}$$

i.e., u is a solution of problem (P). □

Example 3.6. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, $w(x, y) = (x^2 + y^2)^{-1/2}$ ($\omega \in A_4$, $p = 4$ and $q = 3$), $f(x, y) = \frac{\cos(xy)}{(x^2 + y^2)^{1/6}}$ and $G(x, y) = \left(\frac{\sin(x+y)}{(x^2 + y^2)^{1/6}}, \frac{\sin(xy)}{(x^2 + y^2)^{1/6}} \right)$. By Theorem 3.3, the problem

$$\begin{cases} \Delta \left[(x^2 + y^2)^{-1/2} (|\Delta u|^2 \Delta u + |\Delta u| \Delta u) \right] \\ - \operatorname{div} \left[(x^2 + y^2)^{-1/2} (|\nabla u|^2 \nabla u + |\nabla u| \nabla u) \right] \\ = f(x) - \operatorname{div}(G(x)), \quad \text{in } \Omega \\ u(x) = \Delta u = 0, \quad \text{in } \partial\Omega \end{cases}$$

has a unique solution $u \in W^{2,4}(\Omega, \omega) \cap W_0^{1,4}(\Omega, \omega)$.

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