Hacettepe Journal of Mathematics and Statistics
Volume 47 (4) (2018), 793 – 804

Bivariate Cheney-Sharma operators on simplex

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Abstract

In this paper, we consider bivariate Cheney-Sharma operators which are not the tensor product construction. Precisely, we show that these operators preserve Lipschitz condition of a given Lipschitz continuous function f and also the properties of the modulus of continuity function when f is a modulus of continuity function.

Keywords: Lipschitz continuous function, modulus of continuity function, bivariate Cheney-Sharma operators.

Mathematics Subject Classification (2010): 41A36

Received : 07.03.2017 *Accepted* : 02.05.2017 *Doi* : 10.15672/HJMS.2017.468

1. Introduction

The most celebrated linear positive operators for the uniform approximation of continuous real valued functions on $[0, 1]$ are Bernstein polynomials. As it is well known, besides approximation results, Bernstein polynomials have some nice retaining properties. The most referred study in this direction was due to Brown, Elliott and Paget [7] where they gave an elementary proof for the preservation of the Lipschitz constant and order of a Lipschitz continuous function by the Bernstein polynomials. Whereas, Lindvall previously obtained this result in terms of probabilistic methods in [20]. Moreover, in [19] Li proved that Bernstein polynomials also preserve the properties of the function of modulus of continuity. The same problems for some other type univariate or multivariate linear positive operators were solved by either using an elementary or probabilistic way (see, e.g. [3]-[6], [8], [9], [14], [15], [17], [18], [28]).

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In Abel-Jensen identity (see [2], p.326)

$$(u+v)\,(u+v+m\beta)^{m-1}=\sum_{k=0}^{m}\binom{m}{k}u\,(u+k\beta)^{k-1}\,v\,[v+(m-k)\beta]^{m-k-1} \tag{1.1}$$

where u, v and $\beta\in\mathbb{R}$, by taking $u=x$, $v=1-x$ and $m=n$, Cheney-Sharma [11] introduced the following Bernstein type operators for $f\in C[0,1]$, $x\in[0,1]$ and $n\in\mathbb{N}$,

$$\begin{aligned}Q_n^{\beta}(f;x)=&(1+n\beta)^{1-n}\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}x\,(x+k\beta)^{k-1}\\&\times(1-x)\big[1-x+(n-k)\beta\big]^{n-k-1},\end{aligned}$$

where β is a nonnegative real parameter. For these operators, tensor product of them and their some generalizations we can cite the papers [1], [10], [12], [21]-[27] and the monograph [2]. Remark that from [11] and [21] we know that Q_n^{β} operators reproduce constant functions and linear functions. Very recently, in [6] the authors showed that univariate Cheney-Sharma operators preserve the Lipschitz constant and order of a Lipschitz continuous function as well as the properties of the function of modulus of continuity.

We now introduce the notations, some needful definitions and the construction of the bivariate operators.

Throughout the paper, we shall use the standard notations given below.

Let $\mathbf{x}=(x_1,x_2)\in\mathbb{R}^2$, $\mathbf{k}=(k_1,k_2)\in\mathbb{N}_0^2$, $\mathbf{e}=(1,1)$, $\mathbf{0}=(0,0)$, $0\leq\beta\in\mathbb{R}$ and $n\in\mathbb{N}$. We denote as usual

$$|\mathbf{x}|:=x_1+x_2,\quad \mathbf{x}^{\mathbf{k}}:=x_1^{k_1}x_2^{k_2},\quad |\mathbf{k}|:=k_1+k_2,\quad \mathbf{k}!:=k_1!k_2!,\quad \beta\mathbf{x}=(\beta x_1,\beta x_2)$$

and

$$\binom{n}{\mathbf{k}}:=\frac{n!}{\mathbf{k}!(n-|\mathbf{k}|)!},\quad \sum_{|\mathbf{k}|\leq n}:=\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}.$$

We also denote the two dimensional simplex by

$$S:=\left\{\mathbf{x}=(x_1,x_2)\in\mathbb{R}^2:x_1,x_2\geq 0,|\mathbf{x}|\leq 1\right\}.$$

Moreover, $\mathbf{x}\leq\mathbf{y}$ stands for $x_i\leq y_i$, $i=1,2$.

We now construct the non-tensor product Cheney-Sharma operators. From (1.1) it is clear that

$$\begin{aligned}(1+n\beta)^{n-1}=&\sum_{k_1=0}^{n}\binom{n}{k_1}x_1\,(x_1+k_1\beta)^{k_1-1}\,(1-x_1)\\&\times[1-x_1+(n-k_1)\beta]^{n-k_1-1}.\end{aligned}$$

In (1.1), taking $u=x_2$, $v=1-x_1-x_2$ and $m=n-k_1$ we have

$$\begin{aligned}&(1-x_1)\,[1-x_1+(n-k_1)\beta]^{n-k_1-1}\\&=\sum_{k_2=0}^{n-k_1}\binom{n-k_1}{k_2}x_2\,(x_2+k_2\beta)^{k_2-1}\,(1-x_1-x_2)\\&\quad\times[1-x_1-x_2+(n-k_1-k_2)\beta]^{n-k_1-k_2-1}.\end{aligned}$$

