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$q-$ Harmonic mappings for which analytic part is $q-$ convex functions of complex order

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Abstract

We introduce a new class of harmonic function f, that is subclass of planar harmonic mapping associated with $q-$ difference operator. Let h and g are analytic functions in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$. If $f = h + \overline{g}$ is the solution of the non-linear partial differential equation $w_q(z) = \frac{D_q g(z)}{D_q h(z)} = \frac{\overline{f_{\overline{z}}}}{f_z}$ with $|w_q(z)| < 1$, $w_q(z) \prec b_1 \frac{1+z}{1-qz}$ and h is $q-$ convex function of complex order, then the class of such functions are called $q-$ harmonic functions for which analytic part is $q-$ convex functions of complex order denoted by $\mathcal{S}_{\mathcal{HC}_q(b)}$. Obviously that the class $\mathcal{S}_{\mathcal{HC}_q(b)}$ is the subclass of $\mathcal{S}_{\mathcal{H}}$. In this paper, we investigate properties of the class $\mathcal{S}_{\mathcal{HC}_q(b)}$ by using subordination techniques.

Keywords: $q-$ difference operator, $q-$ harmonic mapping, $q-$ convex function of complex order.

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1. Introduction

A planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex valued harmonic function f, which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain, the mapping f has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ and have the following power series expansions

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

where $a_n, b_n \in \mathbb{C}$, $n = 0, 1, 2, 3, \cdots$. As usual, we call h the analytic part of f and g the co-analytic part of f, respectively. An elegant and complete treatment theory of the harmonic mapping is given in Duren's monograph [3]. Lewy [11] proved in 1936 that the harmonic mapping f is locally univalent in $\mathbb{D}$ if and only if its Jacobian $J_f = |h'(z)|^2 - |g'(z)|^2$ is different from zero in $\mathbb{D}$. In view of this result, locally univalent harmonic mappings in the open unit disc are either sense-preserving if $|g'(z)| < |h'(z)|$ or sense-reversing if $|g'(z)| > |h'(z)|$ in $\mathbb{D}$. Throughout this paper, we will restrict ourselves to the study of sense-preserving harmonic mappings. We also note that $f = h + \overline{g}$ is sense-preserving in $\mathbb{D}$ if and only if h' does not vanish in $\mathbb{D}$ and the second dilatation $w(z) = \frac{g'(z)}{h'(z)}$ has the property $|w(z)| < 1$ for all $z \in \mathbb{D}$. Therefore the class of all sense-preserving harmonic mappings in $\mathbb{D}$ with $a_0 = b_0 = 0$ and $a_1 = 1$ will be denoted by $\mathcal{S}_{\mathcal{H}}$. Thus $\mathcal{S}_{\mathcal{H}}$ contains standard class $\mathcal{S}$ of analytic univalent functions. The family of all mappings $f \in \mathcal{S}_{\mathcal{H}}$ with the additional property that $g'(0) = 0$, i.e., $b_1 = 0$ are denoted by $\mathcal{S}_{\mathcal{H}}^0$. Hence it is clear that $\mathcal{S} \subset \mathcal{S}_{\mathcal{H}}^0 \subset \mathcal{S}_{\mathcal{H}}$.

In 1908 and 1910 Jackson [8, 9] initiated a study of $q-$ difference operator by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad \text{for} \quad z \in B \backslash \{0\} \tag{1.1}$$

where B is a subset of complex plane $\mathbb{C}$, called $q-$ geometric set if $qz \in B$, whenever $z \in B$. Note that if a subset B of $\mathbb{C}$ is $q-$ geometric, then it contains all geometric sequences $\{zq^n\}_0^\infty$, $zq \in B$. Obviously, $D_q f(z) \to f'(z)$ as $q \to 1^-$. The $q-$ difference operator (1.1) is sometimes called Jackson $q-$ difference operator. Note that such an operator plays an important role in the theory of hypergeometric series and quantum physics (see for instance [1, 4, 5, 10]).

Also, note that $D_q f(0) \to f'(0)$ as $q \to 1^-$ and $D_q^2 f(z) = D_q(D_q f(z))$. In fact, $q-$ calculus is ordinary classical calculus without the notion of limits. Recent interest in $q-$ calculus is because of its applications in various branches of mathematics and physics. For definition and properties of $q-$ difference operator and $q-$ calculus, one may refer to [1, 4, 5, 10].

