On Grill $S_p$-Open Set in Grill Topological Spaces

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Abstract - In this paper, we introduce a new type of grill set namely; $G_{S_p}$-open sets, which is analogous to the $G$-semiopen sets in a grill topological space $(X, \tau, G)$. Further, we define $G_{S_p}$-continuous and $G_{S_p}$-open functions by using a $G_{S_p}$-open set and we investigate some of their important properties.

Keywords - $G_{S_p}$-open set, $G_{S_p}O(X)$, $G_{S_p}$-continuous function, $G_{S_p}$-open function.

1. Introduction and Preliminaries

Choquet [2] introduced the concept of grill on a topological space and the idea of grills has shown to be a essential tool for studying some topological concepts. A collection $G$ of nonempty subsets of a topological space $(X, \tau)$ is called a grill on $X$ if (i) $A \in G$ and $A \subseteq B$ implies that $B \in G$, and (ii) $A, B \subseteq X$ and $A \cup B \in G$ implies that $A \in G$ or $B \in G$. A triple $(X, \tau, G)$ is called a grill topological space.

Roy and Mukherjee [17] defined a unique topology by a grill and they studied topological concepts. For any point $x$ of a topological space $(X, \tau)$, $\tau(x)$ denotes the collection of all open neighborhoods of $x$. A mapping $\varphi : P(X) \rightarrow P(X)$ is defined as $\varphi(A) = \{ x \in X : A \cap U \in G \text{ for all } U \in \tau(x) \}$ for each $A \in P(X)$. A mapping $\psi : P(X) \rightarrow P(X)$ is defined as $\psi(A) = A \cup \varphi(A)$ for all $A \in P(X)$. The map $\psi$ satisfies Kuratowski closure axioms:

(i) $\psi(\emptyset) = \emptyset$,
(ii) if $A \subseteq B$, then $\psi(A) \subseteq \psi(B)$,
(iii) if $A \subseteq X$, then $\psi(\psi(A)) = \psi(A)$, and

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In this paper, we define a $G \subseteq (X, \tau)$ of its basic properties. Moreover, we define $G$-open, $G$-semiopen, $G$-preopen, $G$-closed and $G$-continuous sets in a grill topological space $(X, \tau, G)$ is denoted by $G\alpha O(X)$ (resp. $G\beta O(X)$). A function $f : (X, \tau, G) \rightarrow (Y, \sigma)$ is said to be $G$-semicontinuous if $f^{-1}(V) \in G\sigma O(X)$ for each $V \in \sigma$.

Mashhour et al. [14] introduced a class of preopen sets and he defined pre interior and pre closure in a topological space. A subset $A$ in $X$ is said to be preopen if $A \subseteq \text{int}(\text{cl}(A))$ and $PO(X)$ denotes the family of preopen sets. For any subset $A$ of $X$, (i) $\text{pint}(A) = \cap \{U : U \in PO(X) \text{ and } U \subseteq A\}$; (ii) $\text{pcl}(A) = \cap \{F : X \setminus F \in PO(X) \text{ and } A \subseteq F\}$.

In this paper, we define a $G_p$-open set in a grill topological space $(X, \tau, G)$ and we study some of its basic properties. Moreover, we define $G_p$-continuous, $G_p$-open, $G_p^\ast$-closed and $G_p^\ast$-continuous functions on a grill topological space $(X, \tau, G)$ and we discuss some of their essential properties.

Proposition 1.1. [17] Let $(X, \tau, G)$ be a grill topological space. Then for all $A, B \subseteq X$:

1. $A \subseteq B$ implies that $\varphi(A) \subseteq \varphi(B)$;
2. $\varphi(A \cup B) = \varphi(A) \cup \varphi(B)$;
3. $\varphi(\varphi(A)) \subseteq \varphi(A) = \text{cl}(\varphi(A)) \subseteq \text{cl}(A)$.

2. $G_p$-Open Sets

Definition 2.1. Let $(X, \tau, G)$ be a grill topological space and let $A$ be a subset $A$ of $X$. Then $A$ is said to be $G_p$-open if and only if there exist a $U \in PO(X)$ such that $U \subseteq A \subseteq \text{ps}(U)$. A set $A$ of $X$ is $G_p$-closed if its complement $X \setminus A$ is $G_p$-open. The family of all $G_p$-open (resp. $G_p$-closed) sets is denoted by $G_p O(X)$ (resp. $G_p C(X)$).

