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Some properties of the total graph and regular graph of a commutative ring

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Abstract

Let R be a commutative ring with unity. The total graph of R, $T(\Gamma(R))$, is the simple graph with vertex set R and two distinct vertices are adjacent if their sum is a zero-divisor in R. Let $\mathrm{Reg}(\Gamma(\mathrm{R}))$ and $Z(\Gamma(R))$ be the subgraphs of $T(\Gamma(R))$ induced by the set of all regular elements and the set of zero-divisors in R, respectively. We determine when each of the graphs $T(\Gamma(R))$, $\mathrm{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ is locally connected, and when it is locally homogeneous. When each of $\mathrm{Reg}(\Gamma(R))$ and $Z(\Gamma(R))$ is regular and when it is Eulerian.

Keywords: Total graph of a commutative ring, Regular graph of a commutative ring, Locally connected, Locally homogeneous, Regular graph, Eulerian graph.

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1. Introduction

Throughout this paper R will be used to denote a commutative ring with unity $1 \neq 0$. Let $Z(R)$ be the set of all zero-divisors of R. The total graph of R is the simple graph with vertex set R where two distinct vertices x and y are adjacent if $x + y \in Z(R)$. This graph, denoted by $T(\Gamma(R))$, was introduced by Anderson and Badawi in [1], the authors gave full description for the case when $Z(R)$ is an ideal. On the other hand, they computed some graphical invariants such as the diameter and the girth of $T(\Gamma(R))$. Akbari and et al. [3], proved that if R is a finite ring, then a connected total graph is Hamiltonian. Maimani and et al. [12] investigated the genus of $T(\Gamma(R))$. The radius of $T(\Gamma(R))$ was computed in [13]. The domination number of $T(\Gamma(R))$ is determined independently in both [7] and [16]. For a finite commutative ring R, a characterization of

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Eulerian $T(\Gamma(R))$ is given in [16]. Minimum zero -sum k-flows for $T(\Gamma(R))$ are considered in [15]. The complement of $T(\Gamma(R))$ is investigated in [5]. Vertex-connectivity and edge-connectivity of $T(\Gamma(R))$, where R is a finite commutative ring, are discussed in [14]. Some properties of the regular graph $\text{Reg}(\Gamma(R))$ are studied in [4]. The line graph of $T(\Gamma(R))$ is investigated in [8]. Furthermore, the generalized total graph of R is defined in [2]. For a survey on the total graph of a commutative ring, the reader may refer to [6] or [10].

The following theorem gives full description of the graph $T(\Gamma(R))$ when $Z(R)$ is an ideal of R.

1.1. Theorem. [1] Let R be a ring such that $Z(R)$ is an ideal of R. Let $|Z(R)| = \lambda$, $|R/Z(R)| = \mu$.

(i) If $2 \in Z(R)$, then $\text{Reg}(\Gamma(R))$ is the union of $\mu - 1$ disjoint $K_\lambda{}'s$.
(ii) If $2 \in \text{Reg}(R)$, then $\text{Reg}(\Gamma(R))$ is the union of $(\mu - 1)/2$ disjoint $K_{\lambda,\lambda}{}'s$.
(iii) $Z(\Gamma(R))$ is the complete graph, K_λ.
(v) $\text{Reg}(\Gamma(R))$ is connected if and only if $R/Z(R) \cong \mathbb{Z}_2$ or $R/Z(R) \cong \mathbb{Z}_3$.

Several structural properties of $T(\Gamma(R))$, $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ will be considered. Section 2 addresses the problems "when is each of the graphs $T(\Gamma(R))$, $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ locally connected?". Section 3 answers the problem "when is each of the graphs $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ regular?". In Section 4, Eulerian $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ are characterized, where R is a finite commutative ring. Section 5 addresses the problem "when is each of the graphs $T(\Gamma(R))$, $\text{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ locally homogeneous?"

2. When *are* $T(\Gamma(R))$, $\text{Reg}(\Gamma(R))$, *and* $Z(\Gamma(R))$ Locally Connected?

Let G be a graph with vertex set and edge set $V(G)$ and $E(G)$ respectively. Let $v \in V(G)$, the open neighborhood, $N(v)$, of v is defined by $N(v) = \{u \in V(G) : uv \in E(G)\}$. The graph G is said to be locally connected if for all $v \in V(G)$, $N(v)$ induces a connected graph in G. Thus, if G is a union of complete graphs, then G is locally connected and if a graph G has a bipartite component, other than $K_{1,1}$, then it is not locally connected. This, together with Theorem 1.1 give the following theorem.

2.1. Theorem. Let R be a ring and $Z(R)$ be an ideal of R.

(i) $Z(\Gamma(R))$ is a locally connected graph.
(ii) $\text{Reg}(\Gamma(R))$ and $T(\Gamma(R))$ are locally connected graphs if and only if $2 \in Z(R)$, or R is an integral domain.

