

## Terminal value problems with causal operators

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### Abstract

The well-known techniques of monotone iterative have been investigated and expanded for the causal terminal value problem (CTVP). This method construct the monotone sequences of the solutions of linear CTVPs by using the upper and lower solutions. Moreover, these sequence of functions are uniformly and monotonically converge to the extremal solutions of the CTVP.

**Keywords:** causal operator, causal terminal value problem, monotone iterative technique, extremal solutions, theoretical approximation method.

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### 1. Introduction

The monotone iterative technique have been investigated and expanded for the causal terminal value problem. This method construct the monotone sequence through the solutions of linear causal terminal value problems for which these monotone sequences by using the upper and lower solutions. These sequence of functions converge uniformly and monotonically to the extremal solutions of the causal terminal value problem. In this work, we have expanded and refined the monotone iterative technique [9] for the causal terminal value problem that implies monotone sequence through the solutions of linear causal terminal value problem for which these sequence of functions converge uniformly to the extremal solutions of the causal terminal value problem.

This constructive method offers a way of proving existence of maximal and minimal solutions in addition to obtaining solutions in closed sectors as in [11]. It has been shown recently that causal differential equations [11] provide an excellent models for the real world problems [6] and its real time applications in a variety of disciplines. This is the

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main advantage of causal differential equations [11] in comparison with traditional models [10]. There has been a growing interest in this new area to study the concept of causal systems in the qualitative behaviors [3, 11, 22, 24].

The monotone iterative technique [4, 9, 14, 16, 17, 20-23] coupled with the method of upper and lower solutions offers monotone sequences that converge uniformly and monotonically to the extremal solutions of the given nonlinear causal differential equations. Since each member of these sequences are the solution of a certain linear causal differential equations which can be explicitly computed, the advantage and the importance of this technique needs no special emphasis. Moreover, this method can successfully be employed to generate two sides pointwise bounds on solutions of initial value problems of causal differential equations from which qualitative and quantitative behavior of the solution can be investigated explicitly.

The study of differential equations [1, 9, 10, 21] with causal operators [2-5, 7, 11-13, 19, 22-24] has a rapid development in the recent years and some results are assembled in a recent monograph [11]. The term of causal operators is adopted from engineering literature and the theory these operators has the powerful quality of unifying namely the fractional differential equations [2, 11, 14, 15, 24], ordinary differential equations [9, 10], integro-differential equations [18], differential equations with finite or infinite delay, Volterra integral equations and neutral functional equations [3, 11].

## 2. Preliminaries

Let  $E = C[J, X]$  where  $J$  is an appropriate time interval,  $X$  represents either finite or infinite dimensional space, depending on the requirement of the context, so that  $E$  is a function space.

The operator  $Q : E \rightarrow E$  is said to be a causal operator if, for each couple of elements  $x, y$  in  $E$  such that  $x(s) = y(s)$  for  $0 \leq t_0 \leq s \leq t$  the equality  $(Qx)(s) = (Qy)(s)$  holds for  $0 \leq t_0 \leq s \leq t$ ,  $t < T$ ,  $T$  is a given number.

If  $E$  is a space of measurable functions on  $[t_0, T)$  for  $t_0 \geq 0$ , then the definition needs a slight modification, requiring property to be valid almost everywhere on  $[t_0, T)$ . One can point out that for causal operators, a notation identical with what is encountered for a general equation with memory can be stated as follows. A representation of the form

$$x(t) = (Qx)(t)$$

where for each  $t \in [t_0, T)$ ,  $(Qx)(t)$  is a functional on  $E$  which takes values in  $X$ , for each  $t$ , while the whole family of functionals,  $t \in [t_0, T)$ , define the operator from  $E = C([t_0, T), X)$  to itself.

For illustration, let us take  $E = C([t_0, T), \mathbb{R}^n)$  as the underlying space. Let  $\{Q_n\}$  be a sequence of causal operators on  $E$  such that

$$(2.1) \quad \lim_{n \rightarrow \infty} (Q_n x)(t) = (Qx)(t)$$

for each  $(t, x) \in [t_0, T) \times E$ . The question is whether we can infer that the limit  $Q : E \rightarrow E$  is also a causal operator. The answer is yes because the causality of  $\{Q_n\}$  implies that

$$(Q_n x)(s) = (Q_n y)(s), \quad s \in [t_0, T).$$

If we let  $n \rightarrow \infty$  on both sides, in the above relation and use (2.1) for each fixed  $s \in [t_0, T)$ , we obtain the causality of  $Q$ .