Under the hypothesis of the definition of $q-$ difference operator, then we have the following rules:

(1) For a function $f(z) = z^n$, we observe that

$$D_q z^n = \frac{1-q^n}{1-q} z^{n-1}.$$

Therefore we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} a_n \frac{1-q^n}{1-q} z^{n-1}.$$

(2) If functions f and g are defined on a $q-$ geometric set $B \subset \mathbb{C}$ such that $q-$ derivatives of f and g exist for all $z \in B$, then

(i) $D_q(af(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z)$ where a and b are real or complex constants.

(ii) $D_q(f(z).g(z)) = g(z)D_qf(z) + f(qz)D_qg(z)$.

(iii) $D_q\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{g(qz)D_qf(z) - f(qz)D_qg(z)}{g(z)g(qz)}, \quad g(z)g(qz) \neq 0.$

(iv) As a right inverse, Jackson introduced $q-$ integral

$$\int_0^z f(t)d_qt = z(1-q)\sum_{n=0}^{\infty} q^n f(zq^n)$$

provided that the series converges.

The following theorem is an analogue of the fundamental theorem of calculus.

A. Theorem. ([10]) Let f be a $q-$ regular at zero, defined on $q-$ geometric set B containing zero. Define

$$F(z) = \int_c^z f(\zeta)d_q\zeta, \quad (\zeta \in B)$$

where c is a fixed point in B, then F is $q-$ regular at zero. Furthermore $D_qF(z)$ exists for every $z \in B$ and

$$D_qF(z) = f(z)$$

for every $z \in B$.

Conversely, if a and b are two points in B, then

$$\int_a^b D_qf(\zeta)d_q\zeta = f(b) - f(a).$$

(3) The $q-$ differential is defined as

$$d_qf(z) = f(z) - f(qz).$$

Therefore we can write

$$d_qf(z) = \frac{f(z) - f(qz)}{(1-q)z}d_qz.$$

(4) The partial $q-$ derivative of a multivariable real continous functions $f(x_1, x_2, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n)$ to a variable x_i is defined by

$$D_{q,x_i}f(\vec{x}) = \frac{f(\vec{x}) - \varepsilon_{q,x_i}f(\vec{x})}{(1-q)x_i}, \quad x_i \neq 0, q \in (0,1)$$

$$\left[D_{q,x_i}f(\vec{x})\right]_{x_i=0} = \lim_{x_i\to 0} D_{q,x_i}f(\vec{x})$$

where $\varepsilon_{q,x_i}f(\vec{x}) = f(x_1, x_2, ..., x_{i-1}, qx_i, x_{i+1}, ..., x_n)$ and we use $D^k_{k,x}$ instead of operator $\frac{\partial^k_q}{\partial_q x^k}$ for some simplification.

Finally, let Ω be the family of functions ϕ analytic in $\mathbb{D}$, and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Denote by $\mathcal{P}_q$ the family of functions p of the form $p(z) = 1 + p_1z + p_2z^2 + \cdots$, analytic in $\mathbb{D}$ and satisfy the condition

(1.2) $$\left| p(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}$$

where $q \in (0,1)$ is a fixed real number. Let $\mathcal{A}$ be the family of functions f, defined by $f(z) = z + a_2z^2 + a_3z^3 + ...$, that are analytic in $\mathbb{D}$ and satisfy the conditions $f(0) = 0, f'(0) = 1$. If f satisfies the condition

$$1 + \frac{1}{b}\left(qz\frac{D_q(D_qf(z))}{D_qf(z)}\right) \prec \frac{1+z}{1-qz},$$

where $b \in \mathbb{C}, b \neq 0$, then f is called $q-$ convex function of complex order, and the class of such functions are denoted by $\mathcal{C}_q(b)$. If f_1 and f_2 are analytic functions in $\mathbb{D}$, then we say that f_1 is subordinate to f_2, written as $f_1 \prec f_2$ if there exists a Schwarz function $\phi \in \Omega$ such that $f_1(z) = f_2(\phi(z)), z \in \mathbb{D}$. We also note that if f_2 univalent in $\mathbb{D}$, then $f_1 \prec f_2$ if and only if $f_1(0) = f_2(0)$ and $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$. This implies that $f_1(\mathbb{D}_r) \subset f_2(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z : |z| < r \ , 0 < r < 1\}$ (Subordination principle [6]).