The next theorem considers the case when R is a product of two rings.

2.2. Theorem. Let R be a product of two rings R_1 and R_2. Then $T(\Gamma(R))$ is locally connected if and only if either R_1 or R_2 is not an integral domain.

Proof. First, we study the case when both R_1 and R_2 are integral domains. Suppose that $2 \in \text{Reg}(R)$ (i.e. $2 \in \text{Reg}(R_1)$ and $2 \in \text{Reg}(R_2)$), then $(-1, 1)$ and $(-1, -1)$ are only adjacent to each other in $N((1, 0))$ and hence there is no path between $(-1, 0)$ and $(-1, 1)$ in $N((1, 0))$. If $2 \in Z(R_1)$ and $2 \in \text{Reg}(R_2)$, then $(0, -1)$ is an isolated vertex in $N((1, 1))$. And if $2 \in Z(R_1)$ and $2 \in Z(R_2)$, then there is no path joining $(1, 0)$ and $(0, 1)$ in $N((1, 1))$. So, $T(\Gamma(R))$ is not locally connected.

Now, we may assume that either R_1 or R_2 is not an integral domain. Let $N_i(u)$, denotes the open neighborhood of u in $T(\Gamma(R_i))$. Let $(a, b) \in R$ and $(x, y), (z, w) \in N((a, b))$. If (x, y) and (z, w) are non-adjacent in $N((a, b))$, then we have four cases:

Case 1: $x \in N_1(a)$ and $w \in N_2(b)$ or ($z \in N_1(a)$ and $y \in N_2(b)$).

Assume that $x \in N_1(a)$ and $w \in N_2(b)$. Then $(x, y) - (a, w) - (x, b) - (z, w)$ is a path in $N((a, b))$.

Case 2: $x, z \in N_1(a)$ or $(y, w \in N_2(b))$.

Assume that $x, z \in N_1(a)$. Then we have three cases.

Case 2.1: $2 \in Z(R_1)$.

Choose $t \in R_2 \backslash \{b\}$, then $(a, t) \in N((a, b))$. So, $(x, y) - (a, t) - (z, w)$ is a path in $N((a, b))$.

Case 2.2: $2 \in \mathrm{Reg}(R_1)$ and $2 \in Z(R_2)$.

If R_1 is not an integral domain, then there exist $t, s \in Z(R_1)$ such that $-x+t \neq a$ and $-z+r \neq a$. Then if $-x+t \neq -z+r$, the path $(x, y) - (-x+t, b) - (-z+r, b) - (z, w)$ is obtained. Otherwise, $(x, y) - (-x+t, b) - (z, w)$ is a path in $N((a, b))$. Now, if R_2 is not an integral domain, then there exists $r \in Z(R_2)$ such that $-b+r \neq b$. So, $(x, y) - (a, -b+r) - (z, w)$ is a path in $N((a, b))$.

Case 2.3: $2 \in \mathrm{Reg}(R_2)$.

If R_2 is not an integral domain, then there exists $r \in Z(R_2)$ such that $-b+r \neq b$. So, $(x, y) - (a, -b+r) - (z, w)$ is a path in $N((a, b))$. If R_1 is not an integral domain, then there exist $t, s \in Z(R_1)$ such that $-x+t \neq a$ and $-z+r \neq a$. So, when $-x+t \neq -z+r$, we get the path $(x, y) - (-x+t, -b) - (x, b) - (-z+r, -b) - (z, w)$ in $N((a, b))$. Otherwise, we get the path $(x, y) - (-x+t, -b) - (z, w)$.

Case 3: $x \in N_1(a)$, $z \in R_1 - N_1(a)$ and $w = b$ or ($x = a$, $y \in R_2 - N_2(b)$ and $w \in N_2(b)$).

Assume that $x \in N_1(a)$, $z \in R_1 - N_1(a)$ and $w = b$. Then $2b \in Z(R_2)$. So, R_1 is not an integral domain, gives $-x+t \neq a$ for some $t \in Z(R_1)$. Therefore, $(x, y) - (-x+t, b) - (z, w)$ is a path in $N((a, b))$. While R_2 is not an integral domain, implies that $-b+r \neq b$ for some $r \in Z(R_2)$. So, $(x, y) - (a, -b+r) - (z, w)$ is a path in $N((a, b))$.

Case 4: $x = a$, $w = b$, $2a \in Z(R_1)$, and $2b \in Z(R_2)$ or ($y = b$, $x = a$, $2a \in Z(R_1)$ and $2b \in Z(R_2)$).

