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On multivariate Lupaş operators

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Abstract

This paper is primarily concerned with multivariate Lupaş operator. We demonstrate that multivariate Lupaş operator preserves the properties of the general function of modulus of continuity, Lipschitz's constant and order of a Lipschitz continuous function. Moreover, we obtain monotonicity of the multivariate Lupaş operator under the condition that the original function is convex. Lastly, two modified extensions are constructed.

Keywords: Lupaş operator, convexity, Lipschitz continuity, function of modulus of continuity, monotonic convergence.

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1. Introduction

Recall the following identity

$$(1-a)^{-\alpha}=\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}a^k,\ |\alpha|<1,$$

where $(.)_k$ is the Pochammer's symbol defined as

$$\begin{aligned}(\alpha)_0 &= 1\ (\alpha\neq 0)\,,\\ (\alpha)_k &= (\alpha)(\alpha+1)\dots(\alpha+k-1)\,,\ k\in\mathbb{N}.\end{aligned}\tag{1.1}$$

Taking $\alpha=nx$ $(x\geq 0)$ in this identity, Lupaş constructed the linear positive operators given by

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$$L_n(f;x) = (1-a)^{nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} a^k f(\frac{k}{n}),\ x \geq 0,$$

for suitable real valued functions f defined on $[0,\infty)$, $n \in \mathbb{N}$ [9]. In [1], Agratini considered the special case of the operators L_n by taking $a = \frac{1}{2}$ in its formula to get the property of preservation of linear functions. Namely, he considered the following linear positive operators

$$(1.2) \qquad L_{n,1}(f;x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f(\frac{k}{n}),\ n \in \mathbb{N},$$

for real valued, continuous and bounded functions on $[0,\infty)$ so that $L_{n,1}(1;x) = 1$, and $L_{n,1}(t;x) = x$. So, it is clear that

$$2^{nx} = \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!}.$$

We simply call the operatos $L_{n,1}$ as (univariate) Lupaş operators. For some classical approximation results related to Lupaş operators (1.2) we refer to [1], [6], [7], [11]. In this work, among others, we especially are interested in the multivariate extension of the Lupaş operators.

Let $D \subset \mathbb{R}^m,\ m \in \mathbb{N}$, denote the set

$$D = \{\mathbf{x} = (x_1, x_2, ..., x_m) \in \mathbb{R}^m : 0 \leq x_i < \infty,\ 1 \leq i \leq m\}.$$

Throughout, for a function $f : D \to \mathbb{R}$, we shall adopt the representation $f(\mathbf{x}) = f(x_1, x_2, ..., x_m)$ for $f((x_1, x_2, ..., x_m))$, $\mathbf{x} = (x_1, x_2, ..., x_m) \in D$, and also use the following standard notation of multivariate setting:

Let $\mathbf{0} = (0, 0, ..., 0)$, $\mathbf{k} = (k_1, k_2, ..., k_m) \in \mathbb{N}_{\mathbf{0}}^m := \mathbb{N}^m \cup \{\mathbf{0}\}$ and $\mathbf{x} \in D$,

$$\begin{aligned} |\mathbf{k}| &: = k_1 + k_2 + ... + k_m, \\ |\mathbf{x}| &: = x_1 + x_2 + ... + x_m, \\ \mathbf{k}! &= k_1! k_2! ... k_m!, \\ \binom{n}{\mathbf{k}} &: = \frac{n!}{\mathbf{k}!\,(n-|\mathbf{k}|)!}, \\ \sum_{\mathbf{k}=\mathbf{0}}^{\infty} &: = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} ... \sum_{k_m=0}^{\infty}. \end{aligned}$$

For our purposes, we denote

$$(\mathbf{x})_{\mathbf{k}} := (x_1)_{k_1} (x_2)_{k_2} ... (x_m)_{k_m},$$

where each $(x_i)_{k_i}$, $i = 1, 2, ..., m$, is given by (1.1).

Moreover, for $\mathbf{u} = (u_1, u_2, ..., u_m)$ *and* $\mathbf{v} = (v_1, v_2, ..., v_m) \in D$,

$\mathbf{u} \leq \mathbf{v}$ *means that* $u_i \leq v_i$ *for all* $1 \leq i \leq m$.

We will use the consecutive definitions which depend on [5].

1.1. Definition. *A continuous, real valued function f is said to be convex in D, if*

$$f\left(\sum_{i=1}^{r} \alpha_i \mathbf{x}_i\right) \leq \sum_{i=1}^{r} \alpha_i f(\mathbf{x}_i)$$

for every $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_r \in D$ and for every nonnegative numbers $\alpha_1, \alpha_2, ..., \alpha_r$ such that $\alpha_1 + \alpha_2 + ... + \alpha_r = 1$.