Example 2.1. Let $X = \{a, b, c, d\}$, $\tau = \emptyset, X, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}$ and $G = \{\{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}\}$. Then $G_p O(X) = \emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.
**Theorem 2.1.** Let \((X, \tau, G)\) be a grill topological space and let \(A \subseteq X\). Then \(A \in G_{Sp}O(X)\) if and only if \(A \subseteq \psi(pint(A))\).

Proof. If \(A \in G_{Sp}O(X)\), then there exist a \(U \in PO(X)\) such that \(U \subseteq A \subseteq \psi(U)\). But \(U \subseteq A\) implies that \(U \subseteq pint(A)\). Hence \(\psi(U) \subseteq \psi(pint(A))\). Therefore \(A \subseteq \psi(pint(A))\). Conversely, let \(A \subseteq \psi(pint(A))\). To prove that \(A \in G_{Sp}O(X)\), take \(U = pint(A)\), then \(U \subseteq A \subseteq \psi(U)\). Hence \(A \in G_{Sp}O(X)\).

**Corollary 2.1.** If \(A \subseteq X\), then \(A \in G_{Sp}O(X)\) if and only if \(\psi(A) = \psi(pint(A))\).

Proof. Let \(A \in G_{Sp}O(X)\). Then as \(\psi\) is monotonic and idempotent, \(\psi(A) \subseteq \psi(\psi(pint(A))) = \psi(pint(A))\). Hence \(\psi(A) = \psi(pint(A))\). The converse is obvious.

**Theorem 2.2.** Let \((X, \tau, G)\) be a grill topological space. If \(A \in G_{Sp}O(X)\) and \(B \subseteq X\) such that \(A \subseteq B \subseteq \psi(pint(A))\), then \(B \in G_{Sp}O(X)\).

Proof. Given \(A \in G_{Sp}O(X)\). Then by Theorem 2.1, \(A \subseteq \psi(pint(A))\). But \(A \subseteq B\) implies that \(pint(A) \subseteq pint(B)\) and hence by Theorem 2.4[17], \(\psi(pint(A)) \subseteq \psi(pint(B))\). Therefore \(B \subseteq \psi(pint(A)) \subseteq \psi(pint(B))\). Hence \(B \in G_{Sp}O(X)\).

**Corollary 2.2.** If \(A \in G_{Sp}O(X)\) and \(B \subseteq X\) such that \(A \subseteq B \subseteq \psi(A)\), then \(B \in G_{Sp}O(X)\).

Proof. Follows from the Theorem 2.2 and Corollary 2.1.

**Proposition 2.1.** If \(U \in PO(X)\), then \(U \in G_{Sp}O(X)\).

Proof. Let \(U \in PO(X)\), it implies that \(U = pint(U) \subseteq \psi(pint(U))\). Hence \(U \in G_{Sp}O(X)\).

Note that the converse of the above proposition need not be true. Let \(X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\} and G = \{\{a\}, \{b\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}. Then \(PO(X) = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}\). Here \(\{b, d\}\) and \(\{a, b, d\}\) are \(G_{Sp}\)-open sets but not preopen.

**Theorem 2.3.** Let \((X, \tau, G)\) be a grill topological space. If \(A \in GSO(X)\), then \(A \in G_{Sp}O(X)\).

Proof. Given \(A \in GSO(X)\). Then \(A \subseteq \psi(\text{int}(A))\). Since \(\text{int}(A) \subseteq \text{pint}(A)\), we have that \(\psi(\text{int}(A)) \subseteq \psi(\text{pint}(A))\) (by Theorem 2.4[17]). Hence \(A \subseteq \psi(\text{pint}(A))\) and thus \(A \in G_{Sp}O(X)\).

Note that the converse of the above theorem need not be true. By Example 2.1, we have that \(GSO(X) = \{\emptyset, X, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}\). Therefore \(\{a\}, \{b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, c, d\}\) and \(\{b, c, d\}\) are \(G_{Sp}\)-open sets but not \(G\)-semiopen.
Proposition 2.2. If $PO(X) = \tau$, then $G_{sp}O(X) = GSO(X)$.