Assume that $x = a$, $w = b$, $2a \in Z(R_1)$, and $2b \in Z(R_2)$. Then R_1 is not an integral domain, implies that $-x+t \neq a$ for some $t \in Z(R_1)$ and R_2 is not an integral domain implies that $-b+r \neq b$ for some $r \in Z(R_2)$. Thus, $(x, y) - (-x+t, b) - (z, w)$ or $(x, y) - (a, -b+r) - (z, w)$ is a path in $N((a, b))$. □

If R is a local ring, then $Z(R)$ is an ideal and hence $Z(\Gamma(R))$ is a complete graph which is obviously locally connected. When R is a product of two rings, we have the following theorem.

2.3. Theorem. Let R be a product of two rings R_1 and R_2. Then $Z(\Gamma(R))$ is locally connected if and only if either R_1 or R_2 is not an integral domain.

Proof. Observe that if R is a product of two integral domains, then there is no path joining $(1, 0)$ and $(0, 1)$ in $N((0, 0))$. So $Z(\Gamma(R))$ is not locally connected. Assume that either R_1 or R_2 is not an integral domain. Since $(0, 0) \in N((a, b))$ for any non-zero zero-divisors (a, b), we have $(x, y) - (0, 0) - (z, w)$ is a path joining (x, y) and (z, w) in $N((a, b))$. Thus $N((a, b))$ is locally connected for all $(a, b) \in Z(R) - \{0\}$. So it remains to study connectivity of the graph induced by $N((0, 0))$. Assume that (x, y) and (z, w) are two non-adjacent vertices in $N((0, 0))$, then $x \in Z(R_1) \backslash \{0\}$ implies that $(x, y) - (-x, -w) - (z, w)$ is a path in $N((0, 0))$ and $y \in Z(R_2) \backslash \{0\}$ implies that $(x, y) - (-z, -y) - (z, w)$ is a path in $N((0, 0))$. □

Next, we will investigate when $\mathrm{Reg}(\Gamma(R))$ is locally connected. If R is a local ring, then $\mathrm{Reg}(\Gamma(R))$ is locally connected if R is an integral domain or $2 \in Z(R)$. If R is a product of two rings, then we have the following.

2.4. Theorem. Let R be a product of two rings and $2 \in \mathrm{Reg}(R)$. Then $\mathrm{Reg}(\Gamma(R))$ is locally connected.

Proof. Assume that $(a,b) \in \mathrm{Reg}(R)$ and (x,y), (z,w) are two non-adjacent vertices in $N((a,b))$. Then $x \in N(a)$ gives the path $(x,y)-(a,-b)-(-a,-w)-(z,w)$ in $N((a,b))$, and $y \in N(b)$ gives the path $(x,y)-(-a,b)-(-z,-b)-(z,w)$ in $N((a,b))$. □

Let $R = R_1 \times R_2$, then it is easy to see that if $|\mathrm{Reg}(R_1)| = 1$, then $2 \in Z(R)$ and $\mathrm{Reg}(\Gamma(R))$ is a complete graph and hence it is locally connected.

A Boolean ring provides an example of a ring R with only one regular element, this is due to the fact that for all $r \in R$, $r = r^2$. So, we get the following.

2.5. Theorem. If R is a Boolean ring or R is a product of rings with at least one Boolean factor, then $\mathrm{Reg}(\Gamma(R))$ is a complete graph.

At this point it makes sense to require that $|\mathrm{Reg}(R_i)| \geq 2$, for all i.

2.6. Theorem. Let R be a product of two local rings R_1 and R_2 such that $2 \in Z(R)$ and $|\mathrm{Reg}(R_i)| \geq 2$ for $i = 1, 2$. Then $\mathrm{Reg}(\Gamma(R))$ is locally connected if and only if R_1 or R_2 is not an integral domain.

Proof. Suppose that $R = R_1 \times R_2$ where R_1 and R_2 are integral domains, $2 \in Z(R)$ and $|\mathrm{Reg}(R_i)| \geq 2$ for $i = 1, 2$. Choose $(t,s) \in \mathrm{Reg}(R)\backslash\{(1,1)\}$, then $2 \in Z(R_1)$ and $2 \in Z(R_2)$ imply that $(1,s)$ and $(t,1)$ are two non-adjacent vertices in $\mathrm{Reg}(\Gamma(R))$ and there is no path joining them in $N((1,1))$. If $2 \in Z(R_1)$ and $2 \in \mathrm{Reg}(R_2)$, then $(1,-1)$ and $(t,-1)$,where $t \in \mathrm{Reg}(R_1)\backslash\{1\}$, are non-adjacent vertices in $N((1,1))$, with no path joining them in $N((1,1))$. So $\mathrm{Reg}(\Gamma(R))$ is not locally connected.