1.2. Definition. *A continuous, real valued, nonnegative function ω defined in D is said to be a modulus of continuity, if the ensuing are satisfied.*

i) $\omega(\mathbf{0}) = \omega(0,0,...,0) = 0$,
ii) $\omega(\mathbf{u})$ *is nondecreasing, namely* $\omega(\mathbf{u}) \leq \omega(\mathbf{v})$ whenever $\mathbf{u} \leq \mathbf{v}$,
iii) $\omega(\mathbf{u})$ *is sub-additive, namely* $\omega(\mathbf{u}+\mathbf{v}) \leq \omega(\mathbf{u}) + \omega(\mathbf{v})$,
for all $\mathbf{u} = (u_1, u_2, ..., u_m)$, *and* $\mathbf{v} = (v_1, v_2, ..., v_m) \in D$.

1.3. Definition. *A continuous function from $D \subset \mathbb{R}^m$ into $\mathbb{R}$ is said to be Lipschitz continuous of order α, $\alpha \in (0,1]$, if there exists a constant $M > 0$ such that f satisfies*

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq M \sum_{i=1}^{m} |x_i - y_i|^{\alpha}$$

for every $\mathbf{x} = (x_1, x_2, ..., x_m)$, $\mathbf{y} = (y_1, y_2, ..., y_m) \in D$. The set of Lipschitz continuous functions is denoted by $Lip_M(\alpha, D)$.

Let f be a real valued continuous function on D. Then $n-$th multivariate Lupaş operator $L_{n,m}f$ (with $m-$dimension) can be defined as

$$L_{n,m}(f;\mathbf{x}) = 2^{-n|\mathbf{x}|} \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \frac{(n\mathbf{x})_{\mathbf{k}}}{2^{|\mathbf{k}|}\mathbf{k}!} f\left(\frac{\mathbf{k}}{n}\right) \tag{1.3}$$

for $\mathbf{x} \in D$ and $n \in \mathbb{N}$. It is not to difficult to see that $L_{n,m}(1;\mathbf{x}) = 1$ and for $f(\mathbf{t}) = t_i$, $\mathbf{t} \in D$, $L_{n,m}(t_i;\mathbf{x}) = x_i$, $i = 1, 2, ..., m$.
Motivated from the excellent work of Cao, Ding and Xu as for multivariate Baskakov operator [5], in this work, apart from the approximation properties of $L_{n,m}$, we deal with some classical shape properties that the multivariate Lupaş operator $L_{n,m}$ satisfies. We show that the operator $L_{n,m}$ preserves the properties of a general function of modulus of continuity, Lipschitz's constant and order of a Lipschitz continuous function, and also that the monotonic convergence of the sequence of the multivariate Lupaş operators $L_{n,m}$, when the attached function is convex. It is clear that the case $m = 1$ gives the results of [6]. On the other hand, similar results for multivariate Szász-Mirakyan operator, which has a close similarity with the Lupaş operators, were obtained in [10]. Note that a brief history for this kind of approach for some univariate operators can also be reached in [5].

2. Shape Preserving Properties

Firstly, we study the monotonicity of the sequence of multivariate Lupaş operators $L_{n,m}(f;\mathbf{x})$ defined by (1.3) from inspiring [6].

2.1. Theorem. *Let f be a convex function defined on D. Then $L_{n,m}f$ is monotonically nonincreasing in n for all n.*

Proof. For simplicity, we take $m = 2$. The proof for higher dimensions will be similar. From the definition of $L_n(f;\mathbf{x})$, with taking into account of multivariate notation, we have

$$
\begin{aligned}
& L_{n,m}(f;\mathbf{x}) - L_{n+1,m}(f;\mathbf{x}) \\
= \; & 2^{-n|\mathbf{x}|-|\mathbf{x}|}\left\{2^{|\mathbf{x}|}\sum_{\mathbf{k}=\mathbf{0}}^{\infty}\frac{(n\mathbf{x})_{\mathbf{k}}}{2^{|\mathbf{k}|}\ \mathbf{k}!}f\left(\frac{\mathbf{k}}{n}\right) - \sum_{\mathbf{k}=\mathbf{0}}^{\infty}\frac{((n+1)\,\mathbf{x})_{\mathbf{k}}}{2^{|\mathbf{k}|}\ \mathbf{k}!}f\left(\frac{\mathbf{k}}{n+1}\right)\right\} \\
= \; & 2^{-n|\mathbf{x}|-|\mathbf{x}|}\left\{2^{|\mathbf{x}|}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(nx_1)_{k_1}}{2^{k_1}\ k_1!}\frac{(nx_2)_{k_2}}{2^{k_2}\ k_2!}f\left(\frac{k_1}{n},\frac{k_2}{n}\right)\right. \\
& \left. -\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{((n+1)\,x_1)_{k_1}}{2^{k_1}\ k_1!}\frac{((n+1)\,x_2)_{k_2}}{2^{k_2}\ k_2!}f\left(\frac{k_1}{n+1},\frac{k_2}{n+1}\right)\right\}.
\end{aligned}
\tag{2.1}
$$