Proof. By Theorem 2.3, $GSO(X) \subseteq G_{sp}O(X)$. Let $A \in G_{sp}O(X)$. Then by Theorem 2.1, $A \subseteq \psi(pint(A))$. Since $PO(X) = \tau$, we have that $pint(A) = \text{int}(A)$ implies that $A \subseteq \psi(\text{pint}(A)) = \psi(\text{int}(A))$ and hence $A \in GSO(X)$. Thus $G_{sp}O(X) \subseteq GSO(X)$.

Theorem 2.4. Let $(X, \tau, G)$ be a grill topological space.
(i) If $A_i \in G_{sp}O(X)$ for each $i \in J$, then $\bigcup_{i \in J} A_i \in G_{sp}O(X)$;
(ii) If $A \in G_{sp}O(X)$ and $U \in PO(X)$, then $A \cap U \in G_{sp}O(X)$.

Proof. (i) Since $A_i \in G_{sp}O(X)$, we have that $A_i \subseteq \psi(pint(A_i))$ for each $i \in J$. Thus, we obtain $A_i \subseteq \psi(pint(A_i)) \subseteq \psi(pint(\bigcup_{i \in J} A_i))$ and hence $\bigcup_{i \in J} A_i \subseteq \psi(pint(\bigcup_{i \in J} A_i))$. This shows that $\bigcup_{i \in J} A_i \in G_{sp}O(X)$.

(ii) Let $A \in G_{sp}O(X)$ and $U \in PO(X)$. Then $A \subseteq \psi(pint(A))$ and $pint(U) = U$. Now, $A \cap U \subseteq \psi(pint(A)) \cap U = (\psi(pint(A)) \cap \phi(pint(A))) \cap U = (\phi(pint(A)) \cap U) \subseteq \psi(\phi(pint(A)) \cap U)$ for each $A \cap U \subseteq \phi(pint(A) \cap U)$ (by Theorem 2.10[17]) = $\psi(pint(A) \cap U) \cup \phi(pint(A) \cap U)) = \psi(pint(A \cap U))$. Therefore $A \cap U \in G_{sp}O(X)$.

Remark 2.1. The following example shows that if $A, B \in G_{sp}O(X)$, then $A \cap B \notin G_{sp}O(X)$.

From Example 2.1, take $A = \{b, c\}$ and $B = \{c, d\}$, then $A, B \in G_{sp}O(X)$ but $A \cap B = \{c\} \notin G_{sp}O(X)$.

Theorem 2.5. Let $(X, \tau, G)$ be a grill topological space and $A \subseteq X$. If $A \in G_{sp}C(X)$, then $pint(\psi(A)) \subseteq A$.

Proof. Suppose $A \in G_{sp}C(X)$. Then $X - A \in G_{sp}O(X)$ and hence $X - A \subseteq \psi(pint(X - A)) \subseteq \text{pcl}(\text{pcl}(X - A)) = X - \text{pct}(\text{pcl}(X - A)) \subseteq X - \psi(\psi(A))$, implies that $\psi(\psi(A)) \subseteq A$.

Theorem 2.6. Let $(X, \tau, G)$ be a grill topological space and $A \subseteq X$ such that $X - \psi(pint(A)) = \psi(pint(X - A))$. Then $A \in G_{sp}C(X)$ if and only if $\psi(pint(A)) \subseteq A$.

Proof. Necessary part is proved by Theorem 2.5. Conversely, suppose that $\psi(pint(A)) \subseteq A$. Then $X - A \subseteq X - \psi(pint(A)) = \psi(pint(X - A))$, implies that $X - A \in G_{sp}O(X)$. Hence $A \in G_{sp}C(X)$.