Conversely, let $R = R_1 \times R_2$ where R_1 and R_2 are two local rings such that $2 \in Z(R)$ and $|\mathrm{Reg}(R_i)| \geq 2$, for $i = 1, 2$. Without loss of generality, assume that $2 \in Z(R_1)$. Let $(a,b) \in \mathrm{Reg}(R)$ and $(x,y), (z,w)$ be two non-adjacent vertices in $N((a,b))$. If R_1 is not an integral domain, then there exists $t \in Z(R_1)$ such that $t + a \neq a$. Since $Z(R_1)$ is an ideal of R, $t + a \in \mathrm{Reg}(R_1)$. Therefore, $(x,y)-(a+t,-y)-(a+t,-w)-(z,w)$ is a path in $N((a,b))$. And if R_2 is not an integral domain, then $t - y \neq b$ and $s - w \neq b$ for some $t, s \in Z(R_2)$, so $(x,y)-(a,t-y)-(a,s-w)-(z,w)$ is a path in $N((a,b))$ when $t - y \neq s - w$, otherwise, we have the path $(x,y)-(a,t-y)-(z,w)$ in $N((a,b))$. □

2.7. Theorem. If $R = \Pi_{i=1}^{n} R_i, n \geq 3$, then $\mathrm{Reg}(\Gamma(R))$ is locally connected.

Proof. Let $a = (a_i) \in \mathrm{Reg}(R)$ and $u = (u_i)$ and $v = (v_i)$ be two non-adjacent vertices in $N(a)$. Since $u \in N(a)$, $a_i + u_i \in Z(R_i)$, for some i, say for $i = 1$. Define $w = (w_i)$ such that $w_1 = u_1$, $w_2 = -u_2$, $w_3 = -v_3$ and $w_i = 1$ for all $i \geq 4$, then $u - w - v$ is a path in $N(a)$. □

An Artinian ring is a ring that satisfies the descending chain condition on ideals. An Artinian ring R can be written uniquely (up to isomorphism) as a finite direct product of Artinian local rings. Since $Z(R)$ is an ideal of R when R is local, we may conclude the following.

2.8. Theorem. Let R be an Artinian ring, then

(i) $T(\Gamma(R))$ is not locally connected if and only if R is a local ring satisfying $2 \in \mathrm{Reg}(R)$ and R is not an integral domain or R is a product of integral domains.

(ii) $Z(\Gamma(R))$ is not locally connected if and only if R is a product of two integral domains.

(iii) $\mathrm{Reg}(\Gamma(R))$ is not locally connected if and only if R is a local ring satisfying $2 \in \mathrm{Reg}(R)$ and R is not an integral domain or $R = R_1 \times R_2$, $2 \in Z(R)$, and $|\mathrm{Reg}(R_i)| \geq 2$ and R_i is an integral domain for $i = 1, 2$.

2.9. Corollary. (i) $T(\Gamma(\mathbb{Z}_n))$ is not locally connected if and only if $n = t^m$, where t is an odd prime and $m \geq 2$ or $n = t_1t_2$, where t_1, and t_2 are distinct primes.

(ii) $Z(\Gamma(\mathbb{Z}_n))$ is not locally connected if and only if $n = t_1t_2$ where t_1 and , t_2 are two distinct primes.

(iii) $\mathrm{Reg}(\Gamma(\mathbb{Z}_n))$ is not locally connected if and only if $n = t^m$, where t is an odd prime and $m \geq 2$.

3. When *are* $T(\Gamma(R))$, $\mathrm{Reg}(\Gamma(R))$, *and* $Z(\Gamma(R))$ regular?

In this section, we study regularity of the graphs $T(\Gamma(R))$, $\mathrm{Reg}(\Gamma(R))$, and $Z(\Gamma(R))$ for any ring R. Maimani et al. [12] proved that in $T(\Gamma(R))$, $\deg(u) = |Z(R)| - 1$ if $2 \in Z(R)$ or $u \in Z(R)$, and $\deg(u) = |Z(R)|$ otherwise. So, $T(\Gamma(R))$ is regular graph only if $2 \in Z(R)$ or R is an infinite non integral domain ring.

Now, we examine regularity of $\mathrm{Reg}(\Gamma(R))$. Clearly, if $Z(R)$ is an ideal, then $\mathrm{Reg}(\Gamma(R))$ is regular of degree $|Z(R)| - 1$, when $2 \in Z(R)$ and it is regular graph of degree $|Z(R)|$ when $2 \in \mathrm{Reg}(R)$.

The following theorems address the case when R is a product of two rings.