Using this result in the above equality we find that

$$\begin{aligned}
&(1+n\beta)^{n-1}\\
&=\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}x_1x_2\,(x_1+k_1\beta)^{k_1-1}\,(x_2+k_2\beta)^{k_2-1}\\
&\quad\times(1-x_1-x_2)\,[1-x_1-x_2+(n-k_1-k_2)\beta]^{n-k_1-k_2-1}\\
&=\sum_{|\mathbf{k}|\le n}\binom{n}{\mathbf{k}}\mathbf{x}^{\mathbf{e}}(\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}\,(1-|\mathbf{x}|)\,[1-|\mathbf{x}|+(n-|\mathbf{k}|)\beta]^{n-|\mathbf{k}|-1}
\end{aligned}$$

or

$$\begin{aligned}
1=(1+n\beta)^{1-n}&\sum_{|\mathbf{k}|\le n}\binom{n}{\mathbf{k}}\mathbf{x}^{\mathbf{e}}(\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}\,(1-|\mathbf{x}|)\\
&\times[1-|\mathbf{x}|+(n-|\mathbf{k}|)\beta]^{n-|\mathbf{k}|-1}.
\end{aligned}$$

In this paper, for a continuous real valued function f defined on S we consider the non-tensor product bivariate extension of the operators $Q_n^{\beta}(f;x)$ defined by

$$\begin{aligned}
G_n^{\beta}(f;\mathbf{x})=(1+n\beta)^{1-n}&\sum_{|\mathbf{k}|\le n}f\left(\frac{\mathbf{k}}{n}\right)\binom{n}{\mathbf{k}}\mathbf{x}^{\mathbf{e}}(\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}\\
&\times(1-|\mathbf{x}|)\,[1-|\mathbf{x}|+(n-|\mathbf{k}|)\beta]^{n-|\mathbf{k}|-1}
\end{aligned}\tag{1.2}$$

where β is a nonnegative real parameter, $\mathbf{x}\in S$ and $n\in\mathbb{N}$. We observe that for $\beta=0$ these operators reduce to non-tensor product bivariate Bernstein polynomials (see [13],[16]).

1.1. Definition. (see, e.g.[9]) A continuous real valued function f defined on $A\subseteq\mathbb{R}^2$ is said to be Lipschitz continuous function of order μ , $0<\mu\le 1$ on A, if there exists $M>0$ such that

$$|f(\mathbf{x})-f(\mathbf{y})|\le M\sum_{i=1}^{2}|x_i-y_i|^{\mu}$$

for all $\mathbf{x},\mathbf{y}\in A$. The set of Lipschitz continuous functions of order μ with Lipschitz constant M on A is denoted by $Lip_M(\mu,A)$.

1.2. Definition. (see, e.g.[8]) If a bivariate non-negative and continuous function $\omega(\mathbf{t})$ satisfies the following conditions, then it is called a function of modulus of continuity.
(a) $\omega(\mathbf{0})=0$,
(b) $\omega(\mathbf{t})$ is a non-decreasing function in $\mathbf{t}$, i.e., $\omega(\mathbf{t})\ge\omega(\mathbf{v})$ for $\mathbf{t}\ge\mathbf{v}$,
(c) $\omega(\mathbf{t})$ is semi-additive, i.e., $\omega(\mathbf{t}+\mathbf{v})\le\omega(\mathbf{t})+\omega(\mathbf{v})$.

2. Main results

In this section, inspired by the paper of Cao, Ding and Xu [9], including preservation properties of multivariate Baskakov operators, we show that non-tensor product bivariate Cheney-Sharma operators defined by $G_n^{\beta}(f;\mathbf{x}):=G_n^{\beta}(f)(\mathbf{x})$ preserve the Lipschitz condition of a given Lipschitz continuous function f and properties of the function of modulus of continuity when the attached function f is a modulus of continuity function.

2.1. Theorem. If $f\in Lip_M(\mu,S)$, then $G_n^{\beta}(f)\in Lip_M(\mu,S)$ for all $n\in\mathbb{N}$.

Proof. Let $\mathbf{x}, \mathbf{y} \in S$ such that $\mathbf{y} \geq \mathbf{x}$. From (1.2) we have

$$\begin{aligned} G_n^{\beta}(f;\mathbf{y}) &= (1+n\beta)^{1-n} \sum_{i_1=0}^{n} \sum_{i_2=0}^{n-i_1} f\left(\frac{\mathbf{i}}{n}\right) \binom{n}{\mathbf{i}} \mathbf{y}^{\mathbf{e}} (\mathbf{y}+\mathbf{i}\beta)^{\mathbf{i}-\mathbf{e}} \\ &\quad \times (1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{i}|)\beta\right]^{n-|\mathbf{i}|-1} \\ &= (1+n\beta)^{1-n} \sum_{i_1=0}^{n} \sum_{i_2=0}^{n-i_1} f\left(\frac{\mathbf{i}}{n}\right) \binom{n}{\mathbf{i}} y_1 (y_1+i_1\beta)^{i_1-1} \\ &\quad \times y_2 (y_2+i_2\beta)^{i_2-1} (1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{i}|)\beta\right]^{n-|\mathbf{i}|-1}. \end{aligned}$$