We give some basic definitions and lemmas for the development of main result.

**2.1. Definition.** The causal terminal value problem (*CTVP*) is defined as follows

$$(2.2) \quad u' = (Qu)(t), \quad u(\infty) = u_\infty, \quad t \in \mathbb{R}_+$$

where  $Q : E \rightarrow E$ ,  $E = C(\mathbb{R}_+, \mathbb{R})$  given a causal operator.

**2.2. Definition.** Let  $u \in C[(t_0, t_0 + a), \mathbb{R}]$  be a function. Then its Dini derivatives as follow

$$D^+u(t) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [u(t+h) - u(t)]$$

$$D^-u(t) = \liminf_{h \rightarrow 0^-} \frac{1}{h} [u(t+h) - u(t)]$$

where  $u \in C[(t_0, t_0 + a), \mathbb{R}]$ . When  $D^+u(t) = D_+u(t)$ , the right derivative is denoted by  $u'_+(t)$ . Similarly,  $u'_-(t)$  denotes the left derivative when  $D^-u(t) = D_-u(t)$ . When  $u'_+(t) = u'_-(t) \neq \pm\infty$ , the derivative is denoted by  $u'(t)$ .

Sometimes, it is enough to have the differential inequality satisfied relative to only Dini derivatives.

**2.3. Definition.** (i) Let  $r(t)$  be a solution (2.2) of the *CTVP* on  $t \in \mathbb{R}_+$ . Then  $r(t)$  is said to be a maximal solution of *CTVP* if, for every solution  $u(t)$  of *CTVP* existing on  $\mathbb{R}_+$  the inequality

$$u(t) \leq r(t), \quad t \in \mathbb{R}_+$$

holds.

(ii) Let  $\rho(t)$  be a solution (2.2) of the *CTVP* on  $t \in \mathbb{R}_+$ . Then  $\rho(t)$  is said to be a minimal solution of *CTVP* if, for every solution  $u(t)$  of *CTVP* existing on  $\mathbb{R}_+$  the inequality

$$\rho(t) \leq u(t), \quad t \in \mathbb{R}_+$$

holds.

**2.4. Definition.** The functions  $v, w \in C^1[\mathbb{R}_+, \mathbb{R}]$  are said to be uncoupled-natural type lower and upper solutions (2.2) of *CTVP* if  $v$  and  $w$  satisfy the differential inequalities

$$v'(t) \geq (Qv)(t), \quad v(\infty) \leq u_\infty \text{ for } t \in \mathbb{R}_+$$

$$w'(t) \leq (Qw)(t), \quad w(\infty) \geq u_\infty \text{ for } t \in \mathbb{R}_+$$

where the causal operator  $Q \in E = C(\mathbb{R}_+, \mathbb{R})$ ,  $Q : E \rightarrow E$  is continuous.

**2.5. Definition.** The functions  $v, w \in C^1[\mathbb{R}_+, \mathbb{R}]$  are said to be coupled lower and upper solutions (2.2) of *CTVP* if  $v$  and  $w$  satisfy the differential inequalities

$$v'(t) \geq (Qw)(t), \quad v(\infty) \leq u_\infty \text{ for } t \in \mathbb{R}_+$$

$$w'(t) \leq (Qv)(t), \quad w(\infty) \geq u_\infty \text{ for } t \in \mathbb{R}_+$$

where the causal operator  $Q \in E = C(\mathbb{R}_+, \mathbb{R})$ ,  $Q : E \rightarrow E$  is continuous.