3.1. Theorem. Let R be a product of two rings R_1 and R_2 where R_1 and R_2 are two rings such that $|\mathrm{Reg}(R_1)| = n_1$ and $|\mathrm{Reg}(R_2)| = n_2$. Let $(u_1, u_2) \in \mathrm{Reg}(R)$ and $\deg_1(u_1) = r_1$ and $\deg_2(u_2) = r_2$, where $\deg_i(u_i)$ is the degree of u_i in $\mathrm{Reg}(\Gamma(R_i))$. Then the degree of the vertex (u_1, u_2) in $\mathrm{Reg}(\Gamma(R))$ is given by,

$$\deg((u_1,u_2)) = \begin{cases} n_2r_1 + n_1r_2 - r_1r_2, & \text{if } 2 \in \mathrm{Reg}(R); \\ n_1r_2 + n_2r_1 + (n_1+n_2) - (r_1+r_2) - r_1r_2 - 2, & \text{if } 2 \in Z(R_1) \text{ and } 2 \in Z(R_2); \\ n_1r_2 + n_2r_1 - r_2 + n_2 - r_1r_2 - 1, & \text{if } 2 \in Z(R_1) \text{ and } 2 \in \mathrm{Reg}(R_2)\ . \end{cases}$$

Proof. Note that if $2 \in \mathrm{Reg}(R)$, then $N((u_1, u_2)) = \{(a, b) \in \mathrm{Reg}(R) : a \in N(u_1) \text{ or } b \in N(u_2)\}$. So, $|N((u_1, u_2))| = r_1n_2 + n_1r_2 - r_1r_2$. If $2 \in Z(R_1)$ and $2 \in Z(R_2)$, then $N((u_1, u_2)) = \{(a, b) \in \mathrm{Reg}(R)\backslash\{(u_1, u_2)\} : a \in N(u_1) \cup \{u_1\} \text{ or } b \in N(u_2) \cup \{u_2\}\}$. So, $|N((u_1, u_2))| = (r_2+1)n_1+(r_1+1)n_2-(r_1+1)(r_2+1)-1$. If $2 \in Z(R_1)$ and $2 \in \mathrm{Reg}(R_2)$, then $N((u_1, u_2)) = \{(a, t) \in \mathrm{Reg}(R)\backslash\{(u_1, u_2)\} : a \in N(u_1) \cup \{u_1\} \text{ or } b \in N(u_2)\}$. So, $|N((u_1, u_2))| = (r_1 + 1)n_2 + n_1r_2 - (r_1 + 1)r_2 - 1$. □

Since for any local ring R the graph $\mathrm{Reg}(\Gamma(R))$ is regular and every finite ring is a product of local rings by using Theorem 3.1 we get the following.

3.2. Theorem. If R is a finite ring, then $\mathrm{Reg}(\Gamma(R))$ is a regular graph.

The following two lemmas will be useful in the subsequent work.

3.3. Lemma. Let R be a finite ring. Then

(i) if $|R|$ is even, then $|Z(R)|$ and $|\mathrm{Reg}(R)|$ are both odd when R is a field or a product of fields of even orders, and they are both even otherwise.

(ii) if $|R|$ is odd, then $|\mathrm{Reg}(R)|$ is even and $|Z(R)|$ is odd.

If R is a ring, then $2 \in Z(R)$ if and only if $|r| = 2$ in $(R, +)$, for some $r \in R\backslash\{0\}$. If R is a finite ring, then $2 \in Z(R)$ if and only if $|R|$ is even.

Using Theorem 3.1 and the same notation, it is easy to conclude the following.

3.4. Lemma. Let R be a product of two local rings R_1 and R_2 and $(u_1, u_2) \in \mathrm{Reg}(R)$. Then the degree of the vertex (u_1, u_2) in $\mathrm{Reg}(\Gamma(R))$ is even if and only if $|\mathrm{Reg}(R_1)|$, $|\mathrm{Reg}(R_2)|$ are both odd and $\deg_1(u_1)$, $\deg_2(u_2)$ are both even.

Now, we are ready to prove the following theorem.

3.5. Theorem. Let R be a finite ring. Then $\mathrm{Reg}(\Gamma(R))$ is a regular graph of even degree if and only if R is a field or a product of two or more fields of even orders.

Proof. Let $R = \Pi_{i=1}^{n} R_i, n \geq 2$, where R_i is a finite local ring for all i. First, we will study the three special cases: (i) $|R|$ is odd or (ii) R_i is a field of even order for all i, or (iii) R_i is not a field of even order for all i. Using induction in each case, Theorem 3.1 and the above two lemmas, we get $\mathrm{Reg}(\Gamma(R))$ is a regular graph of odd order and even degree when R is a product of fields of even orders, and it is a regular graph of even order and odd degree otherwise. Now, we move to the case where R is a product of fields of even orders and local rings that are not fields of even orders, note that $R \cong S \times T$, where S is the product of all fields $R_i's$ and T is the product of all not fields local rings $R_i's$. Then $\mathrm{Reg}(\Gamma(R))$ is a regular graph of even order and odd degree. Finally if $|R| = 2^m t$, where $t > 1$ is odd integer, we may write $R \cong S \times T$, where $|S| = 2^m$, and $|T| = t$. Therefore, $\mathrm{Reg}(\Gamma(R))$ is a regular graph of even order and odd degree.