Setting $u = x_1$, $v = y_1 - x_1$, $m = i_1$ and $u = x_2$, $v = y_2 - x_2$, $m = i_2$, respectively, in (1.1), we find

$$\begin{aligned} y_1 (y_1+i_1\beta)^{i_1-1} &= \sum_{k_1=0}^{i_1} \binom{i_1}{k_1} x_1 (x_1+k_1\beta)^{k_1-1} (y_1-x_1) \\ &\quad \times \left[y_1-x_1+(i_1-k_1)\beta\right]^{i_1-k_1-1} \end{aligned}$$

and

$$\begin{aligned} y_2 (y_2+i_2\beta)^{i_2-1} &= \sum_{k_2=0}^{i_2} \binom{i_2}{k_2} x_2 (x_2+k_2\beta)^{k_2-1} (y_2-x_2) \\ &\quad \times \left[y_2-x_2+(i_2-k_2)\beta\right]^{i_2-k_2-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} &G_n^{\beta}(f;\mathbf{y}) \\ &= (1+n\beta)^{1-n} \sum_{i_1=0}^{n} \sum_{i_2=0}^{n-i_1} f\left(\frac{\mathbf{i}}{n}\right) \binom{n}{\mathbf{i}} \sum_{k_1=0}^{i_1} \sum_{k_2=0}^{i_2} \binom{i_1}{k_1} \binom{i_2}{k_2} x_1 x_2 \\ &\quad \times (x_1+k_1\beta)^{k_1-1} (x_2+k_2\beta)^{k_2-1} (y_1-x_1)(y_2-x_2) \\ &\quad \times \left[y_1-x_1+(i_1-k_1)\beta\right]^{i_1-k_1-1} \left[y_2-x_2+(i_2-k_2)\beta\right]^{i_2-k_2-1} \\ &\quad \times (1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{i}|)\beta\right]^{n-|\mathbf{i}|-1} \\ &= (1+n\beta)^{1-n} \sum_{i_1=0}^{n} \sum_{k_1=0}^{i_1} \sum_{i_2=0}^{n-i_1} \sum_{k_2=0}^{i_2} f\left(\frac{\mathbf{i}}{n}\right) \frac{n!}{\mathbf{k}!(n-|\mathbf{i}|)!(i_1-k_1)!(i_2-k_2)!} \\ &\quad \times \mathbf{x}^{\mathbf{e}} (\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}} (\mathbf{y}-\mathbf{x})^{\mathbf{e}} \left[\mathbf{y}-\mathbf{x}+(\mathbf{i}-\mathbf{k})\beta\right]^{\mathbf{i}-\mathbf{k}-\mathbf{e}} \\ &\quad \times (1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{i}|)\beta\right]^{n-|\mathbf{i}|-1}. \end{aligned}$$

Changing the order of the above summations and then letting $\mathbf{i}-\mathbf{k}=\mathbf{l}$ we obtain

$$
\begin{aligned}
&G_n^{\beta}(f;\mathbf{y})\\
&=(1+n\beta)^{1-n}\sum_{k_1=0}^{n}\sum_{i_1=k_1}^{n}\sum_{k_2=0}^{n-i_1}\sum_{i_2=k_2}^{n-i_1} f\left(\frac{\mathbf{i}}{n}\right)\frac{n!}{\mathbf{k}!(n-|\mathbf{i}|)!(i_1-k_1)!(i_2-k_2)!}\\
&\quad\times\mathbf{x}^{\mathbf{e}}(\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}}\left[\mathbf{y}-\mathbf{x}+(\mathbf{i}-\mathbf{k})\beta\right]^{\mathbf{i}-\mathbf{k}-\mathbf{e}}\\
&\quad\times(1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{i}|)\beta\right]^{n-|\mathbf{i}|-1} \qquad (2.1)\\
&=(1+n\beta)^{1-n}\sum_{k_1=0}^{n}\sum_{l_1=0}^{n-k_1}\sum_{k_2=0}^{n-k_1-l_1}\sum_{l_2=0}^{n-|\mathbf{k}|-l_1} f\left(\frac{\mathbf{k}+\mathbf{l}}{n}\right)\frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!}\\
&\quad\times\mathbf{x}^{\mathbf{e}}(\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}}(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}}\\
&\quad\times(1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.
\end{aligned}
$$