We also need the following lemmas:

1.1. Lemma. Let ϕ be analytic in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1, z \in \mathbb{D}$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0 , then we have

$$z_0\phi'(z_0) = m\phi(z_0), \quad m \geq 1.$$

For more details of Jack's lemma, one may refer to [7].

1.2. Lemma. ([12]) If h is an element of $\mathcal{C}_q(b)$, then

$$F_2(|b|, Reb, q, r) \leq |D_q h(z)| \leq F_1(|b|, Reb, q, r)$$

where

$$F_1(|b|, Reb, q, r) = \left[(1 - qr)^{Reb+|b|}.(1 + qr)^{Reb-|b|}\right]^{-\frac{1-q^2}{2q^2 logq^{-1}}},$$

$$F_2(|b|, Reb, q, r) = \left[(1 - qr)^{Reb-|b|}.(1 + qr)^{Reb+|b|}\right]^{-\frac{1-q^2}{2q^2 logq^{-1}}}.$$

The aim of this paper is to investigate properties of the class of $q-$ harmonic functions for which analytic part is $q-$ convex functions of complex order defined by

$$\mathcal{S}_{\mathcal{HC}_q(b)} = \left\{ f = h + \overline{g} : w_q(z) = \frac{D_q g(z)}{D_q h(z)} = \frac{\overline{f_{\overline{z}}}}{f_z}, w_q(z) \prec b_1 \frac{1+z}{1-qz}, |w_q(z)| < 1, h \in \mathcal{C}_q(b) \right\},$$

where

$$D_q h(z) = \frac{h(z) - h(qz)}{(1-q)z} = f_z \quad \text{and} \quad D_q g(z) = \frac{g(z) - g(qz)}{(1-q)z} = \overline{f_{\overline{z}}}.$$

2. Main Results

In this section, we first assume that the function f is sense-preserving $q-$ harmonic function if and only if $w_q(z) = \frac{\overline{f_{\overline{z}}}}{f_z}$ is analytic. To show that

($\Rightarrow$) Let $f = h + \overline{g}$ be sense-preserving $q-$ harmonic function, then we will show that w_q is analytic. Since $h(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=1}^{\infty} b_n z^n$ are analytic functions, then we can write $q-$ derivatives of these functions as

$$D_q h(z) = 1 + \sum_{n=2}^{\infty} \frac{1-q^n}{1-q} a_n z^{n-1} \quad \text{and} \quad D_q g(z) = b_1 + \sum_{n=2}^{\infty} \frac{1-q^n}{1-q} b_n z^{n-1}.$$

We must note that when $q \to 1^-$, $D_q h(z)$ reduces to $h'(z)$ and $D_q g(z)$ reduces to $g'(z)$. The second $q-$ dilatation and $q-$ Jakobian are defined by

$$w_q(z) = \frac{D_q g(z)}{D_q h(z)} = \frac{\overline{f_{\overline{z}}}}{f_z},$$

$$J_{fq}(z) = |D_q h(z)|^2 - |D_q g(z)|^2.$$

Also, the total $q-$ differential of $f(\vec{x})$ can be written in the following manner,

$$d_q f(\vec{x}) = D_{q,x_1} d_q x_1 + D_{q,x_2} d_q x_2 + D_{q,x_3} d_q x_3 + \cdots + D_{q,x_n} d_q x_n.$$

Therefore the $q-$ differential can be written as

$$d_q f = D_{q,z} d_q z + D_{q,\overline{z}} d_q \overline{z}.$$

Consequently, f is locally univalent and sense-preserving if $|D_qh(z)| > |D_qg(z)|$ and sense-reversing if $|D_qg(z)| > |D_qh(z)|$. Note that $f_z \neq 0$ whenever $J_{fq}(z) > 0$. For sense-preserving f, one sees that

$$(|D_qh(z)| - |D_qg(z)|)|d_qz| \leq |d_qf| \leq (|D_qh(z)| + |D_qg(z)|)|d_qz|.$$