□

Note that $Z(\Gamma(R))$ is a regular graph, of degree $|Z(R)| - 1$, when R is a local ring since $Z(\Gamma(R)) \cong K_{|Z(R)|}$. However, $Z(\Gamma(R))$ is not regular if R is a product of two rings, since $N((0,0)) = Z(R)/\{(0,0)\}$ and $N((0,1)) \subseteq Z(R)/\{(1,0),(0,1)\}$. So, we get the following.

3.6. Theorem. Let R be a finite ring, then

(i) $Z(\Gamma(R))$ is a regular graph if and only if R is a local ring
(ii) $Z(\Gamma(R))$ is a regular graph of even degree if and only if R is a field or R is a local ring of odd order.

3.7. Corollary. (i) $T(\Gamma(\mathbb{Z}_n))$, and $\mathrm{Reg}(\Gamma(\mathbb{Z}_n))$ are regular graphs of even degrees if and only if $n = 2$.
(ii) $Z(\Gamma(\mathbb{Z}_n))$ is regular graph of even degree if and only if $n = 2$ or $n = p^m$, p is odd prime and $m \geq 1$.

4. When *are* $\mathrm{Reg}(\Gamma(R))$ *and* $Z(\Gamma(R))$ Eulerian?

A graph is said to be Eulerian if it has a closed trail containing all of its edges. Or equivalently, a connected graph G is Eulerian if and only if the degree of each vertex in $V(G)$ is even.

Clearly, if R is a finite local ring, then $T(\Gamma(R))$ is non Eulerian, and $\mathrm{Reg}(\Gamma(R))$ is Eulerian if and only if $R \cong \mathbb{Z}_2$, while $Z(\Gamma(R))$ is Eulerian if and only if $|R|$ is odd or R is a field.

The next theorem, which is due to Shekarriz et al. [16], characterizes Eulerian $T(\Gamma(R))$ when R is a finite ring.

4.1. Theorem. Let R be a finite ring, then the graph $T(\Gamma(R))$ is Eulerian if and only if R is a product of two or more fields of even orders.

Let R be a direct product of two rings. Then $\mathrm{Reg}(\Gamma(R))$ is connected, since for any two vertices (a,b) and (x,y) in $\mathrm{Reg}(\Gamma(R))$, $(a,b) - (-a,-y) - (x,y)$ is a path joining the two non-adjacent vertices, [1]. So, for any finite non local ring R, $\mathrm{Reg}(\Gamma(R))$ is connected.

Using Theorem 3.5, the following theorem is obtained.

4.2. Theorem. Let R be a finite ring. Then the graph $\mathrm{Reg}(\Gamma(R))$ is Eulerian if and only if $R \cong \mathbb{Z}_2$ or R is a product of two or more fields of even orders.

Finally, we investigate when $Z(\Gamma(R))$ is Eulerian.

4.3. Theorem. Let R be a finite ring. Then
$Z(\Gamma(R))$ is Eulerian if and only if R is a field or $|R|$ is odd.

Proof. Clearly, if R is is a local ring, then $Z(\Gamma(R))$ is Eulerian if and only if R is a field or $|R|$ is odd. Suppose that $R = \prod_{i=1}^{n} R_i$, where R_i is a finite local ring for all i. Then we have two cases.

Case 1: $|R|$ is even. If $Z(\Gamma(R))$ is Eulerian, then $\deg((0,0,...,0)) = |Z(R)| - 1$ is even. From Lemma 3.3, R is a product of fields of even orders. So $\deg((1,0,0,...,0) = |Z(R)| - 1 - \prod_{i=2}^{n} |Reg(R_i)|$ is odd, a contradiction.

Case 2: $|R|$ is odd. Then $|R_i|$ is odd for all i. Take $w = (w_i) \in Z(R)$. Define $T = \{t \in \{1,2,..,n\} : w_t \in Z(R_t)\}$ and $J = \{1,2,..,n\} \backslash T$. Now, to compute the degree of w in $Z(\Gamma(R))$, note that for any finite local ring of odd order S, the sum of any two elements is a zero-divisor if and only if both elements are zero-divisors or one of them belongs to the coset $x + Z(S)$ and the other belongs to the coset $-x + Z(S)$, where $x \in \mathrm{Reg}(S)$. So, the vertex $a = (a_i) \in Z(R)\backslash\{w\}$ is non-adjacent to w when $a_i \in \mathrm{Reg}(R_i)$ for all $i \in T$, and $a_i \in R_i \backslash - w_i + Z(R_i))$ for all $i \in J$ and $a_i \in Z(R_i)$ for some $i \in J$. Since $|-w_i + Z(R_i)| = |Z(R_i)|$ for all i, we have $\deg(w) = (|Z(R)| - 1) - (\prod_{i \in T} |\mathrm{Reg}(R_i)|(\prod_{i \in J} |\mathrm{Reg}(R_i)| - \prod_{i \in J}(|\mathrm{Reg}(R_i)| - |Z(R_i)|))$. Since $|Z(R)|$ is odd and $|\mathrm{Reg}(R_i)|$ is even for all i, we get $\deg(w)$ is even. Moreover $Z(\Gamma(R))$ is connected graph since 0 adjacent s to all other vertices in $Z(\Gamma(R))$. Thus $Z(\Gamma(R))$ is Eulerian.