Now we consider

$$
\begin{aligned}
G_n^{\beta}(f;\mathbf{x})=(1+n\beta)^{1-n}&\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1} f\left(\frac{\mathbf{k}}{n}\right)\binom{n}{\mathbf{k}}\mathbf{x}^{\mathbf{e}}(\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}\\
&\times(1-|\mathbf{x}|)\left[1-|\mathbf{x}|+(n-|\mathbf{k}|)\beta\right]^{n-|\mathbf{k}|-1}.
\end{aligned}
$$

In (1.1), if we put y_1-x_1, $1-y_1-x_2$ and $n-|\mathbf{k}|$ in place of u, v and m, respectively, one has

$$
\begin{aligned}
&(1-|\mathbf{x}|)\left[1-|\mathbf{x}|+(n-|\mathbf{k}|)\beta\right]^{n-|\mathbf{k}|-1}\\
&=\sum_{l_1=0}^{n-|\mathbf{k}|}\binom{n-|\mathbf{k}|}{l_1}(y_1-x_1)(y_1-x_1+l_1\beta)^{l_1-1}(1-y_1-x_2)\\
&\quad\times\left[1-y_1-x_2+(n-|\mathbf{k}|-l_1)\beta\right]^{n-|\mathbf{k}|-l_1-1}.
\end{aligned}
$$

Again in the equality (1.1), we replace u, v and m by y_2-x_2, $1-|\mathbf{y}|$ and $m=n-|\mathbf{k}|-l_1$, respectively, we find

$$
\begin{aligned}
&(1-y_1-x_2)\left[1-y_1-x_2+(n-|\mathbf{k}|-l_1)\beta\right]^{n-|\mathbf{k}|-l_1-1}\\
&=\sum_{l_2=0}^{n-|\mathbf{k}|-l_1}\binom{n-|\mathbf{k}|-l_1}{l_2}(y_2-x_2)(y_2-x_2+l_2\beta)^{l_2-1}\\
&\quad\times(1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.
\end{aligned}
$$

Making use of this in the above equality leads to

$$
\begin{aligned}
&(1-|\mathbf{x}|)\left[1-|\mathbf{x}|+(n-|\mathbf{k}|)\beta\right]^{n-|\mathbf{k}|-1}\\
&=\sum_{l_1=0}^{n-|\mathbf{k}|}\sum_{l_2=0}^{n-|\mathbf{k}|-l_1}\binom{n-|\mathbf{k}|}{\mathbf{l}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}}(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}}\\
&\quad\times(1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.
\end{aligned}
$$

Thus we conclude that

$$\begin{aligned}
&G_n^{\beta}(f;\mathbf{x})\\
&=(1+n\beta)^{1-n}\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\sum_{l_1=0}^{n-|\mathbf{k}|}\sum_{l_2=0}^{n-|\mathbf{k}|-l_1} f\left(\frac{\mathbf{k}}{n}\right)\frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!}\\
&\quad\times\mathbf{x}^{\mathbf{e}}(\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}}(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}}\\
&\quad\times(1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.
\end{aligned}$$

Now changing the order of the two summations in the middle, we obtain

$$\begin{aligned}
&G_n^{\beta}(f;\mathbf{x})\\
&=(1+n\beta)^{1-n}\sum_{k_1=0}^{n}\sum_{l_1=0}^{n-k_1}\sum_{k_2=0}^{n-k_1-l_1}\sum_{l_2=0}^{n-|\mathbf{k}|-l_1} f\left(\frac{\mathbf{k}}{n}\right)\frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!}\\
&\quad\times\mathbf{x}^{\mathbf{e}}(\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}}(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}}\\
&\quad\times(1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.
\end{aligned}\tag{2.2}$$

So, from (2.1) and (2.2) it follows that

$$\begin{aligned}
&G_n^{\beta}(f;\mathbf{y})-G_n^{\beta}(f;\mathbf{x})\\
&=(1+n\beta)^{1-n}\sum_{k_1=0}^{n}\sum_{l_1=0}^{n-k_1}\sum_{k_2=0}^{n-k_1-l_1}\sum_{l_2=0}^{n-|\mathbf{k}|-l_1}\left[f\left(\frac{\mathbf{k}+\mathbf{l}}{n}\right)-f\left(\frac{\mathbf{k}}{n}\right)\right]\\
&\quad\times\frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!}\mathbf{x}^{\mathbf{e}}(\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}}(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}}\\
&\quad\times(1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.
\end{aligned}$$