With aid of these definitions, let $f = h+\overline{g}$ be the solution of the non-linear elliptic partial differential equation

$$w_q(z)f_z = \overline{f}_{\overline{z}}$$

under the condition $|w_q(z)| < 1$ for all $z \in \mathbb{D}$. A non-constant complex -valued function f is $q-$ harmonic and orientation sense-preserving mapping on $\mathbb{D}$ if and only if f is the solution of the non-linear elliptic partial differential equation

$$w_q(z)f_z = \overline{f}_{\overline{z}} \tag{2.1}$$

where

$$f_z = D_qh(z) = \frac{h(z) - h(qz)}{(1-q)z} \quad \text{and} \quad \overline{f}_{\overline{z}} = D_qg(z) = \frac{g(z) - g(qz)}{(1-q)z}.$$

If we take the $q-$ derivative of equation (2.1) with respect to $\overline{z}$, we get

$$\overline{f}_{\overline{z}z} = f_{z\overline{z}}w_q(z) + f_z\frac{\partial w_q}{\partial \overline{z}}. \tag{2.2}$$

On the other hand, since f is $q-$ harmonic, then we have $\triangle f = 4\frac{\partial^2 f}{\partial z \partial \overline{z}} = 4f_{z\overline{z}} = 0$ and $\overline{f}_{\overline{z}z} = 0$. Therefore the equality (2.2) reduces to

$$f_z\frac{\partial w_q}{\partial \overline{z}} = 0 \tag{2.3}$$

and this shows that $\frac{\partial w_q}{\partial \overline{z}} = 0$, that is, w_q is analytic.

($\Leftarrow$) Conversely, if w_q is analytic in $\mathbb{D}$, then $\frac{\partial w_q}{\partial \overline{z}} = 0$. Therefore equality (2.2) reduces to

$$\overline{f}_{\overline{z}z} = f_{z\overline{z}}w_q(z). \tag{2.4}$$

On the other hand, using the definition of w_q, we have $|w_q(z)| < 1$. Thus, we get

$$1 - |w_q(z)| \neq 0. \tag{2.5}$$

Considering (2.4) and (2.5), we obtain

$$\overline{f}_{\overline{z}z} = f_{z\overline{z}}w_q(z) \Rightarrow f_{z\overline{z}} = 0 \tag{2.6}$$

and the equality (2.6) shows that f is $q-$ harmonic. This proves our assumption.

We now investigate properties of the class $\mathcal{S}_{\mathcal{HC}_q(b)}$. For Theorem 2.4, we need the following results. The first theorem is very important in order to obtain subordination of the analytic functions involving $q-$ difference operator.

2.1. Theorem. ([2]) p is an element of $\mathcal{P}_q$ if and only if $p(z) \prec \dfrac{1+z}{1-qz}$. This result is sharp for the functions $p(z) = \dfrac{1+\phi(z)}{1-q\phi(z)}$, where ϕ is a Schwarz function.

Proof. If p is an element of $\mathcal{P}_q$, then we have

$$\left| p(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q} \Leftrightarrow |p(z) - m| \leq m,$$

where $m = \dfrac{1}{1-q} > 1$. Therefore we can write

$$\left| \frac{1}{m}p(z) - 1 \right| \leq 1.$$

Thus the function $\psi(z)=\frac{1}{m}p(z)-1$ is analytic and has modulus at most 1 in $\mathbb{D}$, and so

$$\phi(z)=\frac{\psi(z)-\psi(0)}{1-\overline{\psi(0)}\psi(z)}=\frac{(\frac{1}{m}p(z)-1)-(\frac{1}{m}-1)}{1-(\frac{1}{m}-1)(\frac{1}{m}p(z)-1)}$$

satisfies the conditions of Schwarz lemma. This shows that we can write

$$p(z)=\frac{1+\phi(z)}{1-(1-\frac{1}{m})\phi(z)}\Rightarrow\ p(z)\prec\frac{1+z}{1-qz}.$$

Conversely, suppose that the function p is analytic in $\mathbb{D}$ and satisfies the condition $p(0)=1$ and

$$p(z)\prec\frac{1+z}{1-qz}\Rightarrow p(z)=\frac{1+\phi(z)}{1-(1-\frac{1}{m})\phi(z)}$$

$$p(z)-m=m\frac{\frac{1-m}{m}+\phi(z)}{1+\frac{1-m}{m}\phi(z)}.$$

On the other hand the function $\frac{\frac{1-m}{m}+\phi(z)}{1+\frac{1-m}{m}\phi(z)}$ maps the unit circle onto itself, then we have

$$|p(z)-m|\leq m\Leftrightarrow\left|\ p(z)-\frac{1}{1-q}\right|\ \leq\frac{1}{1-q}.$$

This shows that $p\in\mathcal{P}_q$. □

We must note that the linear tranformation $\frac{1+z}{1-qz}$ maps $|z|=r$ onto the disc with centre $C(r)=\frac{1+qr^2}{1-q^2r^2}$ and radius $\rho(r)=\frac{(1+q)r}{1-q^2r^2}$.

