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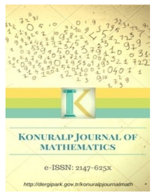

Harmonically (s,P)-Functions and Related Inequalities

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Abstract

In this paper, we introduce and investigate the concept of harmonically (s,P)-functions and establish Hermite-Hadamard type inequalities for this class of functions. In addition, we derive new Hermite-Hadamard type inequalities for functions whose first derivatives in absolute value are harmonically (s,P)-functions, by employing Hölder's inequality and the power-mean inequality. Furthermore, we present some new inequalities related to special means of real numbers.

Keywords: *s-convex function, (s,P)-function, harmonically(s,P)-function, Hermite-Hadamard inequality*
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1. Preliminaries

Let $f:I\to\mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2} \tag{1.1}$$

for all $a,b\in I$ with $a<b$. Both inequalities hold in the reversed direction if the function f is concave. This double inequality is well known as the Hermite-Hadamard inequality [8]. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping f.
In [7], Dragomir et al. gave the following definition and related Hermite-Hadamard integral inequalities as follows:

Definition 1.1. *A nonnegative function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be a P-function if the inequality*

$$f(tx+(1-t)y)\le f(x)+f(y)$$

holds for all $x,y\in I$ and $t\in(0,1)$.

Theorem 1.2. *Let $f\in P(I)$, $a,b\in I$ with $a<b$ and $f\in L[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\le\frac{2}{b-a}\int_a^b f(x)\,dx\le 2\left[f(a)+f(b)\right]. \tag{1.2}$$

Definition 1.3. *[18] Let $h:J\to\mathbb{R}$ be a non-negative function, $h\neq 0$. We say that $f:I\to\mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h,I)$, if f is non-negative and for all $x,y\in I$, $\alpha\in(0,1)$ we have*

$$f(\alpha x+(1-\alpha)y)\le h(\alpha)f(x)+h(1-\alpha)f(y).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f\in SV(h,I)$. It is clear that, if we choose $h(\alpha)=\alpha$ and $h(\alpha)=1$, then the h-convexity reduces to convexity and definition of P-function, respectively.

Readers can look at [3, 18] for studies on h-convexity.
In [4], Breckner defined s-convexity in the second sense as follows.

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Definition 1.4. *A function $f:[0,\infty)\to\mathbb{R}$ is said to be $s-$convex in the second sense if*

$$f(\alpha x+\beta y)\leq \alpha^s f(x)+\beta^s f(y)$$

for all $x,y\in[0,\infty)$, $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ and for some fixed $s\in(0,1]$. This class of s-convex functions in the second sense is usually denoted by K_s^2.

It can be easily seen that for $s=1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.
In [9] Hudzik and Maligranda presented some properties of $s-$convex functions in the second sense and various examples as the following.

Example 1.5. *Let $0<s<1$ and $a,b,c\in\mathbb{R}$. Define $f:\mathbb{R}^+\to\mathbb{R}$ by,*

$$f(t)=\begin{cases} a, & \text{if } t=0,\\ bt^s+c, & \text{if } t>0\end{cases}$$

we have the following.

i. If $b\geq 0$ and $0\leq c<a$ then $f\in K_s^2$.
ii. If $b>0$ and $c<0$ then $f\notin K_s^2$.

In [6], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 1.6. *Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1)$, and let $a,b\in[0,\infty)$, $a<b$. If $f\in L[a,b]$ then the following inequalities hold*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

Both inequalities hold in the reversed direction if f is s-concave. The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.3).

Definition 1.7 ([17]). *Let $s\in(0,1]$. A function $f:I\to\mathbb{R}$ is called (s,P)-function if*

$$f(tx+(1-t)y)\leq\left(t^s+(1-t)^s\right)[f(x)+f(y)]$$

for every $x,y\in I$ and $t\in[0,1]$.