**2.6. Lemma.** Suppose  $m(t)$  is continuous on  $(a, b)$ . Then  $m(t)$  is nondecreasing (non-increasing) on  $(a, b)$  if and only if  $D^+m(t) \geq 0$  ( $\leq 0$ ) for every  $t \in (a, b)$  where

$$D^+m(t) = \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [m(t+h) - m(t)].$$

*Proof.* For the detail of the proof please see [10]. The condition is obviously necessary. Let us prove that it is sufficient. Assume first that  $D^+m(t) > 0$  on  $(a, b)$ . If there exists two points  $\alpha, \beta \in (a, b)$ ,  $\alpha < \beta$ , such that  $m(\alpha) > m(\beta)$ , then there exists a  $\mu$  with  $m(\alpha) > \mu > m(\beta)$  and some points  $t \in [\alpha, \beta]$  such that  $m(t) > \mu$ . Let  $\zeta = \sup \{t; m(t) > \mu, t \in [\alpha, \beta]\}$ . Clearly,  $\zeta \in (\alpha, \beta)$  and  $m(\zeta) = \mu$ . Therefore, for every  $t \in (\zeta, \beta)$ , we have

$$\frac{m(t) - m(\zeta)}{t - \zeta} < 0$$

which implies  $D^+m(\zeta) \leq 0$ . This is a contradiction and therefore the proof is complete.  $\square$

**2.7. Lemma.** *Let  $v, w \in C([t_0, T], \mathbb{R})$  and for some fixed Dini derivative,  $Dv(t) \leq w(t)$ ,  $t \in [t_0, T]$ . Then  $D_-v(t) \leq w(t)$ ,  $t \in [t_0, T]$ .*

*Proof.* For the detail of the proof please see [10]. Define the function

$$m(t) = v(t) - \int_{t_0}^t w(s) ds.$$

It then follows, from the assumption, that

$$Dm(t) = Dv(t) - w(t) \leq 0, \quad t \in [t_0, T].$$

Hence by Lemma 2.1,  $m(t)$  is nonincreasing in  $t$  on  $[t_0, T]$ . Consequently,

$$D_-m(t) = D_-v(t) - w(t) \leq 0, \quad t \in [t_0, T],$$

and the lemma is proved.  $\square$

### 3. Comparison Results

In this section, we give some basic comparison theorems and existence results for the improvement of the main results. The proof of the theorems are very much similar to the proof of the comparison results in monotone iterative techniques for ODEs in [9].

**3.1. Theorem.** *Let us assume that  $(Qu)(t) \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ , where the causal operator  $Q \in E = C(\mathbb{R}_+, \mathbb{R})$ ,  $Q : E \rightarrow E$  is continuous. In addition to  $w, v \in C[\mathbb{R}_+, \mathbb{R}]$  such that  $w(\infty), v(\infty)$  exist and*

- (i)  $D_+w(t) \leq (Qw)(t)$ ,  $t \in [0, \infty)$ ;
- (ii)  $D_+v(t) \geq (Qv)(t)$ ,  $t \in [0, \infty)$ ;
- (iii)  $(Qu)(t) \leq (Qv)(t)$  whenever  $u \leq v$  for each  $t$ .

*Then  $v(t) \leq w(t)$  for  $t \in [0, \infty)$  provided that  $v(\infty) \leq w(\infty)$ .*

*Proof.* Initially, we prove the theorem for strict inequalities, let us assume that  $v(\infty) < w(\infty)$  and one of the inequalities either in (i) or (ii) be strict. Let  $D_+v(t) > (Qv)(t)$ ,  $t \in [0, \infty)$ .

Suppose that the  $v(t) \leq w(t)$  is not true, then there exists a  $t_1 \in [0, \infty)$  such that  $v(t_1) = w(t_1)$  and  $v(t) < w(t)$  for  $t \in (t_1, \infty)$ . For sufficiently small  $h > 0$ , we have

$$v(t_1 + h) < w(t_1 + h)$$

and consequently

$$\begin{aligned} h^{-1} [v(t_1 + h) - v(t_1)] &< h^{-1} [w(t_1 + h) - w(t_1)] \\ \liminf_{h \rightarrow 0^+} h^{-1} [v(t_1 + h) - v(t_1)] &\leq \liminf_{h \rightarrow 0^+} h^{-1} [w(t_1 + h) - w(t_1)] \\ D_+ v(t_1) &\leq D_+ w(t_1). \end{aligned}$$