2.2. Lemma. If f is a function (real or complex valued) defined on $q-$ geometric set $\mathbb{B}$ with $|q|\neq 1$, then

$$D_q(logf(z))=\frac{D_qf(z)}{f(z)}.$$

Proof. Using the definition of $q-$ difference operator, then we have

$$D_q(logf(z))=\frac{logf(qz)-logf(z)}{qz-z}=\log\left(1+h\frac{D_qf(z)}{f(z)}\right)^{\frac{1}{h}}.$$

If we take limit for $h\to 0$, we obtain the desired result. □

2.3. Lemma. ($q-$Jack's Lemma) Let ϕ be analytic in $\mathbb{D}$ with $\phi(0)=0$. If $|\phi(z)|$ attains its maximum value on the circle $|z|=r$ at a point $z_0\in\mathbb{D}$, then we have

$$z_0D_q\phi(z_0)=m\phi(z_0)$$

where $m\geq 1$ is a real number.

Proof. Using the definition of $q-$ difference operator and Jack's lemma, then we can write

$$D_q\phi(z)=\frac{\phi(z)-\phi(qz)}{z-qz}=\frac{\phi(z)-\phi(z_0)}{z-z_0},\quad qz=z_0.$$

If we take limit for $z\to z_0$, we get

$$\lim_{z\to z_0}D_q\phi(z)=D_q\phi(z_0)=\lim_{z\to z_0}\frac{\phi(z)-\phi(z_0)}{z-z_0}=\phi'(z_0).$$

Therefore we have

$$z_0D_q\phi(z_0)=m\phi(z_0).$$

□

2.4. Theorem. If $f = h + \overline{g}$ is an element of $\mathcal{S}_{\mathcal{HC}_q(b)}$, then

$$\frac{g(z)}{h(z)} \prec b_1 \frac{1+z}{1-qz}. \tag{2.7}$$

Proof. Since $f = h + \overline{g} \in \mathcal{S}_{\mathcal{HC}_q(b)}$, then we have

$$\frac{D_q g(z)}{D_q h(z)} \prec b_1 \frac{1+z}{1-qz}.$$

The linear transformation $w = b_1 \frac{1+z}{1-qz}$ maps $|z| = r$ onto the disc with centre $C(r) = \left(\frac{\alpha_1(1+qr^2)}{1-q^2r^2}, \frac{\alpha_2(1+qr^2)}{1-q^2r^2}\right)$ and radius $\rho(r) = \frac{|b_1|(1+q)r}{1-q^2r^2}$, where $\alpha_1 = Reb_1$ and $\alpha_2 = Reb_2$. Thus using the subordination principle and the definition of the class $\mathcal{S}_{\mathcal{HC}_q(b)}$, we can write

$$w_q(\mathbb{D}_r) = \left\{ \frac{D_q g(z)}{D_q h(z)} : \left| \frac{D_q g(z)}{D_q h(z)} - \frac{b_1(1+qr^2)}{1-q^2r^2} \right| \le \frac{|b_1|(1+q)r}{1-q^2r^2}, q \in (0,1) \right\}. \tag{2.8}$$

In order to verify Schwarz function conditions, we define the function ϕ by

$$\frac{g(z)}{h(z)} = b_1 \frac{1+\phi(z)}{1-q\phi(z)}. \tag{2.9}$$

Note that ϕ is a well defined analytic function and

$$\left.\frac{g(z)}{h(z)}\right|_{z=0} = b_1 = b_1 \frac{1+\phi(0)}{1-q\phi(0)}.$$

This proves that $\phi(0) = 0$. We now need to show that $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. If we take $q-$ derivative of both sides of (2.9) and simplify, we get

$$\frac{D_q g(z)}{h(z)} - \frac{g(qz) D_q h(z)}{h(z) h(qz)} = b_1 \frac{D_q\phi(z) - q\phi(qz) D_q\phi(z) + q D_q\phi(z) + q\phi(qz) D_q\phi(z)}{(1-\phi(z))(1-\phi(qz))}.$$