Using the facts $2^{x_i} = \sum_{l_i=0}^{\infty}\frac{(x_i)_{l_i}}{2^{l_i}\ l_i!}$, $i = 1, 2$, then the summations in the bracket in (2.1) reduce to

$$
\begin{aligned}
& \sum_{l_1=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(x_1)_{l_1}}{2^{l_1}\ l_1!}\frac{(nx_1)_{k_1}}{2^{k_1}\ k_1!}\frac{(x_2)_{l_2}}{2^{l_2}\ l_2!}\frac{(nx_2)_{k_2}}{2^{k_2}\ k_2!}f\left(\frac{k_1}{n},\frac{k_2}{n}\right) \\
& -\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{((n+1)\,x_1)_{k_1}}{2^{k_1}\ k_1!}\frac{((n+1)\,x_2)_{k_2}}{2^{k_2}\ k_2!}f\left(\frac{k_1}{n+1},\frac{k_2}{n+1}\right).
\end{aligned}
\tag{2.2}
$$

Replacing k_i with $k_i - l_i$, $i = 1, 2$, then (2.2) gives rise to

$$
\begin{aligned}
& \sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{k_1=l_1}^{\infty}\sum_{k_2=l_2}^{\infty}\frac{(x_1)_{l_1}\ (x_2)_{l_2}}{l_1!\ l_2! 2^{l_1+l_2}}\frac{(nx_1)_{k_1-l_1}\ (nx_2)_{k_2-l_2}}{2^{k_1+k_2-l_1-l_2}\,(k_1-l_1)!\ (k-l_2)!} \\
& \times f\left(\frac{k_1-l_1}{n},\frac{k_2-l_2}{n}\right) \\
& -\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{((n+1)\,x_1)_{k_1}\ ((n+1)\,x_2)_{k_2}}{2^{k_1+k_2}\ k_1!\ k_2!}f\left(\frac{k_1}{n+1},\frac{k_2}{n+1}\right).
\end{aligned}
$$

Changing the order of the summations and replacing $k_i - l_i$, with l_i, $i = 1, 2$, then the last formula reduces to

$$
\begin{aligned}
& \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\frac{1}{2^{k_1+k_2}}\left\{\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\frac{(nx_1)_{l_1}\,(nx_2)_{l_2}\,(x_1)_{k_1-l_1}\,(x_2)_{k_2-l_2}}{l_1!\ l_2!\ (k_1-l_1)!\ (k_2-l_2)!}f\left(\frac{l_1}{n},\frac{l_2}{n}\right)\right. \\
& \left. -\frac{((n+1)\,x_1)_{k_1}\,((n+1)\,x_2)_{k_2}}{k_1!\ k_2!}f\left(\frac{k_1}{n+1},\frac{k_2}{n+1}\right)\right\} \\
& +\sum_{k_1=1}^{\infty}\frac{1}{2^{k_1}}\left\{\sum_{l_1=0}^{k_1}\frac{(nx_1)_{l_1}\,(x_1)_{k_1-l_1}}{l_1!\ (k_1-l_1)!}f\left(\frac{l_1}{n},0\right) - \frac{((n+1)\,x_1)_{k_1}}{k_1!}f\left(\frac{k_1}{n+1},0\right)\right\} \\
& +\sum_{k_2=1}^{\infty}\frac{1}{2^{k_2}}\left\{\sum_{l_2=0}^{k_2}\frac{(nx_2)_{l_2}\,(x_2)_{k_2-l_2}}{l_2!\ (k_2-l_2)!}f\left(0,\frac{l_2}{n}\right) - \frac{((n+1)\,x_2)_{k_2}}{k_2!}f\left(0,\frac{k_2}{n+1}\right)\right\} \\
& +f((0,0)) - f((0,0)).
\end{aligned}
$$