Definition 2.2. Let $(X, \tau, G)$ be a grill topological space and $A \subseteq X$. Then
(i) $G_{sp}$-interior of $A$ is defined as union of all $G_{sp}$-open sets contained in $A$.
Thus $G_{sp}\text{int}(A) = \cup\{U : U \in G_{sp}O(X) \text{ and } U \subseteq A\}$;
(ii) $G_{sp}$-closure of $A$ is defined as intersection of all $G_{sp}$-closed sets containing $A$.
Thus $G_{sp}\text{cl}(A) = \cap\{F : X - F \in G_{sp}O(X) \text{ and } A \subseteq F\}$. 
Theorem 2.7. Let \((X, \tau, G)\) be a grill topological space and \(A \subseteq X\). Then

(i) \(G_{sp}\text{-int}(A)\) is a \(G_{sp}\)-open set contained in \(A\);
(ii) \(G_{sp}\text{-cl}(A)\) is a \(G_{sp}\)-closed set containing \(A\);
(iii) \(A\) is \(G_{sp}\)-closed if and only if \(G_{sp}\text{-cl}(A) = A\);
(iv) \(A\) is \(G_{sp}\)-open if and only if \(G_{sp}\text{-int}(A) = A\);
(v) \(G_{sp}\text{-int}(A) = X - G_{sp}\text{-cl}(X - A)\);
(vi) \(G_{sp}\text{-cl}(A) = X - G_{sp}\text{-int}(X - A)\).

Proof. Follows from the Definition 2.15 and Theorem 2.4(i).

Theorem 2.8. Let \((X, \tau, G)\) be a grill topological space and \(A, B \subseteq X\). Then the following are hold:

(i) If \(A \subseteq B\), then \(G_{sp}\text{-int}(A) \subseteq G_{sp}\text{-int}(B)\);
(ii) \(G_{sp}\text{-int}(A \cup B) \supseteq G_{sp}\text{-int}(A) \cup G_{sp}\text{-int}(B)\);
(iii) \(G_{sp}\text{-int}(A \cap B) = G_{sp}\text{-int}(A) \cap G_{sp}\text{-int}(B)\).

Proof. Follows from the Theorem 2.8.

Definition 2.3. A function \(f: (X, \tau, G) \rightarrow (Y, \sigma)\) is said to be \(G_{sp}\)-continuous if \(f^{-1}(V) \in G_{sp}\text{-O}(X)\) for each \(V \in \text{PO}(Y)\).

Example 2.2. Let \(X = \{a, b, c, d\}, Y = \{1, 2, 3, 4\}, \tau = \{\emptyset, X, \{a, b\}, \{c, d\}\}, \sigma = \{\emptyset, Y, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\} and G = \{\{a, b, c\}, X\}. Then \(G_{sp}\text{-O}(X) = P(X)\) and \(\text{PO}(Y) = \{\emptyset, Y, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}.

Define \(f: (X, \tau, G) \rightarrow (Y, \sigma)\) by \(f(a) = 2, f(b) = 1, f(c) = 4\) and \(f(d) = 3\). Then inverse image of every preopen sets in \(Y\) is \(G_{sp}\)-open in \(X\). Hence \(f\) is \(G_{sp}\)-continuous.

Remark 2.2. The concepts of G-semicontinuous and \(G_{sp}\)-continuous are independent.

(i) From Example 2.2, we have that \(G_{SO}(X) = \{\emptyset, X, \{a, b\}, \{c, d\}\}\) and the function \(f\) is \(G_{sp}\)-continuous. Also \(f^{-1}([1, 2, 3]) = \{a, b, d\}\) is not G-semiopen in \(X\) for the open set \([1, 2, 3]\) of \(Y\). Hence \(f\) is not G-semicontinuous.

(ii) Let \(X = \{a, b, c, d\}, Y = \{1, 2, 3, 4\}, \tau = \{\emptyset, X, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}, \sigma = \{\emptyset, Y, \{1, 2\}, \{3, 4\}\} and G = \{\{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}. Then \(G_{SO}(X) = \tau, G_{sp}\text{-O}(X) = \{\emptyset, X, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}\) and \(\text{PO}(Y) = P(Y)\). Define \(f: (X, \tau, G) \rightarrow (Y, \sigma)\) by \(f(a) = 4, f(b) = 3, f(c) = 2\) and \(f(d) = 1\). Then the function \(f\) is G-semicontinuous. Also the inverse image \(f^{-1}(\{3\}) = \{b\}\) is not \(G_{sp}\)-open in \(X\) for the preopen set \(\{3\}\) of \(Y\). Hence \(f\) is not \(G_{sp}\)-continuous.