In [17], Numan and İşcan proved some variants of Hermite-Hadamard's inequality which holds for (s,P)-functions as follows:

Theorem 1.8. *Let $s\in(0,1]$ and $f:[a,b]\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be an (s,P)-function. If $a<b$ and $f\in L[a,b]$, then the following Hermite-Hadamard type inequalities hold:*

$$2^{s-2}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{2}{s+1}[f(a)+f(b)]. \tag{1.4}$$

Theorem 1.9. *Let $a<b$, $s\in(0,1]$ and $f:[a,b]\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be an (s,P)-function. If f is symmetric with respect to $\frac{a+b}{2}$(i.e. $f(x)=f(a+b-x)$ for all $x\in[a,b]$), then the following inequalities hold:*

$$2^{s-2}f\left(\frac{a+b}{2}\right)\leq f(x)\leq 2^{1-s}[f(a)+f(b)] \tag{1.5}$$

for all $x\in I$.

The harmonic convex function class and some related function classes are defined as follows:

Definition 1.10 ([10]). *Let $I\subset\mathbb{R}\backslash\{0\}$ be an interval. A function $f:I\to\mathbb{R}$ is called harmonically convex if*

$$f\left(\frac{xy}{ty+(1-t)x}\right)\leq tf(x)+(1-t)f(y)$$

for every $x,y\in I$ and $t\in[0,1]$.

Definition 1.11 ([11]). *Let $s\in(0,1]$ and $I\subset(0,\infty)$ be an interval. A function $f:I\to\mathbb{R}$ is called harmonically s-convex if*

$$f\left(\frac{xy}{ty+(1-t)x}\right)\leq t^s f(x)+(1-t)^s f(y)$$

for every $x,y\in I$ and $t\in[0,1]$.

Definition 1.12 ([12]). *Let $I\subset(0,\infty)$ be an interval. A function $f:I\to\mathbb{R}$ is called harmonically P-function if*

$$f\left(\frac{xy}{ty+(1-t)x}\right)\leq f(x)+f(y)$$

for every $x,y\in I$ and $t\in[0,1]$.

Theorem 1.13 ([12]). *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be an harmonically P-function such that $f \in L[a,b]$, where $a,b \in I$ with $a < b$ and then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{2ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq 2\left[f(a)+f(b)\right]. \tag{1.6}$$

Definition 1.14 ([16]). *Let $h : [0,1] \subseteq J \to \mathbb{R}$ be a nonnegative function. A function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically h-convex function if*

$$f\left(\frac{xy}{ty+(1-t)x}\right) \leq h(t)f(x)+h(1-t)f(y)$$

for every $x,y \in I$ and $t \in (0,1)$.

We recall the following special functions:
(1) The Beta function:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x,y > 0.$$

(2) The hypergeometric function:

$${}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}dt,\ c > b > 0,\ |z| < 1,$$

(see [15]).
The main purpose of this paper is to introduce the concept of harmonically (s,P)-function which is connected with the concepts of P-function, s-convex function, (s,P)-function, harmonically P-function and harmonically s-convex function and to establish some new Hermite-Hadamard type inequalities for these classes of functions. In recent years many authors have studied error estimations of Hermite-Hadamard type inequalities; for refinements, counterparts, generalizations, for some related papers see [1, 2, 5, 7, 14, 13].

2. The definition of harmonically (s,P)-function

In this section, we introduce a new concept, which we called harmonically (s,P)-function and we give by setting some algebraic properties for harmonically (s,P)-function, as follows:

Definition 2.1. *Let $s \in (0,1]$ and $I \subset (0,\infty)$ be an interval. A function $f : I \to \mathbb{R}$ is called harmonically (s,P)-function if*

$$f\left(\frac{xy}{ty+(1-t)x}\right) \leq \left(t^s+(1-t)^s\right)\left[f(x)+f(y)\right] \tag{2.1}$$

for every $x,y \in I$ and $t \in [0,1]$.