Denoting

$$\begin{aligned} I \quad &: \quad = \left\{ \sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_2} \frac{(nx_1)_{l_1} (nx_2)_{l_2} (x_1)_{k_1-l_1} (x_2)_{k_2-l_2}}{l_1!\ l_2!\ (k_1-l_1)!\ (k_2-l_2)!} f\left(\frac{l_1}{n}, \frac{l_2}{n}\right) \right. \\ &\qquad \left. - \frac{((n+1)\,x_1)_{k_1} ((n+1)\,x_2)_{k_2}}{k_1!\ k_2!} f\left(\frac{k_1}{n+1}, \frac{k_2}{n+1}\right) \right\}, \\ I_1 \quad &: \quad = \sum_{l_1=0}^{k_1} \frac{(nx_1)_{l_1} (x_1)_{k_1-l_1}}{l_1!\ (k_1-l_1)!} f\left(\frac{l_1}{n}, 0\right) - \frac{((n+1)\,x_1)_{k_1}}{k_1!} f\left(\frac{k_1}{n+1}, 0\right), \\ I_2 \quad &: \quad = \sum_{l_2=0}^{k_2} \frac{(nx_2)_{l_2} (x_2)_{k_2-l_2}}{l_2!\ (k_2-l_2)!} f\left(0, \frac{l_2}{n}\right) - \frac{((n+1)\,x_2)_{k_2}}{k_2!} f\left(0, \frac{k_2}{n+1}\right), \end{aligned}$$

it sufficies to show that $I,\ I_1$, and I_2 are nonnegative.

For I_1, let

$$\alpha_{l_1} := \binom{k_1}{l_1} \frac{(nx_1)_{l_1}\ (x_1)_{k_1-l_1}}{((n+1)\,x_1)_{k_1}} \geq 0$$

and

$$\mathbf{x}_{l_1} = \left(\frac{l_1}{n}, 0\right),\ l_1 = 0, 1, ..., k_1.$$

Then, as in [6], using the formula $((n+1)\,x_i)_{k_i} = \sum\limits_{l_i=0}^{k_i} \binom{k_i}{l_i} (nx_i)_{l_i} (x_i)_{k_i-l_i}$, $i = 1, 2$, It readily follows that

$$\sum_{l_1=0}^{k_1} \alpha_{l_1} = 1$$

and

$$\sum_{l_1=0}^{k_1} \alpha_{l_1} \mathbf{x}_{l_1} = \left(\frac{k_1}{n+1}, 0\right).$$

Hence, convexity of f gives that

$$f\left(\frac{k_1}{n+1}, 0\right) \leq \frac{1}{((n+1)\,x_1)_{k_1}} \sum_{l_1=0}^{k_1} (nx_1)_{l_1}\ \ (x_1)_{k_1-l_1} \binom{k_1}{l_1} f\left(\frac{l_1}{n}, 0\right),$$

which implies that $I_1 \geq 0$. The case $I_2 \geq 0$ is obtained in a similar way by taking

$$\beta_{l_2} := \binom{k_2}{l_2} \frac{(nx_2)_{l_2} (x_2)_{k_2-l_2}}{((n+1)\,x_2)_{k_2}} \geq 0.$$

and

$$\mathbf{x}_{l_2} = \left(0, \frac{l_2}{n}\right), \quad l_2 = 0, 1, ..., k_2.$$

Similarly, it holds

$$\sum_{l_2=0}^{k_2} \beta_{l_2} = 1,$$

and

$$\sum_{l_2=0}^{k_2} \beta_{l_2} \mathbf{x}_{l_2} = \left(0, \frac{k_2}{n+1}\right),$$

so, the result $I_2 \geq 0$ follows from convexity of f. Finally, for I, we may take

$$\alpha'_{l_1} := \alpha_{l_1} \sum_{l_2=0}^{k_2} \beta_{l_2} \geq 0$$

as nonnegative constants satisfying $\sum_{l_1=0}^{k_1} \left(\alpha_{l_1} \sum_{l_2=0}^{k_2} \beta_{l_2} \right) = 1$, and $x'_{l_1} = \left(\frac{l_1}{n}, \frac{l_2}{n}\right)$, where α_{l_1} and β_{l_2} are the same as given above. So, from the convexity of f we reach to the following.

$$\begin{aligned}
& \sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_2} \binom{k_1}{l_1} \binom{k_2}{l_2} \frac{(nx_1)_{l_1} (nx_2)_{l_2} (x_1)_{k_1-l_1} (x_2)_{k_2-l_2}}{((n+1)\,x_1)_{k_1} ((n+1)\,x_2)_{k_2}} f\left(\frac{l_1}{n}, \frac{l_2}{n}\right) \\
\geq \quad & f\left(\sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_2} \binom{k_1}{l_1} \binom{k_2}{l_2} \frac{(nx_1)_{l_1} (nx_2)_{l_2} (x_1)_{k_1-l_1} (x_2)_{k_2-l_2}}{((n+1)\,x_1)_{k_1} ((n+1)\,x_2)_{k_2}} \left(\frac{l_1}{n}, \frac{l_2}{n}\right) \right) \\
= \quad & f\left(\frac{k_1}{n+1}, \frac{k_2}{n+1}\right),
\end{aligned}$$

which shows that $I \geq 0$. This completes the proof. □

2.2. Theorem. *Let ω be a function of modulus of continuity. Then, for each $n \in \mathbb{N}$, $L_{n,m}\omega$ is also a function of modulus of continuity.*