Again interchanging the order of the summations two times successively, we find

$$\begin{aligned}
&G_n^{\beta}(f;\mathbf{y})-G_n^{\beta}(f;\mathbf{x})\\
&=(1+n\beta)^{1-n}\sum_{l_1=0}^{n}\sum_{k_1=0}^{n-l_1}\sum_{l_2=0}^{n-k_1-l_1}\sum_{k_2=0}^{n-k_1-|\mathbf{l}|}\left[f\left(\frac{\mathbf{k}+\mathbf{l}}{n}\right)-f\left(\frac{\mathbf{k}}{n}\right)\right]\\
&\quad\times\frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!}\mathbf{x}^{\mathbf{e}}(\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}}(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}}\\
&\quad\times(1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1}\\
&=(1+n\beta)^{1-n}\sum_{l_1=0}^{n}\sum_{l_2=0}^{n-l_1}\sum_{k_1=0}^{n-|\mathbf{l}|}\sum_{k_2=0}^{n-k_1-|\mathbf{l}|}\left[f\left(\frac{\mathbf{k}+\mathbf{l}}{n}\right)-f\left(\frac{\mathbf{k}}{n}\right)\right]\\
&\quad\times\frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!}\mathbf{x}^{\mathbf{e}}(\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}}(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}}\\
&\quad\times(1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.
\end{aligned}\tag{2.3}$$

Using the fact $f\in Lip_M(\mu,S)$ and the following equality

$$\frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!}=\binom{n}{\mathbf{l}}\binom{n-|\mathbf{l}|}{k_1}\binom{n-k_1-|\mathbf{l}|}{k_2}$$

one can write

$$\begin{aligned}
&\left|G_n^{\beta}(f;\mathbf{y})-G_n^{\beta}(f;\mathbf{x})\right| \\
\leq& M\left(1+n\beta\right)^{1-n}\sum_{l_1=0}^{n}\sum_{l_2=0}^{n-l_1}\sum_{k_1=0}^{n-|\mathbf{l}|}\sum_{k_2=0}^{n-k_1-|\mathbf{l}|}\left[\left(\frac{l_1}{n}\right)^{\mu}+\left(\frac{l_2}{n}\right)^{\mu}\right] \\
&\times\binom{n}{\mathbf{l}}\binom{n-|\mathbf{l}|}{k_1}\binom{n-k_1-|\mathbf{l}|}{k_2}\mathbf{x}^{\mathbf{e}}(\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}} \\
&\times\left(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta\right)^{\mathbf{l}-\mathbf{e}}\left(1-|\mathbf{y}|\right)\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1} \\
=& M\left(1+n\beta\right)^{1-n}\sum_{l_1=0}^{n}\sum_{l_2=0}^{n-l_1}\left[\left(\frac{l_1}{n}\right)^{\mu}+\left(\frac{l_2}{n}\right)^{\mu}\right]\binom{n}{\mathbf{l}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}} \\
&\times\left(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta\right)^{\mathbf{l}-\mathbf{e}}\sum_{k_1=0}^{n-|\mathbf{l}|}\binom{n-|\mathbf{l}|}{k_1}x_1(x_1+k_1\beta)^{k_1-1} \\
&\times\sum_{k_2=0}^{n-k_1-|\mathbf{l}|}\binom{n-k_1-|\mathbf{l}|}{k_2}x_2(x_2+k_2\beta)^{k_2-1} \\
&\times\left(1-|\mathbf{y}|\right)\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.
\end{aligned}$$

Taking $u=x_2$, $v=1-|\mathbf{y}|$ and $m=n-k_1-|\mathbf{l}|$ in (1.1), it is easily seen that

$$\begin{aligned}
&\left(x_2+1-|\mathbf{y}|\right)\left[x_2+1-|\mathbf{y}|+(n-k_1-|\mathbf{l}|)\beta\right]^{n-k_1-|\mathbf{l}|-1} \\
=&\sum_{k_2=0}^{n-k_1-|\mathbf{l}|}\binom{n-k_1-|\mathbf{l}|}{k_2}x_2(x_2+k_2\beta)^{k_2-1}\left(1-|\mathbf{y}|\right) \\
&\times\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.
\end{aligned}\tag{2.4}$$

Hence we can write

$$\begin{aligned}
&\left|G_n^{\beta}(f;\mathbf{y})-G_n^{\beta}(f;\mathbf{x})\right| \\
\leq& M\left(1+n\beta\right)^{1-n}\sum_{l_1=0}^{n}\sum_{l_2=0}^{n-l_1}\left[\left(\frac{l_1}{n}\right)^{\mu}+\left(\frac{l_2}{n}\right)^{\mu}\right]\binom{n}{\mathbf{l}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}} \\
&\times\left(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta\right)^{\mathbf{l}-\mathbf{e}}\sum_{k_1=0}^{n-|\mathbf{l}|}\binom{n-|\mathbf{l}|}{k_1}x_1(x_1+k_1\beta)^{k_1-1}\left(x_2+1-|\mathbf{y}|\right) \\
&\times\left[x_2+1-|\mathbf{y}|+(n-k_1-|\mathbf{l}|)\beta\right]^{n-k_1-|\mathbf{l}|-1}.
\end{aligned}$$