From (i) and (ii), we got the concepts of G-semicontinuous and \(G_{sp}\)-continuous are independent.
Theorem 2.9. For a function \( f: (X, \tau, G) \to (Y, \sigma), \) the following are equivalent:

(i) \( f \) is \( G_{sp} \)-continuous;

(ii) For each \( F \in PC(Y), f^{-1}(F) \in G_{sp}C(X); \)

(iii) For each \( x \in X \) and each \( V \in PO(Y) \) containing \( f(x), \) there exists a \( U \in G_{sp}O(X) \)

containing \( x \) such that \( f(U) \subseteq V. \)

Proof. (i) \( \Leftrightarrow \) (ii): It is obvious.

(i) \( \Rightarrow \) (iii): Let \( V \in PO(Y) \) and \( f(x) \in V(x \in X). \) Then by (i), \( f^{-1}(V) \in G_{sp}O(X) \) containing \( x. \)

Taking \( f^{-1}(V) = U, \) we have that \( x \in U \) and \( f(U) \subseteq V. \)

(iii) \( \Rightarrow \) (i): Let \( V \in PO(Y) \) and \( x \in f^{-1}(V). \) Then \( f(x) \in V \in PO(Y) \) and hence by (iii), there exists a \( U \in G_{sp}O(X) \) containing \( x \) such that \( f(U) \subseteq V. \) Thus, we obtain \( x \in U \subseteq \psi \left( \text{pint}(U) \right) \subseteq \psi(\text{pint}(f^{-1}(V))). \) This shows that \( f^{-1}(V) \subseteq \psi(\text{pint}(f^{-1}(V))). \) Hence \( f \) is \( G_{sp} \)-continuous.

Theorem 2.10. A function \( f: (X, \tau, G) \to (Y, \sigma) \) is \( G_{sp} \)-continuous if and only if the graph function \( g : X \to X \times Y, \) defined by \( g(x) = (x, f(x)) \) for each \( x \in X, \) is \( G_{sp} \)-continuous.

Proof. Suppose that \( f \) is \( G_{sp} \)-continuous. Let \( x \in X \) and \( W \in PO(X \times Y) \) containing \( g(x). \) Then there exist a \( U \in PO(X) \) and \( V \in PO(Y) \) such that \( g(x) = (x, f(x)) \in U \times V \subseteq W. \) Since \( f \) is \( G_{sp} \)-continuous, there exists a \( G \in G_{sp}O(X) \) containing \( x \) such that \( f(G) \subseteq V. \) By Theorem 2.4(b), \( G \cap U \in G_{sp}O(X) \) and \( g(G \cap U) \subseteq U \times V \subseteq W. \) This shows that \( g \) is \( G_{sp} \)-continuous.

Conversely, suppose that \( g \) is \( G_{sp} \)-continuous. Let \( x \in X \) and \( V \in \alpha(Y) \) containing \( f(x). \) Then \( X \times V \in PO(X \times Y) \) and by \( G_{sp} \)-continuity of \( g, \) there exists a \( U \in G_{sp}O(X) \) containing \( x \) such that \( g(U) \subseteq X \times V. \) Thus we have that \( f(U) \subseteq V \) and hence \( f \) is \( G_{sp} \)-continuous.

Definition 2.3. Let \( (X, \tau) \) be a topological space and \( (Y, \sigma, G) \) a grill topological space. A function \( f: (X, \tau) \to (Y, \sigma, G) \) is said to be \( G_{sp} \)-open (resp. \( G_{sp} \)-closed ) if for each \( U \in PO(X) \) (resp. for each \( U \in PC(X) \)), \( f(U) \) is \( G_{sp} \)-open (resp. \( G_{sp} \)-closed) in \((Y, \sigma, G). \)

Theorem 2.11. A function \( f: (X, \tau) \to (Y, \sigma, G) \) is \( G_{sp} \)-open if and only if for each \( x \in X \) and each pre-neighbourhood \( U \) of \( x, \) there exists a \( V \in G_{sp}O(Y) \) such that \( f(x) \in V \subseteq f(U). \)