This implies, in view of (i), (ii) we get

$$(Qv)(t_1) < D_+ v(t_1) \leq D_+ w(t_1) \leq (Qw)(t_1) = (Qv)(t_1)$$

which is a contradiction. Hence

$$v(t) < w(t), \text{ for } t \in [0, \infty).$$

Now define for  $\varepsilon > 0$  arbitrary

$$\tilde{v}(t) = v(t) - \varepsilon(1 + e^{-t}).$$

Then  $\tilde{v}(t) < v(t)$  for  $t \in [0, \infty)$  and  $\tilde{v}(\infty) < v(\infty)$ . Hence, using (i) and (iii), we get

$$\begin{aligned} D_+ v(t) + \varepsilon e^{-t} &\geq (Qv)(t) + \varepsilon e^{-t} \\ &\geq (Q\tilde{v})(t) + \varepsilon e^{-t} \\ D_+ \tilde{v}(t) &> (Q\tilde{v})(t). \end{aligned}$$

It then follows by the proof of the earlier argument, strict inequality, that implies

$$\tilde{v}(t) < \tilde{w}(t).$$

Now, letting  $t \rightarrow \infty$ , we get  $\tilde{v}(\infty) < v(\infty) < \tilde{w}(\infty)$  and letting  $\varepsilon \rightarrow 0$ , we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} [v(t) - \varepsilon(1 + e^{-t})] &\leq \lim_{\varepsilon \rightarrow 0} [w(t) + \varepsilon(1 + e^{-t})] \\ v(t) &\leq w(t), \quad t \in [0, \infty). \end{aligned}$$

This completes the proof of the Theorem 3.1.  $\square$

**3.2. Theorem.** Let  $w, v \in C[\mathbb{R}_+, \mathbb{R}]$  such that  $v(t) \leq w(t)$ ,  $t \in \mathbb{R}_+$ .

Let  $Q : \Omega \rightarrow \mathbb{R}$  be the continuous causal operator

where  $\Omega = \{(t, u) : v(t) \leq u \leq w(t), t \in \mathbb{R}_+\}$ . Suppose that

- (i)  $v'(t) \geq (Qv)(t)$ ,  $t \in \mathbb{R}_+$ ;
- (ii)  $w'(t) \leq (Qw)(t)$ ,  $t \in \mathbb{R}_+$ ;
- (iii)  $(Qu)(t) \leq \lambda(t)$  on  $\Omega$  such that  $\lambda \in L^1[\mathbb{R}_+, \mathbb{R}]$ .

Then the CTVP of (2.2) has a solution which satisfies to  $v(t) \leq u(t) \leq w(t)$  on  $[a, \infty)$  provided that  $v(\infty) \leq u(\infty) \leq w(\infty)$  for some  $a \geq 0$ .

*Proof.* Consider  $P : C[\mathbb{R}_+, \mathbb{R}] \rightarrow C[\mathbb{R}_+, \mathbb{R}]$  defined by

$$(Pu)(t) = \max[v(t), \min[u(t), w(t)]].$$

$Q$  is a continuous causal operators and by the assumption (iii), we get  $(Qu)(t) \leq \lambda(t)$ . So that  $(QPu)(t)$  defines a continuous extension of  $Q$  to  $\mathbb{R}_+ \times \mathbb{R}$  which is also bounded since  $Q$  is assumed to be bounded on  $\Omega$ . Therefore, the CTVP of

$$(3.1) \quad u'(t) = (QPu)(t), \quad u(\infty) = u_\infty, \quad t \in \mathbb{R}_+$$

has a solution  $u(t)$  on  $[a, \infty) \subset \mathbb{R}_+$ . We need to show that  $v(t) \leq u(t) \leq w(t)$  for  $t \in [a, \infty)$  where  $u(t)$  is a solution of *CTVP*.

In order to show these for any  $\varepsilon > 0$ , consider

$$\begin{aligned} \tilde{v}(t) &= v(t) - \varepsilon(1 + e^{-t}) \\ \tilde{w}(t) &= w(t) + \varepsilon(1 + e^{-t}). \end{aligned}$$

Then  $\tilde{w}(t) > w(t)$ ,  $\tilde{v}(t) < v(t)$  and  $\tilde{v}(\infty) < u_\infty < \tilde{w}(\infty)$ . We claim that  $\tilde{v}(t) < u(t) < \tilde{w}(t)$  on  $[a, \infty)$ . If this is not true, then there would exist  $t_1 \in [a, \infty)$  such that  $\tilde{v}(t) < u(t) < \tilde{w}(t)$  for  $t \in (t_1, \infty)$  and either  $\tilde{v}(t_1) = u(t_1)$  or  $\tilde{w}(t_1) = u(t_1)$ .