Proof. Let $\mathbf{x} = (x_1, x_2, ..., x_m)$, $\mathbf{y} = (y_1, y_2, ..., y_m) \in D$, and $\mathbf{x} \leq \mathbf{y}$. Then, taking multivariate notation into consideration, we have

$$\begin{aligned}
L_{n,m}(\omega\,;\mathbf{y}) &= 2^{-n|\mathbf{y}|} \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \frac{(n\mathbf{y})_{\mathbf{k}}}{2^{|\mathbf{k}|}\mathbf{k}!} \omega\left(\frac{\mathbf{k}}{n}\right) \\
&= 2^{-n|\mathbf{y}|} \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \frac{(n\,(\mathbf{x} + (\mathbf{y} - \mathbf{x})))_{\mathbf{k}}}{2^{|\mathbf{k}|}\mathbf{k}!} \omega\left(\frac{\mathbf{k}}{n}\right) \\
&= 2^{-n|\mathbf{y}|} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} ... \sum_{k_m=0}^{\infty} \sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2} ... \sum_{i_m=0}^{k_m} \frac{1}{\mathbf{i}! 2^{|\mathbf{k}|} (\mathbf{k} - \mathbf{i})!} \\
&\quad \times (n\mathbf{x})_{\mathbf{i}} \, (n(\mathbf{y} - \mathbf{x}))_{\mathbf{k}-\mathbf{i}} \, \omega\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_m}{n}\right)
\end{aligned}$$

for $\mathbf{i} = (i_1, i_2, ..., i_m) \in \mathbb{N}_{\mathbf{0}}^m$. Changing the order of the summations and taking $k_r - i_r = j_r$, $r = 1, 2, ..., m$, we can easily obtain

$$L_{n,m}(\omega\,;\mathbf{y}) = 2^{-n|\mathbf{y}|} \sum_{\mathbf{i}=\mathbf{0}}^{\infty} \sum_{\mathbf{j}=\mathbf{0}}^{\infty} \frac{1}{\mathbf{i}!\mathbf{j}! 2^{|\mathbf{i}+\mathbf{j}|}} (n\mathbf{x})_{\mathbf{i}} \, (n(\mathbf{y} - \mathbf{x}))_{\mathbf{j}} \, \omega\left(\frac{\mathbf{i} + \mathbf{j}}{n}\right), \tag{2.3}$$

where $\mathbf{j}=(j_1,j_2,...,j_m)$. On the other hand, for $\mathbf{x}\in D$ we have

$$\begin{aligned}
L_{n,m}(\omega\ ;\mathbf{x}) &= 2^{-n|\mathbf{x}|}\sum_{\mathbf{i}=\mathbf{0}}^{\infty}\frac{(n\mathbf{x})_{\mathbf{i}}}{2^{|\mathbf{i}|}\mathbf{i}!}\omega\left(\frac{\mathbf{i}}{n}\right)\\
&= 2^{-n|(\mathbf{y}-(\mathbf{y}-\mathbf{x}))|}\sum_{\mathbf{i}=\mathbf{0}}^{\infty}\frac{(n\mathbf{x})_{\mathbf{i}}}{2^{|\mathbf{i}|}\mathbf{i}!}\omega\left(\frac{\mathbf{i}}{n}\right)\\
&= 2^{-n|\mathbf{y}|}2^{n|\mathbf{y}-\mathbf{x}|}\sum_{\mathbf{i}=\mathbf{0}}^{\infty}\frac{(n\mathbf{x})_{\mathbf{i}}}{2^{|\mathbf{i}|}\mathbf{i}!}\omega\left(\frac{\mathbf{i}}{n}\right)\\
&= 2^{-n|\mathbf{y}|}\sum_{\mathbf{i}=\mathbf{0}}^{\infty}\frac{(n\mathbf{x})_{\mathbf{i}}}{2^{|\mathbf{i}|}\mathbf{i}!}\sum_{\mathbf{j}=\mathbf{0}}^{\infty}\frac{(n(\mathbf{y}-\mathbf{x}))_{\mathbf{j}}}{2^{|\mathbf{j}|}\mathbf{j}!}\omega\left(\frac{\mathbf{i}}{n}\right)\\
&= 2^{-n|\mathbf{y}|}\sum_{\mathbf{i}=\mathbf{0}}^{\infty}\frac{(n\mathbf{x})_{\mathbf{i}}}{2^{|\mathbf{i}|}\mathbf{i}!}\sum_{\mathbf{j}=\mathbf{0}}^{\infty}\frac{(n(\mathbf{y}-\mathbf{x}))_{\mathbf{j}}}{2^{|\mathbf{j}|}\mathbf{j}!}\omega\left(\frac{\mathbf{i}}{n}\right). \qquad (2.4)
\end{aligned}$$