We will denote by $HP_s(I)$ the class of all harmonically (s,P)-functions on the interval I. Clearly, the definition of harmonically $(1,P)$-function coincides with the definition of harmonically P-function.
We note that, every harmonically (s,P)-function is a harmonically h-convex function with the function $h(t) = t^s+(1-t)^s$. It is easily seen that, if $f,g \in HP_s(I)$ then $f+g \in HP_s(I)$ and $\alpha f \in HP_s(I)$ for $\alpha \geq 0$.

Remark 2.2. *i.) We note that if f is satisfies (2.1), then f is a nonnegative function. Indeed, if we rewrite the inequality (2.1) by $x = y$ then*

$$f(x) \leq 2(t^s+(1-t)^s)f(x)$$

for every $x \in I$ and $t \in [0,1]$. Therefore, since $[2(t^s+(1-t)^s)-1] \geq 0$ for all $t \in [0,1]$, we obtain $f(x) \geq 0$ for all $x \in I$.
ii.) Let $g : I \subset (0,\infty) \to \mathbb{R}$, $g(x) = x^{-1}$ and $g(I) = \{g(x) : x \in I\}$. By the inequality (2.1), we have

$$\left(f\circ g^{-1}\right)(tg(x)+(1-t)g(y)) \leq (t^s+(1-t)^s)\left[\left(f\circ g^{-1}\right)(g(x))+\left(f\circ g^{-1}\right)(g(y))\right]$$

for every $x,y \in I$ and $t \in [0,1]$. This shows us that; $f : I \subset (0,\infty) \to \mathbb{R}$ is an harmonically (s,P)-function if and only if $f\circ g^{-1}$ is an (s,P)-function on $g(I)$.

Proposition 2.3. *Every harmonically s-convex function is also harmonically (s,P)-function.*

Proof. Let $f : I \subset (0,\infty) \to \mathbb{R}$ be an arbitrary harmonically s-convex function. Then

$$\begin{aligned} f\left(\frac{xy}{ty+(1-t)x}\right) &\leq t^s f(x)+(1-t)^s f(y) \\ &\leq (t^s+(1-t)^s)\left[f(x)+f(y)\right] \end{aligned}$$

for every $x,y \in I$ and $t \in [0,1]$. □

Proposition 2.4. *Every harmonically P-function is also harmonically (s,P)-function.*

Proof. Let $f : I \subset (0,\infty) \to \mathbb{R}$ be an arbitrary P-function. The proof is clear from the following inequalities

$$t \leq t^s \text{ and } 1-t \leq (1-t)^s$$

for all $t \in [0,1]$. In this case, we can write

$$1 \leq t^s + (1-t)^s.$$

Therefore,

$$\begin{aligned} f\left(\frac{xy}{ty+(1-t)x}\right) &\leq f(x)+f(y) \\ &\leq (t^s+(1-t)^s)\left[f(x)+f(y)\right] \end{aligned}$$

for every $x, y \in I$, $t \in [0,1]$ and $s \in (0,1]$. Thus the desired result is obtained. □

We can state the following corollary noting that every nonnegative harmonically convex function is also a harmonically P-function.

Corollary 2.5. *Every nonnegative harmonically convex function on the interval $I \subset (0,\infty)$ is also a harmonically (s,P)-function.*

We can also state the following result, noting that every nonnegative convex function is also an s-convex function.

Proposition 2.6. *Every nonnegative and nondecreasing convex function on the interval $I \subset (0,\infty)$ is also harmonically (s,P)-function.*

Proof. Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a nonnegative convex function. It is well known to be

$$H_t(x,y) = \frac{xy}{ty+(1-t)x} \leq tx+(1-t)y = A_t(x,y)$$

and

$$t \leq t^s,\ 1-t \leq (1-t)^s$$

for all $t \in [0,1]$ and $x, y \in (0,\infty)$. Based on this and considering that f is a non-decreasing convex function, we get

$$\begin{aligned} f\left(H_t(x,y)\right) &\leq f\left(A_t(x,y)\right) \leq tf(x)+(1-t)f(y) \\ &\leq t^s f(x)+(1-t)^s f(y) \\ &\leq (t^s+(1-t)^s)\left[f(x)+f(y)\right]. \end{aligned}$$