If  $u(t_1) = \tilde{w}(t_1)$ , then for sufficiently small  $h > 0$

$$\begin{aligned} [u(t_1 + h) - u(t_1)] &< [\tilde{w}(t_1 + h) - \tilde{w}(t_1)] \\ \lim_{h \rightarrow 0} h^{-1} [u(t_1 + h) - u(t_1)] &\leq \lim_{h \rightarrow 0} h^{-1} [\tilde{w}(t_1 + h) - \tilde{w}(t_1)] \\ u'(t_1) &\leq \tilde{w}'(t_1). \end{aligned}$$

Hence, by using (ii), we have, in view of the fact that  $w(t_1) = (Pu)(t_1)$

$$\begin{aligned} (QPu)(t_1) &= u'(t_1) \leq \tilde{w}'(t_1) = w'(t_1) - \varepsilon e^{-t_1} \\ &\leq (Qw)(t_1) - \varepsilon e^{-t_1} = (QPu)(t_1) - \varepsilon e^{-t_1} \\ &< (QPu)(t_1) \end{aligned}$$

which is a contradiction. If  $\tilde{v}(t_1) = u(t_1)$ , then we arrive at a similar contradiction. Thus, it follows that

$$\begin{aligned} v(t) - \varepsilon(1 + e^{-t}) &\leq u(t) \leq w(t) + \varepsilon(1 + e^{-t}) \\ \tilde{v}(t) &< u(t) < \tilde{w}(t) \quad \text{on } [a, \infty). \end{aligned}$$

Now, letting  $\varepsilon \rightarrow 0$ , we get

$$(3.2) \quad v(t) \leq u(t) \leq w(t) \quad \text{on } [a, \infty)$$

It follows that  $u(t)$  is actually a solution of the *CTVP* of (2.2). Hence, the proof of the theorem is completed.  $\square$

#### 4. Monotone Iterative Technique

In this section, we will prove the main theorem that gives several different conditions to apply the method of monotone iterative technique [9] to the nonlinear causal differential equations [11].

**4.1. Theorem.** Let  $Q : C[\mathbb{R}_+, \mathbb{R}] \rightarrow C[\mathbb{R}_+, \mathbb{R}]$  be a continuous causal operator,  $(Qu)(t) \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ ,

- (i)  $|(Qu)(t)| \leq \lambda(t)|u|$  with  $\lambda \in L^1[0, \infty)$ ;
- (ii)  $v_0, w_0 \in C^1[\mathbb{R}_+, \mathbb{R}]$  with  $v_0(t) \leq w_0(t)$  on  $\mathbb{R}_+$ ,  $v_0(\infty)$ ,  $w_0(\infty)$  exist and
  - (a)  $v_0' \geq (Qv_0)(t)$ ,  $v_0(\infty) \leq u_\infty$  for  $t \in \mathbb{R}_+$ ;
  - (b)  $w_0' \leq (Qw_0)(t)$ ,  $w_0(\infty) \geq u_\infty$  for  $t \in \mathbb{R}_+$ ;

(iii)  $(Qx)(t) - (Qy)(t) \leq M(t)[x(s) - y(s)]$  whenever  $v_0(t) \leq y \leq x \leq w_0(t)$  on  $\mathbb{R}_+$  and  $M \in L^1[0, \infty)$ .

Then there exist monotone sequences  $\{v_n\}$ ,  $\{w_n\}$  such that  $v_n \rightarrow v$ ,  $w_n \rightarrow w$  as  $n \rightarrow \infty$  uniformly and monotonically on  $[a, \infty)$  for some  $a \geq 0$  and that  $v$  and  $w$  are the minimal and the maximal solutions (2.2) of *CTVP*; respectively.