If we subtract (2.3) from (2.4), we arrive at

(2.5)
$$L_{n,m}(\omega\ ;\mathbf{y})-L_{n,m}(\omega\ ;\mathbf{x})=2^{-n|\mathbf{y}|}\sum_{\mathbf{i}=\mathbf{0}}^{\infty}\sum_{\mathbf{j}=\mathbf{0}}^{\infty}\frac{(n\mathbf{x})_{\mathbf{i}}}{2^{|\mathbf{i}|}\mathbf{i}!}\frac{(n(\mathbf{y}-\mathbf{x}))_{\mathbf{j}}}{2^{|\mathbf{j}|}\mathbf{j}!}\left[\omega\left(\frac{\mathbf{i}+\mathbf{j}}{n}\right)-\omega\left(\frac{\mathbf{i}}{n}\right)\right].$$

Using the sub-additivity property of ω, we have

$$\begin{aligned}
L_{n,m}(\omega;\mathbf{y})-L_{n,m}(\omega;\mathbf{x}) &\leq 2^{-n|\mathbf{y}|}\sum_{\mathbf{i}=\mathbf{0}}^{\infty}\sum_{\mathbf{j}=\mathbf{0}}^{\infty}\frac{(n\mathbf{x})_{\mathbf{i}}}{2^{|\mathbf{i}|}\mathbf{i}!}\frac{(n(\mathbf{y}-\mathbf{x}))_{\mathbf{j}}}{2^{|\mathbf{j}|}\mathbf{j}!}\omega\left(\frac{\mathbf{j}}{n}\right)\\
&= 2^{-n|\mathbf{x}|}\sum_{\mathbf{i}=\mathbf{0}}^{\infty}\frac{(n\mathbf{x})_{\mathbf{i}}}{2^{|\mathbf{i}|}\mathbf{i}!}2^{-n|\mathbf{y}-\mathbf{x}|}\sum_{\mathbf{j}=\mathbf{0}}^{\infty}\frac{(n(\mathbf{y}-\mathbf{x}))_{\mathbf{j}}}{2^{|\mathbf{j}|}\mathbf{j}!}\omega\left(\frac{\mathbf{j}}{n}\right)\\
&= 2^{-n|\mathbf{y}-\mathbf{x}|}\sum_{\mathbf{j}=\mathbf{0}}^{\infty}\frac{(n(\mathbf{y}-\mathbf{x}))_{\mathbf{j}}}{2^{|\mathbf{j}|}\mathbf{j}!}\omega\left(\frac{\mathbf{j}}{n}\right)\\
&= L_{n,m}(\omega;\mathbf{y}-\mathbf{x}).
\end{aligned}$$

The last inequality shows the sub-additivity of $L_{n,m}$. Moreover, from (2.5) we easily obtain that $L_{n,m}(\omega;\mathbf{x})\leq L_{n,m}(\omega;\mathbf{y})$ when $\mathbf{x}\leq\mathbf{y}$. Obviously, this gives that $L_{n,m}$ is nondecreasing. Finally, from the definition of the operators $L_{n,m},\ L_{n,m}(\omega;\mathbf{0})=\omega(\mathbf{0})=0$ is obvious. Therefore $L_{n,m}\omega$ is also a function of modulus of continuity, this completes the proof. □

Here, we present the preservation of the Lipschitz constant and order of a Lipschitz continuous function by the multivariate Lupaş operators $L_{n,m}$.