Again in (1.1), we replace u, v and m by x_1, $x_2+1-|\mathbf{y}|$ and $m=n-|\mathbf{l}|$, respectively, to obtain

$$\begin{aligned}
&\left(x_1+x_2+1-|\mathbf{y}|\right)\left[x_1+x_2+1-|\mathbf{y}|+(n-|\mathbf{l}|)\beta\right]^{n-|\mathbf{l}|-1} \\
=&\left(1-|\mathbf{y}-\mathbf{x}|\right)\left[1-|\mathbf{y}-\mathbf{x}|+(n-|\mathbf{l}|)\beta\right]^{n-|\mathbf{l}|-1} \\
=&\sum_{k_1=0}^{n-|\mathbf{l}|}\binom{n-|\mathbf{l}|}{k_1}x_1(x_1+k_1\beta)^{k_1-1}(x_2+1-|\mathbf{y}|) \\
&\times\left[x_2+1-|\mathbf{y}|+(n-k_1-|\mathbf{l}|)\beta\right]^{n-k_1-|\mathbf{l}|-1}.
\end{aligned}\tag{2.5}$$

This leads to

$$
\begin{aligned}
&\left|G_n^\beta(f;\mathbf{y})-G_n^\beta(f;\mathbf{x})\right| \\
\le& M\,(1+n\beta)^{1-n}\sum_{l_1=0}^{n}\sum_{l_2=0}^{n-l_1}\left[\left(\frac{l_1}{n}\right)^\mu+\left(\frac{l_2}{n}\right)^\mu\right]\binom{n}{\mathbf{l}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}} \\
&\times(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}}(1-|\mathbf{y}-\mathbf{x}|)\left[1-|\mathbf{y}-\mathbf{x}|+(n-|\mathbf{l}|)\beta\right]^{n-|\mathbf{l}|-1} \\
=& M\left[G_n^\beta\left(t_1^\mu;\mathbf{y}-\mathbf{x}\right)+G_n^\beta\left(t_2^\mu;\mathbf{y}-\mathbf{x}\right)\right].
\end{aligned}
\tag{2.6}
$$

Now consider the term $G_n^\beta\left(t_1^\mu;\mathbf{y}-\mathbf{x}\right)$. With the help of the equality

$$
\binom{n}{\mathbf{l}}=\binom{n}{l_1}\binom{n-l_1}{l_2}
$$

we can write

$$
\begin{aligned}
&G_n^\beta\left(t_1^\mu;\mathbf{y}-\mathbf{x}\right) \\
=&(1+n\beta)^{1-n}\sum_{l_1=0}^{n}\sum_{l_2=0}^{n-l_1}\left(\frac{l_1}{n}\right)^\mu\binom{n}{\mathbf{l}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}}(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}} \\
&\times(1-|\mathbf{y}-\mathbf{x}|)\left[1-|\mathbf{y}-\mathbf{x}|+(n-|\mathbf{l}|)\beta\right]^{n-|\mathbf{l}|-1} \\
=&(1+n\beta)^{1-n}\sum_{l_1=0}^{n}\left(\frac{l_1}{n}\right)^\mu\binom{n}{l_1}(y_1-x_1)(y_1-x_1+l_1\beta)^{l_1-1} \\
&\times\sum_{l_2=0}^{n-l_1}\binom{n-l_1}{l_2}(y_2-x_2)(y_2-x_2+l_2\beta)^{l_2-1} \\
&\times(1-|\mathbf{y}-\mathbf{x}|)\left[1-|\mathbf{y}-\mathbf{x}|+(n-|\mathbf{l}|)\beta\right]^{n-|\mathbf{l}|-1}.
\end{aligned}
$$

In the equality (1.1), if we take y_2-x_2, $1-|\mathbf{y}-\mathbf{x}|$ and $n-l_1$ in place of u, v and m, respectively, then we get

$$
\begin{aligned}
&\left[1-(y_1-x_1)\right]\left[1-(y_1-x_1)+(n-l_1)\beta\right]^{n-l_1-1} \\
=&\sum_{l_2=0}^{n-l_1}\binom{n-l_1}{l_2}(y_2-x_2)(y_2-x_2+l_2\beta)^{l_2-1}(1-|\mathbf{y}-\mathbf{x}|) \\
&\times\left[1-|\mathbf{y}-\mathbf{x}|+(n-|\mathbf{l}|)\beta\right]^{n-|\mathbf{l}|-1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&G_n^\beta\left(t_1^\mu;\mathbf{y}-\mathbf{x}\right) \\
=&(1+n\beta)^{1-n}\sum_{l_1=0}^{n}\left(\frac{l_1}{n}\right)^\mu\binom{n}{l_1}(y_1-x_1)(y_1-x_1+l_1\beta)^{l_1-1} \\
&\times\left[1-(y_1-x_1)\right]\left[1-(y_1-x_1)+(n-l_1)\beta\right]^{n-l_1-1} \\
=&Q_n^\beta\left(t_1^\mu;y_1-x_1\right).
\end{aligned}
$$

Applying the Hölder inequality with conjugate pairs $p=\frac{1}{\mu}$ and $q=\frac{1}{1-\mu}$, we find

$$
\begin{aligned}
G_n^\beta\left(t_1^\mu;\mathbf{y}-\mathbf{x}\right)=&Q_n^\beta\left(t_1^\mu;y_1-x_1\right) \\
&\le\left[Q_n^\beta(t_1;y_1-x_1)\right]^\mu\left[Q_n^\beta(1;y_1-x_1)\right]^{1-\mu}.
\end{aligned}
$$