*Proof.* For any  $\eta \in C[\mathbb{R}_+, \mathbb{R}]$  such that  $\eta(\infty) = \eta_\infty$  exists and  $v_0 \leq \eta \leq w_0$  on  $\mathbb{R}_+$ , consider the causal terminal value problem

$$(4.1) \quad u' = (Ku)(t), \quad u(\infty) = u_\infty, \quad t \in \mathbb{R}_+$$

where  $(Ku)(t) = (Q\eta)(t) + M(t)(u - \eta)$  and  $v_0(\infty) \leq u_\infty \leq w_0(\infty)$ . Now

$$|(Ku)(t)| \leq |(Q\eta)(t)| + M(t)|u - \eta| \leq (\lambda(t) + M(t))|\eta| + M(t)|u|.$$

Setting  $(Gr)(t) = M(t)r + \sigma(t)$ , where  $\sigma(t) = [\lambda(t) + M(t)]|\eta(t)|$ , since the solution  $r(t)$  of

$$(4.2) \quad r' = (Gr)(t), \quad r(t_0) = r_0$$

are bounded. Hence, the *CTVP* (4.1) has a solution  $u$  on  $[a, \infty)$  for some  $a \geq 0$ . Also, since  $K$  is linear in  $u$ , the solution is unique. Define a mapping  $A$  by  $A\eta = u$ . This mapping will be used to define the sequences  $\{v_n\}$ ,  $\{w_n\}$ . Let us prove that

- (A)  $v_0 \leq Av_0$ ,  $w_0 \geq Aw_0$ ;
- (B)  $A$  is monotone operator on the segment

$$[v_0, w_0] = \{u \in C[\mathbb{R}_+, \mathbb{R}], v(t) \leq u(t) \leq w(t)\}.$$

To prove (A), set  $Av_0 = v_1$ , where  $v_1$  is the unique solution of (4.2) with  $\eta = v_0$ . Setting  $\phi = v_1 - v_0$ , we see that

$$\phi' = v_1' - v_0' \leq (Qv_0)(t) + M(t)[v_1 - v_0] - (Qv_0)(t) = M(t)\phi,$$

and  $\phi(\infty) \geq 0$ . This shows that

$$\phi(t) \geq \phi(\infty) \exp\left(-\int_t^\infty M(s)ds\right) \geq 0$$

and hence  $v_0 \leq v_1$  on some interval  $[a, \infty)$  or equivalently  $v_0 \leq Av_0$ . Similarly, we can prove that  $w_0 \geq Aw_0$ . To prove (B), let  $\eta_1, \eta_2 \in [v_0, w_0]$  such that  $\eta_1 \leq \eta_2$ . Suppose that  $u_1 = A\eta_1$  and  $u_2 = A\eta_2$ . Set  $\phi = u_2 - u_1$  so that

$$\phi' = (Q\eta_2)(t) + M(t)[u_2 - \eta_2] - (Q\eta_1)(t) + M(t)[u_1 - \eta_1].$$

Now, using (iii), we get

$$\phi' \leq M(t)[\eta_2 - \eta_1] + M(t)[(u_2 - u_1) - (\eta_2 - \eta_1)]$$

or

$$\phi' \leq M(t)\phi \text{ and } \phi(\infty) = 0.$$

As before, this implies that  $A\eta_1 \leq A\eta_2$ , proving (B). This shows that  $A$  is a monotone operator. We can now define the sequences

$$v_n = Av_{n-1}, \quad w_n = Aw_{n-1}.$$

$v_0 \leq v_1$  is obtained from the  $v_1 = Av_0$ ,  $v_0 \leq Av_0$  operations.  $A$  is a monotone operator,  $Av_0 \leq Av_1 = v_2$  and  $v_1 \leq v_2$  is obtained. Similarly, if continued  $v_{n-1} = Av_{n-2}$ ,  $v_{n-2} \leq v_{n-1}$  and  $Av_{n-2} \leq Av_{n-1}$ ,  $v_{n-1} \leq v_n$ . Consequently,

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n.$$

The same procedure can also be applied for  $w$ , we have

$$w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0.$$

We conclude that the previous argument implies

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0$$

on  $[a, \infty)$ . It then follows that

$$\lim_{\varepsilon \rightarrow 0} [v_n] = v \text{ and } \lim_{\varepsilon \rightarrow 0} [w_n] = w$$

on  $[a, \infty)$ . We will show that limit functions that are the solutions of linear causal terminal value problems;  $v(t)$  and  $w(t)$  are continuous such that

$$\begin{aligned} v'_n &= (Qv_{n-1})(t) + M(t)[v_n - v_{n-1}], \quad v(\infty) = u_\infty \\ w'_n &= (Qw_{n-1})(t) + M(t)[w_n - w_{n-1}], \quad w(\infty) = u_\infty. \end{aligned}$$