2.3. Theorem. *Let $f\in Lip_M(\alpha,D)$. Then $L_{n,m}f\in Lip_M(\alpha,D)$.*

Proof. Assume that $\mathbf{x} \leq \mathbf{y}$. Then, writing (2.5) for f, we get

$$\begin{aligned}
& \left|L_{n,m}\left(f;\mathbf{y}\right) - L_{n,m}\left(f;\mathbf{x}\right)\right| \\
\leq \quad & 2^{-n|\mathbf{y}|} \sum_{\mathbf{i}=\mathbf{0}}^{\infty} \sum_{\mathbf{j}=\mathbf{0}}^{\infty} \frac{(n\mathbf{x})_{\mathbf{i}}}{2^{|\mathbf{i}|}\mathbf{i}!} \frac{(n(\mathbf{y}-\mathbf{x}))_{\mathbf{j}}}{2^{|\mathbf{j}|}\mathbf{j}!} \left| f\left(\frac{\mathbf{i}+\mathbf{j}}{\mathbf{n}}\right) - f\left(\frac{\mathbf{i}}{\mathbf{n}}\right) \right| \\
\leq \quad & M 2^{-n|\mathbf{x}|} \sum_{\mathbf{i}=\mathbf{0}}^{\infty} \frac{(n\mathbf{x})_{\mathbf{i}}}{2^{|\mathbf{i}|}\mathbf{i}!} \sum_{\mathbf{j}=\mathbf{0}}^{\infty} \frac{(n(\mathbf{y}-\mathbf{x}))_{\mathbf{j}}}{2^{|\mathbf{j}|}\mathbf{j}!} 2^{-n(|\mathbf{y}|-|\mathbf{x}|)} \sum_{k=1}^{m} \left(\frac{j_k}{n}\right)^{\alpha} \\
= \quad & M 2^{-n|\mathbf{y}-\mathbf{x}|} \sum_{\mathbf{j}=\mathbf{0}}^{\infty} \frac{(n(\mathbf{y}-\mathbf{x}))_{\mathbf{j}}}{2^{|\mathbf{j}|}\mathbf{j}!} \sum_{k=1}^{m} \left(\frac{j_k}{n}\right)^{\alpha} \\
= \quad & M\left\{L_{n,1}\left(t_1^{\alpha}; y_1 - x_1\right) + L_{n,1}\left(t_2^{\alpha}; y_2 - x_2\right) + \ldots + L_{n,1}\left(t_m^{\alpha}; y_m - x_m\right)\right\}
\end{aligned}$$

by the hypothesis that $f \in Lip_M(\alpha, D)$, where $\mathbf{i} = (i_1, i_2, ..., i_m)$ and $\mathbf{j} = (j_1, j_2, ..., j_m)$ belong to $\mathbb{N}_{\mathbf{0}}^m$. Following the same steps of the proof of Theorem 2 in [6] for each sum in the last formula, we reach to

$$\begin{aligned}
& \left|L_{n,m}\left(f;\mathbf{y}\right) - L_{n,m}\left(f;\mathbf{x}\right)\right| \\
\leq \quad & M \sum_{k=1}^{m} |y_k - x_k|^{\alpha},
\end{aligned}$$

which shows that $L_{n,m}f \in Lip_M(\alpha, D)$. In a similar way, we can show the case when $\mathbf{x} \geq \mathbf{y}$. Finally, we can consider the case that $x_1 \geq y_1,\ x_2 \geq y_2, ..., x_{i-1} \geq y_{i-1}, x_{i+1} \geq y_{i+1}, ..., x_m \geq y_m$, and $x_i \leq y_i$. Since $(y_1, y_2, ..., y_{i-1}, x_i, y_{i+1}, ..., y_m) \in D$, then we obtain from the above arguments that

$$\begin{aligned}
& \left|L_{n,m}\left(f;\mathbf{y}\right) - L_{n,m}\left(f;\mathbf{x}\right)\right| \\
\leq \quad & \left|L_{n,m}\left(f;(y_1, y_2, ..., y_m)\right) - L_{n,m}\left(f;(y_1, y_2, ..., y_{i-1}, x_i, y_{i+1}, ..., y_m)\right)\right| \\
& + \left|L_{n,m}\left(f;(y_1, y_2, ..., y_{i-1}, x_i, y_{i+1}, ..., y_m)\right) - L_{n,m}\left(f;(x_1, x_2, ..., x_m)\right)\right| \\
\leq \quad & M \sum_{k=1}^{m} |y_k - x_k|^{\alpha}.
\end{aligned}$$

Clearly, if the last case holds for more than one components, then the result follows similarly. □

3. Extensions

So as to gain approximation process in space of integrable functions, we propose new integral modification, i.e. the Kantorovich type [1]:

$$L_{n,m}^{*}(f, \mathbf{x}) := \sum_{\mathbf{k}=\mathbf{0}}^{\infty} P_{n,\mathbf{k}}(\mathbf{x}) \phi_{n,k}(f), \quad (n \in \mathbb{N}).$$

where

$$P_{n,\mathbf{k}}(\mathbf{x}) := 2^{-n|\mathbf{x}|} \frac{(n\mathbf{x})_{\mathbf{k}}}{2^{|\mathbf{k}|}\mathbf{k}!}$$

and

$$\phi_{n,k}(f) := n^m \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \ldots \int_{\frac{k_m}{n}}^{\frac{k_m+1}{n}} f(t_1, t_2, ... t_m) dt_m ... dt_2 dt_1.$$

A number of solution can be found for multivariate Lupaş-Kantorovich operator, which has been identified above, about shape preserving and approximation properties. However, the objective in this study is to give definition only.