As mentioned before, since the univariate Cheney-Sharma operators given by Q_n^β reproduce constant and linear functions we reach to

$$G_n^\beta\left(t_1^\mu;\mathbf{y}-\mathbf{x}\right)\leq(y_1-x_1)^\mu.$$

Now in the following equality

$$\begin{aligned}G_n^\beta\left(t_2^\mu;\mathbf{y}-\mathbf{x}\right)=&(1+n\beta)^{1-n}\sum_{l_1=0}^{n}\sum_{l_2=0}^{n-l_1}\left(\frac{l_2}{n}\right)^\mu\binom{n}{\mathbf{l}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}}(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}}\\&\times(1-|\mathbf{y}-\mathbf{x}|)\left[1-|\mathbf{y}-\mathbf{x}|+(n-|\mathbf{l}|)\beta\right]^{n-|\mathbf{l}|-1},\end{aligned}$$

if we change the order of summations and use the equality $\binom{n}{\mathbf{l}}=\binom{n-l_2}{l_1}\binom{n}{l_2}$, then we can write

$$\begin{aligned}G_n^\beta\left(t_2^\mu;\mathbf{y}-\mathbf{x}\right)=&(1+n\beta)^{1-n}\sum_{l_2=0}^{n}\sum_{l_1=0}^{n-l_2}\left(\frac{l_2}{n}\right)^\mu\binom{n}{\mathbf{l}}(\mathbf{y}-\mathbf{x})^{\mathbf{e}}(\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}}\\&\times(1-|\mathbf{y}-\mathbf{x}|)\left[1-|\mathbf{y}-\mathbf{x}|+(n-|\mathbf{l}|)\beta\right]^{n-|\mathbf{l}|-1}\\=&(1+n\beta)^{1-n}\sum_{l_2=0}^{n}\left(\frac{l_2}{n}\right)^\mu\binom{n}{l_2}(y_2-x_2)(y_2-x_2+l_2\beta)^{l_2-1}\\&\times\sum_{l_1=0}^{n-l_2}\binom{n-l_2}{l_1}(y_1-x_1)(y_1-x_1+l_1\beta)^{l_1-1}\\&\times(1-|\mathbf{y}-\mathbf{x}|)\left[1-|\mathbf{y}-\mathbf{x}|+(n-|\mathbf{l}|)\beta\right]^{n-|\mathbf{l}|-1}.\end{aligned}$$

Taking $u=y_1-x_1$, $v=1-|\mathbf{y}-\mathbf{x}|$ and $m=n-l_2$ in (1.1), it is clear that

$$\begin{aligned}&\left[1-(y_2-x_2)\right]\left[1-(y_2-x_2)+(n-l_2)\beta\right]^{n-l_2-1}\\=&\sum_{l_1=0}^{n-l_2}\binom{n-l_2}{l_1}(y_1-x_1)(y_1-x_1+l_1\beta)^{l_1-1}(1-|\mathbf{y}-\mathbf{x}|)\\&\times\left[1-|\mathbf{y}-\mathbf{x}|+(n-|\mathbf{l}|)\beta\right]^{n-|\mathbf{l}|-1}.\end{aligned}$$

Hence, one gets

$$\begin{aligned}G_n^\beta\left(t_2^\mu;\mathbf{y}-\mathbf{x}\right)=&(1+n\beta)^{1-n}\sum_{l_2=0}^{n}\left(\frac{l_2}{n}\right)^\mu\binom{n}{l_2}(y_2-x_2)(y_2-x_2+l_2\beta)^{l_2-1}\\&\times\left[1-(y_2-x_2)\right]\left[1-(y_2-x_2)+(n-l_2)\beta\right]^{n-l_2-1}\\=&Q_n^\beta\left(t_2^\mu;y_2-x_2\right).\end{aligned}$$

Application of Hölder's inequality with $p=\frac{1}{\mu}$ and $q=\frac{1}{1-\mu}$ gives

$$G_n^\beta\left(t_2^\mu;\mathbf{y}-\mathbf{x}\right)\leq(y_2-x_2)^\mu.$$

Thus from (2.6) it follows that

$$\left|G_n^\beta(f;\mathbf{y})-G_n^\beta(f;\mathbf{x})\right|\leq M\left[(y_1-x_1)^\mu+(y_2-x_2)^\mu\right]$$

which implies that $G_n^\beta(f) \in Lip_M(\mu, S)$. In a similar way the same result can be found for $\mathbf{x} \geq \mathbf{y}$. If $x_1 \geq y_1$, $x_2 \leq y_2$, then we obtain from the above result for $(y_1, x_2) \in S$ that

$$\begin{aligned}\left|G_n^\beta(f;\mathbf{y}) - G_n^\beta(f;\mathbf{x})\right| \leq & \left|G_n^\beta\left(f;(x_1,x_2)\right) - G_n^\beta\left(f;(y_1,x_2)\right)\right| \\ & + \left|G_n^\beta\left(f;(y_1,y_2)\right) - G_n^\beta\left(f;(y_1,x_2)\right)\right| \\ \leq & M\left[(x_1-y_1)^\mu + (y_2-x_2)^\mu\right]\end{aligned}$$

Finally, for the case $x_1 \leq y_1$, $x_2 \geq y_2$ we have the same result. This completes the proof. □

2.2. Theorem. If ω is a modulus of continuity function, then $G_n^\beta(\omega)$ is also a modulus of continuity function for all $n \in \mathbb{N}$.