For each  $n$ , define  $(K_n)(t) = (Qv_{n-1})(t) + M(t)[v_n - v_{n-1}]$ ; then  $\{(K_n)(t)\}_{n=1}^\infty$  is a sequence of continuous functions. Hence

$$\lim_{n \rightarrow \infty} (K_n)(t) = \lim_{n \rightarrow \infty} (Qv_{n-1})(t) = (Qv)(t) = (K)(t).$$

Furthermore, note that  $(K)(t)$  is measurable on  $[a, \infty)$  for some  $a$  and also, in view of the fact that  $v_0(t) \leq v_n(t) \leq w_0(t)$  on  $[0, \infty)$ , we have

$$\begin{aligned} |(K_n)(t)| &= |(Qv_n)(t) + M(t)[v_n - v_{n-1}]| \leq \lambda(t)|v_n(t)| + 2M(t)|v_n(t)| \\ &\leq L[\lambda(t) + M(t)] \end{aligned}$$

for some constant  $L$ . Since

$$\int_t^\infty (\lambda(s) + M(s)) ds < \infty$$

so also

$$\int_t^\infty [(Qv_{n-1})(s) + M(s)(v_n(s) - v_{n-1}(s))] ds < \infty.$$

Now using the Lebesgue dominated convergence theorem, we get

$$\lim_{n \rightarrow \infty} \int_t^\infty (K_n)(s) ds = \int_t^\infty (K)(s) ds$$

or

$$\lim_{n \rightarrow \infty} \int_t^\infty [(Qv_n)(s) + M(s)(v_n(s) - v_{n-1}(s))] ds = \int_t^\infty (Qv)(s) ds,$$

which exists. Since each  $v_n(t)$  is a solution of

$$v_n(t) = v_n(\infty) - \int_t^\infty [(Qv_{n-1})(s) + M(s)(v_n(s) - v_{n-1}(s))] ds$$



taking the limit as  $n \rightarrow \infty$ , we have

$$v(t) = v(\infty) - \int_t^\infty (Qv)(s) ds, \quad t \in [a, \infty).$$

Now, we conclude that  $v(t)$  is continuous, since a function has indefinite integral if and only if it is absolutely continuous. In similar way, we show that  $w(t)$  is continuous as well. It is now easy to show that  $v$  and  $w$  are solutions (2.2) of *CTVP*. To prove that  $v, w$  are minimal and maximal solutions (2.2) of *CTVP*, we need to show that if  $u$  is any solution (2.2) of *CTVP* such that  $v_0 \leq u \leq w_0$  on  $[a, \infty)$ , then  $v_0 \leq v \leq u \leq w \leq w_0$  on  $[a, \infty)$ . To do this, suppose that for some  $n$ ,  $v_n \leq u \leq w_n$  on  $[a, \infty)$  and set  $\phi = u - v_{n+1}$ , so that

$$\phi' = (Qu)(t) - (Qv_n)(t) - M(t)(v_{n+1} - v_n) \leq M(t)(u - v_n) - M(t)(v_{n+1} - v_n) = M(t)(u - v_{n+1})$$

or

$$\phi' \leq M(t)\phi, \quad \phi(\infty) = 0.$$

Thus,  $\phi(t) \geq 0$ , which implies  $u \geq v_{n+1}$  on  $[a, \infty)$ . Similarly,  $u \leq w_{n+1}$  on  $[a, \infty)$  and hence  $v_{n+1} \leq u \leq w_{n+1}$  on  $[a, \infty)$ . Since  $v_0 \leq u \leq w_0$  on  $[a, \infty)$ , this proves by induction that  $v_n \leq u \leq w_n$  on  $[a, \infty)$  for all  $n$ . Taking the limit as  $n \rightarrow \infty$ , we conclude that  $v \leq u \leq w$  on  $[a, \infty)$  and the proof is complete.  $\square$

**4.2. Corollary.** *If in addition to the assumptions of Theorem 4.1, we assume*

$$(Qx)(t) - (Qy)(t) \geq M(t) \max_{0 \leq s \leq t} [x(s) - y(s)]$$

*whenever  $v_0 \leq y \leq x \leq w_0$  on  $\mathbb{R}_+$  and  $M \in L^1[0, \infty)$ . Then we have unique solution of (2.2) such that  $v = u = w$ .*