□

4.4. Corollary. (i) $T(\Gamma(\mathbb{Z}_n))$ is never Eulerian.
(ii) $\mathrm{Reg}(\Gamma(\mathbb{Z}_n))$ is Eulerian if and only if $n = 2$.
(iii) $Z(\Gamma(\mathbb{Z}_n))$ is Eulerian if and only if $n = 2$ or n is an odd number.

5. When *are* $T(\Gamma(R))$, *$Reg(\Gamma(R))$ and* $Z(\Gamma(R))$ locally homogeneous?

A graph G is called locally homogeneous if the graph induced by the neighborhoods of any two vertices are isomorphic. Let H be a given graph. A graph G is called locally H if for each vertex $v \in V(G)$, the subgraph induced by the open neighborhood of v, $N(v)$, is isomorphic to H. Locally H graphs are also called locally homogeneous [17]. Graphs associated with algebraic structures are known to exhibit some symmetrical properties, see for example [17]. In this section, we investigate homogeneity in the total graphs associated with rings.

Let R be a local ring with $|Z(R)| = \alpha$. Then by Theorem 1.1, $T(\Gamma(R))$ is locally H if and only if $2 \in Z(R)$. In this case, $H = K_{\alpha-1}$. So, if R is a finite local ring, then $T(\Gamma(R))$ is locally H if and only if $|R|$ is even, $\mathrm{Reg}(\Gamma(\mathrm{R}))$ is either locally $K_{\alpha-1}$ or $\overline{K_\alpha}$, and $Z(\Gamma(R))$ is locally $K_{\alpha-1}$. The next theorem treats the case for any finite ring R.

5.1. Theorem. Let R be a finite ring. Then

(i) Let x and y be two distinct vertices in $T(\Gamma(R))$. Then, the subgraph of $T(\Gamma(R))$ induced by $N(x)$ is isomorphic to the subgraph induced by $N(y)$ if and only if $|R|$ is even.
(ii) Let x and y be two distinct vertices in $\mathrm{Reg}(\Gamma(R))$. Then, the subgraph of $\mathrm{Reg}(\Gamma(R))$ induced by $N(x)$ is isomorphic to the subgraph induced by $N(y)$.
(iii) $Z(\Gamma(R))$ is locally H if and only if R is a local ring. In this case, $H = K_{|Z(R)|-1}$.

Proof. (1) If $|R|$ is odd, then $2 \notin Z(R)$, and so, $T(\Gamma(R)$ is not regular, hence we may assume that $|R|$ is even. Let $R = \Pi_{i=1}^{n} R_i$. Where each R_i is a local ring. Without loss of generality, we may assume that $2 \in Z(R_1)$. Obviously, for $n = 1$, the result holds. If $S = \Pi_{i=2}^{n} R_i$, then $R = R_1 \times S$. We will prove that the neighborhoods of any two distinct vertices in $T(\Gamma(R))$ are isomorphic. Let (a, b) be an arbitrary element in R. Let $N_1 = \{a\} \times (S/\{b\})$, $N_2 = \{(x, y) \in R : x \neq a, x + a \in Z(R_1)\}$ and $N_3 = \{(x, y) \in R : x + a \in \mathrm{Reg}(R_1)$, and $y + b \in Z(S)\}$. Note that N_1, N_2 and N_3 form a partition for $N((a,b))$. Thus $N((a,b)) = N_1 \cup N_2 \cup N_3$. N_1 induces a complete graph of order $|S|-1$. For each fixed vertex $r \in S$, let $N_{2_r} = \{(x,r) \in R : x \neq a, x+a \in Z(R_1)\}$. Each set N_{2_r} induces a copy of the graph induced by $N(a)$ in the graph $T(\Gamma(R_1)$ which a complete graph. Besides, for each pair of distinct vertices in $r, s \in S$, each vertex (x_1, r) in N_{2_r} is adjacent to each vertex (x_2, s) in N_{2_s}, since $x_1 + x_2 + 2a \in Z(R_1)$ implies that $x_1 + x_2 \in Z(R_1)$. Each vertex in N_1 is adjacent to each vertex in N_2.