Now, we establish $n-$th multivariate generalized Lupaş operator to investigate and understand their properties reckon weighted approximation in [8]. A lot of work has been done in this regard, some are [3], [4], [2].

Let ρ be a function defined on $\mathbb{R}^+ := [0, \infty)$ and have the following properties:

(ρ_1) ρ is a continuously differentiable function on $\mathbb{R}^+$,

(ρ_2) $\rho(0) = 0, \ \inf\limits_{x \in \mathbb{R}^+} \rho'(x) \geq 1.$

These conditions ensure that ρ is strictly increasing and the inverse $\rho^{-1}(x)$ of ρ exists on $\mathbb{R}^+$. For example, $\rho(x) = x + x^2$ is a function which is given from [2] satisfies the conditions (ρ_1) and (ρ_2). Let f be a real valued continuous function defined on D, which is explained above for multivariate Lupaş operator and $\rho(\mathbf{x}), \ \mathbf{x} = (x_1, ..., x_m) \in D$ denote a function acting from the set D onto D such that each component of which is given by $\rho(x_i), \ 1 \leq i \leq m$, namely

$$\rho(\mathbf{x}) := (\rho(x_1), \rho(x_2), ..., \rho(x_m)).$$

Denoting the inverse of ρ by ρ^{-1}, which means that

$$\rho^{-1}(\mathbf{x}) := \left(\rho^{-1}(x_1), \rho^{-1}(x_2), ..., \rho^{-1}(x_m)\right),$$

Then $n-$th multivariate generalized Lupaş operator $L^{\rho}_{n,m} f$ is defined as

$$L^{\rho}_{n,m}(f; \mathbf{x}) = 2^{-n|\rho(\mathbf{x})|} \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \left(f \circ \rho^{-1}\right) \left(\frac{\mathbf{k}}{n}\right) \frac{(n\rho(\mathbf{x}))_{\mathbf{k}}}{2^{|\mathbf{k}|}\mathbf{k}!}, \tag{3.1}$$

where $\left(f \circ \rho^{-1}\right)\left(\frac{\mathbf{k}}{n}\right) := f\left(\rho^{-1}\left(\frac{k_1}{n}\right), \rho^{-1}\left(\frac{k_2}{n}\right), ..., \rho^{-1}\left(\frac{k_m}{n}\right)\right), \ n \in \mathbb{N}.$

Here, we recall the extended version of the notion of the $\rho-$*convexity,* due to Aral et.al [2], to the multivariate case as in [3].

3.1. Definition. *A continuous, real valued function f is called as $\rho-$convex in D, if* $f \circ \rho^{-1}$ is convex in the sense of Definition 1.1.

We need the following definition for the generalized Lipschitz class used, for bivariate case, in [3].

3.2. Definition. *A continuous function from $D \subset \mathbb{R}^m$ into $\mathbb{R}$ is said to be $\rho-$Lipschitz continuous of order α, $\alpha \in (0, 1]$, if there exists a constant $M > 0$ such that f satisfies*

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq M \sum_{i=1}^{m} |\rho(x_i) - \rho(y_i)|^{\alpha}$$

for every $\mathbf{x} = (x_1, x_2, ..., x_m), \ \mathbf{y} = (y_1, y_2, ..., y_m) \in D$.

The set of $\rho-$Lipschitz continuous functions is denoted by $Lip^{\rho}_M(\alpha, D)$.

Note that clearly when $\rho(\mathbf{x}) = \mathbf{x}$ one obtaines the multivariate Lupaş operators given by (1.3).

It can be shown that the multivariate generalized Lupaş operators $L^{\rho}_{n,m} f$ preserve some properties.

The following two theorems can be given in the light of all these;

3.3. Theorem. *Let f be a $\rho-$convex function defined on D. Then $L^{\rho}_{n,m}$ is monotonically nondecreasing in n.*

Proof. The theorem can be proved similar with multivariate Lupaş operator so it can be omitted. $\square$

3.4. Theorem. *Let* $f \in Lip_M^{\rho}(\alpha, D)$, $0 < \alpha \leq 1$. *Then* $L_{n,m}^{\rho}(f; \mathbf{x}) \in Lip_M^{\rho}(\alpha, D)$.

Proof. Also, omitted. □

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