Multiplying both sides of this equation by $h(z)/D_q h(z)$ and simplifying, we obtain

$$\frac{D_q g(z)}{D_q h(z)} = b_1 \left(\frac{1+\phi(qz)}{1-q\phi(qz)} + \frac{(1+q) z D_q\phi(z)}{(1-q\phi(z))(1-q\phi(qz))} . \frac{h(z)}{z D_q h(z)} \right). \tag{2.10}$$

Applying Lemma 2.2 in the equation (2.10), we can write the following form

$$\frac{D_q g(z)}{D_q h(z)} = b_1 \left(\frac{1+\phi(qz)}{1-q\phi(qz)} + \frac{(1+q) z D_q\phi(z)}{(1-q\phi(z))(1-q\phi(qz))} \big(1-q\phi(z)\big)^{b\frac{1-q^2}{q^2 logq^{-1}}} \right). \tag{2.11}$$

Assume to the contrary that there esists a point $z_0 \in \mathbb{D}_r$ such that $|\phi(z_0)| = 1$. In view of Lemma 2.3, equation (2.11) gives

$$\frac{D_q g(z_0)}{D_q h(z_0)} = b_1 \left(\frac{1+\phi(qz_0)}{1-q\phi(qz_0)} + \frac{(1+q) m\phi(z_0)}{(1-q\phi(z_0))(1-q\phi(qz_0))} \big(1-q\phi(z_0)\big)^{b\frac{1-q^2}{q^2 logq^{-1}}} \right) \notin w_q(\mathbb{D}_r).$$

This contradicts our assumption (2.8) and therefore $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. This completes the proof of our theorem. □

2.5. Corollary. If $f = h + \overline{g} \in \mathcal{S}_{\mathcal{HC}_q(b)}$, then we have

$$F_2(|b|, Reb, |b_1|, q, r) \le |D_q g(z)| \le F_1(|b|, Reb, |b_1|, q, r), \tag{2.12}$$

where

$$F_1(|b|, Reb, |b_1|, q, r) = \left[\big(1-qr\big)^{Reb+|b|} \big(1+qr\big)^{Reb-|b|} \right]^{-\frac{1-q^2}{2q^2 logq^{-1}}} \frac{|b_1|(1+r)}{1-qr},$$

$$F_2(|b|, Reb, |b_1|, q, r) = \left[\big(1-qr\big)^{Reb-|b|} \big(1+qr\big)^{Reb+|b|} \right]^{-\frac{1-q^2}{2q^2 logq^{-1}}} \frac{|b_1|(1-r)}{1+qr}.$$

Proof. Since $f = h + \overline{g}$ is an element of $\mathcal{S}_{\mathcal{HC}_q(b)}$, from Theorem 2.4 we write $\frac{D_q g(z)}{D_q h(z)} \prec b_1 \frac{1+z}{1-qz}$, where $h \in \mathcal{C}_q(b)$. Therefore we have

$$\left| \frac{D_q g(z)}{D_q h(z)} - \frac{b_1(1+qr^2)}{1-q^2r^2} \right| \leq \frac{|b_1|(1+q)r}{1-q^2r^2}.$$

This inequality yields

$$|D_q g(z)| \leq |D_q h(z)| \frac{|b_1|(1+r)}{1-qr}.$$

If we use Lemma 1.2, we get the right side of (2.12). Similarly, we can prove the other side of the inequality (2.12). □

2.6. Corollary. If $f = h + \overline{g} \in \mathcal{S}_{\mathcal{HC}_q(b)}$, then we have

$$f = h(z) + \overline{h(z) b_1 \frac{1+\phi(z)}{1-q\phi(z)}}, \tag{2.13}$$

where ϕ is a Schwarz function.

Proof. Using Theorem 2.4, then we can write

$$\frac{g(z)}{h(z)} \prec b_1 \frac{1+z}{1-qz} \Rightarrow \frac{g(z)}{h(z)} = b_1 \frac{1+\phi(z)}{1-q\phi(z)}.$$

Therefore we obtain

$$g(z) = h(z) b_1 \frac{1+\phi(z)}{1-q\phi(z)},$$

which gives (2.13). □

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