Now, we claim that N_3 induces a complete graph. Let $(x_1,y_1),(x_2,y_2) \in N_3$ then $a + x_1 \in \mathrm{Reg}(R_1)$ and $a + x_2 \in \mathrm{Reg}(R_1)$. we study two cases:

Case 1: $a \in Z(R_1)$, then both x_1 and x_2 belong to $\mathrm{Reg}(R_1)$. By Theorem 2.9 of [1], $x_1+x_2 \in Z(R_1)$ or $x_1-x_2 \in Z(R_1)$. Assume that $x_1 - x_2 \in Z(R_1)$, say $x_1 - x_2 = z$ and $x_1 + x_2 = r$, for some $r \in \mathrm{Reg}(R_1)$ and some $z \in Z(R_1)$. This implies that $2x_1 - z = r$ which is a contradiction, thus $x_1 + x_2 \in Z(R_1)$ and hence (x_1,y_1) is adjacent to (x_2,y_2).

Case 2. $a \in \mathrm{Reg}(R_1)$, we have $x_1 + a = r_1$ and $x_2 + a = r_2$, where $r_1, r_2 \in \mathrm{Reg}(R_1)$. Either $r_1+r_2 \in Z(R_1)$ or $r_1-r_2 \in Z(R_1)$. If $r_1+r_2 \in Z(R_1)$, then $x_1+x_2+2a \in Z(R_1)$, and hence $x_1 + x_2 \in Z(R_1)$. If $r_1 - r_2 \in Z(R_1)$,then $x_1 - x_2 \in Z(R_1)$, if $x_1 \in \mathrm{Reg}(R_1)$, then $x_1 - a = z_1$, for some $z_1 \in Z(R_1)$. But $x_1 + a = r_1$, where $r_1 \in \mathrm{Reg}(R_1)$. So, $2x_1 = z_1 + r_1$ which is a contradiction. Similarly, $x_2 \in Z(R_1)$, and hence, (x_1,y_1) is adjacent to (x_2,y_2) .

If a vertex $(x_1,y_1) \in N_2$, is adjacent to a vertex $(x_2,y_2) \in N_3$, then, $x_1+x_2 \in \mathrm{Reg}(R_1)$, To see this write $x_1+a = z$ and $x_2+a = r$, where $z \in Z(R_1)$ and $r \in \mathrm{Reg}(S)$, this implies that $x_1 + x_2 + (2a - z) = r$, and so, $x_1 + x_2 \in \mathrm{Reg}(R_1)$. We may write $Z(S) = \bigcup_{i=1}^{m} I_i$, where each I_i is a maximal ideal of S. Suppose that $b \in b_i + I_i$, if $a_i + b_i \in I_i$, then $y_2 \in \bigcup_{i=1}^{m} a_i + I_i$. Let G be the bipartite subgraph of $T(\Gamma(R))$ with partite sets N_2 and N_3 where two vertices $(x_1,y_1) \in N_2$ and $(x_2,y_2) \in N_3$ are adjacent if $y_1 + y_2 \in Z(S)$. Similarly, $N_1 \cup N_3$ with edges joining N_1 to N_3 form another bipartite graph. Finally, since this description of $N((a,b))$ does not depend on the choice of (a,b), we conclude that the neighborhood of any two vertices in $T(\Gamma(R))$ are isomorphic .

(ii) Considering Theorem 3.2, $\mathrm{Reg}(\Gamma(R))$ is regular. Let $R = \Pi_{i=1}^{n} R_i$. For $i = 1, 2, \ldots, n$, let G_i be the spanning subgraph of $\mathrm{Reg}(\Gamma(R))$ where two vertices $(x_1, x_2, ...x_n)$ and $(y_1, y_2, ...y_n)$ are adjacent in G_i if $x_i + y_i \in Z(R_i)$. The graph $\mathrm{Reg}(\Gamma(R))$ is the overlay of the layers G_i, $i = 1, 2, \ldots, n$. Each layer is a union of complete graphs or a union of complete bipartite graphs. Let x and y be two distinct vertices in $\mathrm{Reg}(\Gamma(R))$. Let $N_i(x)$ and $N_i(y)$ be the open neighborhoods of x and y respectively, in the graph G_i. Then $N(x) = \cup_{i=1}^{i=n} N_i(x)$, and $N(y) = \cup_{i=1}^{i=n} N_i(y)$. So, $N(x)$ is the overlay of the layers induced by $N_i(x)$, $i = 1, 2, \ldots, n$. Similar result holds for $N(y)$. Observe that for each $i = 1, 2, \ldots, n$, $N_i(x)$ and $N_i(y)$ induce isomorphic subgraphs of the graph G_i, consequently, $N(x)$ and $N(y)$ induce isomorphic subgraphs of the graph $\mathrm{Reg}(\Gamma(R))$.

(iii) Direct result of Theorem 3.6 part (1) and the argument before Theorem 6.1. $\square$

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