*Proof.* If we set  $p = |v - w|$  then  $p'(t) \leq |(Qp)(t)| = |Q(v - w)(t)|$   
 $= |(Qv)(t) - (Qw)(t)| = M(t) \max_{0 \leq s \leq t} [|v(s) - w(s)|] \leq M(t)|v(t) - w(t)|$   
 $= M(t)p(t)$  which gives  $p'(t) \leq Mp$  and then by using the comparison result in Theorem 3.1 we get  $p(t) \leq 0$  for  $t \in [0, \infty)$  provided that  $v(\infty) \leq w(\infty)$ . Hence,  $v(t) = w(t)$  we have  $v = u = w$  is the unique solution of (2.2).  $\square$

## 5. An Example

In the following example, we illustrate how to apply and arise some of the results in the terminal value problem.

**5.1. Example.** Assuming that a two-species community model living together and competing with each other for the same limiting resources. For this purpose, it is convenient to formulate a causal terminal value problem, as follows:

$$(5.1) \quad B'_i = B_i g_i(B) = (QB_i)(t), \quad B_i(\infty) = B_{i\infty} \text{ for } i = 1, 2 \text{ and } t \in \mathbb{R}_+$$

where the causal operator  $Q \in E = C(\mathbb{R}_+, \mathbb{R})$ ,  $Q : E \rightarrow E$  is continuous and  $B \in \mathbb{R}^2$ ;  $g_i(B) = a_i - b_{ii}B_i - b_{ij}B_j$  where  $B_i$  is the population density of species for  $i, j = 1, 2$  and  $i \neq j$ ;  $a_i, b_{ii}, b_{ij}$  are positive constants. In order to construct lower and upper functions, we define

$$(5.2) \quad \begin{cases} \bar{g}_1(B) = \sup_{0 \leq \theta \leq B_2} g_1(B_1, \theta) = a_1 - b_{11}B_1, \\ \bar{g}_2(B) = \sup_{0 \leq \theta \leq B_1} g_2(\theta, B_2) = a_2 - b_{22}B_2. \end{cases}$$

Now, consider a causal terminal value problem

$$(5.3) \quad v'_i = (Qv_i)(t) = v_i \bar{g}_i(v) = v_i(a_i - b_{ii}v_i), \quad v_i(\infty) = v_{i\infty} \text{ for } i = 1, 2 \text{ and } t \in \mathbb{R}_+$$

The solution of (5.3) is expressed as follows:

$$(5.4) \quad v_i(t) = k_i [1 + \exp(-a_i t) (k_i - v_{i\infty}) / v_{i\infty}]^{-1}$$

for  $i = 1, 2$ . Here  $k_i = \frac{a_i}{b_{ii}}$  is referred as the carrying capacity of the  $i$ -th species. From (5.1), (5.2), (5.3), (5.4), and choosing  $v_{i\infty} \geq B_{i\infty}$ , we have

$$(5.5) \quad \begin{cases} v'_i \geq v_i g_i(v) \\ v_{i\infty} \geq B_{i\infty} \text{ for } i = 1, 2. \end{cases}$$

This implies that  $v_i(t)$  in (5.4) is the lower solution of system (5.1).

Now, for sufficiently large  $M$ , namely  $M \geq \bar{g}_2(B)$ , we find that  $w_i(t) = M$  for  $i = 1, 2$  that is the candidate for a upper solution of (5.1). Indeed, for all  $t \in \mathbb{R}_+$ ,

$$(5.6) \quad \begin{cases} w'_i(t) = 0 \leq w_i g_i(w) \\ w_{i\infty} \leq B_{i\infty} \text{ for } i = 1, 2. \end{cases}$$

This implies that  $w_i(t)$  in (5.6) is the upper solution of system (5.1). The verification of  $v_i(t) \leq w_i(t)$  for  $i = 1, 2$  follows immediately from (5.4).

We note that the definition of lower and upper solutions relative to CTVPs is in the reverse order to what would be expected.

## 6. Conclusion

In this paper, some existence results and comparison theorems in terms of lower and upper solutions have been investigated of the terminal value problem for causal differential equations as well as the well-known monotone iterative technique has been applied in a closed set for the given causal differential equations.

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