Proof. Let $\mathbf{x}, \mathbf{y} \in S$ such that $\mathbf{y} \geq \mathbf{x}$. Regarding f as a modulus of continuity function ω in (2.3) we have

$$\begin{aligned}& G_n^\beta(\omega;\mathbf{y}) - G_n^\beta(\omega;\mathbf{x}) \\ = & (1+n\beta)^{1-n} \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \sum_{k_1=0}^{n-|\mathbf{l}|} \sum_{k_2=0}^{n-k_1-|\mathbf{l}|} \left[\omega\left(\frac{\mathbf{k}+\mathbf{l}}{n}\right) - \omega\left(\frac{\mathbf{k}}{n}\right)\right] \\ & \times \frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!} \mathbf{x}^{\mathbf{e}} (\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}} (\mathbf{y}-\mathbf{x})^{\mathbf{e}} (\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}} \\ & \times (1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.\end{aligned}$$

Using the property (b) of ω we have $G_n^\beta(\omega;\mathbf{y}) - G_n^\beta(\omega;\mathbf{x}) \geq 0$ when $\mathbf{y} \geq \mathbf{x}$.
Moreover, from the property (c) of modulus of continuity function ω and the equality

$$\frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!} = \binom{n}{\mathbf{l}} \binom{n-|\mathbf{l}|}{k_1} \binom{n-k_1-|\mathbf{l}|}{k_2}$$

we can write

$$\begin{aligned}& G_n^\beta(\omega;\mathbf{y}) - G_n^\beta(\omega;\mathbf{x}) \\ \leq & (1+n\beta)^{1-n} \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \sum_{k_1=0}^{n-|\mathbf{l}|} \sum_{k_2=0}^{n-k_1-|\mathbf{l}|} \omega\left(\frac{\mathbf{l}}{n}\right) \binom{n}{\mathbf{l}} \binom{n-|\mathbf{l}|}{k_1} \\ & \times \binom{n-k_1-|\mathbf{l}|}{k_2} \mathbf{x}^{\mathbf{e}} (\mathbf{x}+\mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}} (\mathbf{y}-\mathbf{x})^{\mathbf{e}} (\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}} (1-|\mathbf{y}|) \\ & \times \left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1} \\ = & (1+n\beta)^{1-n} \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \omega\left(\frac{\mathbf{l}}{n}\right) \binom{n}{\mathbf{l}} (\mathbf{y}-\mathbf{x})^{\mathbf{e}} (\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}} \\ & \times \sum_{k_1=0}^{n-|\mathbf{l}|} \binom{n-|\mathbf{l}|}{k_1} x_1 (x_1+k_1\beta)^{k_1-1} \sum_{k_2=0}^{n-k_1-|\mathbf{l}|} \binom{n-k_1-|\mathbf{l}|}{k_2} x_2 \\ & \times (x_2+k_2\beta)^{k_2-1} (1-|\mathbf{y}|)\left[1-|\mathbf{y}|+(n-|\mathbf{k}|-|\mathbf{l}|)\beta\right]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.\end{aligned}$$

Using the equalities (2.4) and (2.5) respectively, one has

$$\begin{aligned} G_n^{\beta}(\omega;\mathbf{y}) - G_n^{\beta}(\omega;\mathbf{x}) \leq & (1+n\beta)^{1-n} \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \omega\left(\frac{\mathbf{l}}{n}\right) \binom{n}{\mathbf{l}} \\ & \times (\mathbf{y}-\mathbf{x})^{\mathbf{e}} (\mathbf{y}-\mathbf{x}+\mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}} (1-|\mathbf{y}-\mathbf{x}|) \\ & \times [1-|\mathbf{y}-\mathbf{x}|+(n-|\mathbf{l}|)\beta]^{n-|\mathbf{l}|-1} \\ = & G_n^{\beta}(\omega;\mathbf{y}-\mathbf{x}). \end{aligned}$$

This shows that $G_n^{\beta}(\omega)$ is semi-additive. Finally, from the definition of G_n^{β} it is obvious that $G_n^{\beta}(\omega;\mathbf{0}) = \omega(\mathbf{0}) = 0$. Therefore $G_n^{\beta}(\omega)$ itself is a function of modulus of continuity when ω is so. □

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