Proof. Suppose that \( f \) is a \( G_{sp} \)-open function and let \( x \in X. \) Also let \( U \) be any pre-neighbourhood of \( x. \) Then there exists \( G \in PO(X) \) such that \( x \in G \subseteq U. \) Since \( f \) is \( G_{sp} \)-open, \( f(G) = V \) (say) \( \in G_{sp}O(Y) \) and \( f(x) \in V \subseteq f(U). \) Conversely, suppose that \( U \in PO(X). \) Then for each \( x \in U, \) there exists a \( V_x \in G_{sp}O(X) \) such that \( f(x) \in V_x \subseteq f(U). \) Thus \( f(U) = \bigcup \{ V_x : x \in U \} \) and hence by Theorem 2.4(a), \( f(U) \in G_{sp}O(Y). \) This shows that \( f \) is \( G_{sp} \)-open.

Theorem 2.12. Let \( f: (X, \tau) \to (Y, \sigma, G) \) be a \( G_{sp} \)-open function. If \( V \subseteq Y \) and \( F \in PC(X) \)

containing \( f^{-1}(V), \) then there exists a \( H \in G_{sp}O(Y) \) containing \( V \) such that \( f^{-1}(H) \subseteq F. \)
Proof. Suppose that $f$ is $G_{sp}$-open. Let $V \subseteq Y$ and $F \in PC(X)$ containing $f^{-1}(V)$. Then $X - F \in PO(X)$ and by $G_{sp}$-openness of $f$, $f(X - F) \in G_{sp}O(X)$. Thus $H = Y - f(X - F) \in G_{sp}C(Y)$ consequently $f^{-1}(V) \subseteq F$ implies that $V \subseteq H$. Further, we obtain that $f^{-1}(H) \subseteq F$.

**Theorem 2.13.** For any bijection $f: (X, \tau) \rightarrow (Y, \sigma, G)$, the following are equivalent:
(i) $f^{-1}: (Y, \sigma, G) \rightarrow (X, \tau)$ is $G_{sp}$-continuous;
(ii) $f$ is $G_{sp}$-open;
(iii) $f$ is $G_{sp}$-closed.

Proof. It is obvious.

**Definition 2.4.** Let $(X, \tau, G)$ be a grill topological space. A subset $A$ of $X$ is said to be a $G_{sp}^*$-set if $A = U \cap V$, where $U \in PO(X)$ and $\psi(pint(V)) = pint(V)$.

**Theorem 2.14.** Let $(X, \tau, G)$ be a grill topological space and let $A \subseteq X$. Then $A \in PO(X)$ if and only if $A \in G_{sp}O(X)$ and $A$ is $G_{sp}^*$-set in $(X, \tau, G)$.

Proof. Let $A \in PO(X)$. Then $A \in G_{sp}O(X)$, implies that $A \subseteq \psi(pint(A))$. Also $A$ can be expressed as $A = A \cap X$, where $A \in PO(X)$ and $\psi(pint(X)) = pint(X)$. Thus $A$ is a $G_{sp}^*$-set.

Conversely, Let $A \in G_{sp}O(X)$ and $A$ be a $G_{sp}^*$-set. Thus $A \subseteq \psi(pint(A)) = \psi(pint(U \cap V))$, where $U \in PO(X)$ and $\psi(pint(V)) = pint(V)$. Now $A \subseteq \psi(U \cap V) \subseteq \psi(U) \cap \psi(pint(V)) = U \cap \psi(U \cap V) \subseteq U \cap \psi(U) \cap \psi(pint(V)) = U \cap \psi(V) = \psi(pint(A))$. Hence $A \in PO(X)$.

**Definition 2.5.** A function $f: (X, \tau, G) \rightarrow (Y, \sigma)$ is $G_{sp}^*$-continuous if for each $V \in PO(Y)$, $f^{-1}(V)$ is a $G_{sp}^*$-set in $(X, \tau, G)$.

**Theorem 2.15.** Let $(X, \tau, G)$ be a grill topological space. Then for a function $f: (X, \tau, G) \rightarrow (Y, \sigma)$, the following are equivalent:
(i) $f$ is precontinuous;
(ii) $f$ is $G_{sp}$-continuous and $G_{sp}^*$-continuous.

Proof. Straightforward.

**References**