Thus desired result is obtained. □

Theorem 2.7. *If $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is a harmonically (s,P)-function, then f is bounded on $[a,b]$.*

Proof. Let $M = \max\{f(a), f(b)\}$. For any $x \in [a,b]$, there exists a $t_0 \in [0,1]$ such that $x = ab/(t_0 a + (1-t_0)b)$. Since f is an harmonically (s,P)-function on $[a,b]$, and $t^s + (1-t)^s \leq 2^{1-s}$ for all $t \in [0,1]$ and $s \in (0,1]$, we have

$$f(x) \leq (t^s+(1-t)^s)\left[f(a)+f(b)\right] \leq 2^{2-s}M \leq 4M.$$

This shows that f is bounded from above. Moreover, since f is an harmonically (s,P), f is also a non negative function. This completes the proof. □

Theorem 2.8. *Let $b > a$ and $f_\alpha : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an arbitrary family of harmonically (s,P)-functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in [a,b] : f(u) < \infty\}$ is nonempty, then J is an interval and f is a harmonically (s,P)-function on J.*

Proof. Let $t \in [0,1]$ and $x, y \in J$ be arbitrary. Then

$$\begin{aligned} f\left(\frac{xy}{ty+(1-t)x}\right) &= \sup_\alpha f_\alpha\left(\frac{xy}{ty+(1-t)x}\right) \\ &\leq \sup_\alpha \left\{(t^s+(1-t)^s)\left[f_\alpha(x)+f_\alpha(y)\right]\right\} \\ &\leq (t^s+(1-t)^s)\left[\sup_\alpha f_\alpha(x)+\sup_\alpha f_\alpha(y)\right] \\ &= (t^s+(1-t)^s)\left[f(x)+f(y)\right] < \infty. \end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is a harmonically (s,P)-function on J. This completes the proof of theorem. □

3. Hermite-Hadamard's inequality for harmonically (s,P)-functions

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for harmonically (s,P)-functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on $[a,b]$.

Theorem 3.1. *Let $s\in(0,1]$, $a<b$ and $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an harmonically (s,P)-function. If $f\in L[a,b]$, then the following Hermite-Hadamard type inequalities hold:*

$$2^{s-2}f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\leq\frac{2}{s+1}\left[f(a)+f(b)\right]. \tag{3.1}$$

Proof. Since f is a harmonically (s,P)-function, we get

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &= f\left(\frac{2\left[\frac{ab}{ta+(1-t)b}\right]\left[\frac{ab}{(1-t)a+tb}\right]}{\frac{ab}{ta+(1-t)b}+\frac{ab}{(1-t)a+tb}}\right)\\ &\leq 2^{1-s}\left[f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{(1-t)a+tb}\right)\right].\end{aligned}$$

By taking integral in the last inequality with respect to $t\in[0,1]$, we deduce that

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{2^{2-s}ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx.$$

By using the property of harmonically (s,P)-function of f, if the variable is changed as $x=ab/\left[ta+(1-t)b\right]$, then

$$\begin{aligned}\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx &= \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt\\ &\leq [f(a)+f(b)]\int_0^1 t^s+(1-t)^s dt\\ &= \frac{2}{s+1}\left[f(a)+f(b)\right].\end{aligned}$$

This completes the proof of the theorem. □

Remark 3.2. *i.) In Theorem 3.1, if we choose $s=1$, then inequality (3.1) reduces to inequality (1.6).*
ii.) We can give a different proof of the Theorem 3.1 as follows:
Let $g:I\subset(0,\infty)\to\mathbb{R}$, $g(x)=x^{-1}$ and $g(I)=\{g(x):x\in I\}$. If $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an harmonically (s,P)-function then $f\circ g^{-1}$ is an (s,P)-function on $g(I)$. Thus, by Theorem 1.8, we have

$$\begin{aligned}2^{s-2}\left(f\circ g^{-1}\right)\left(\frac{g(a)+g(b)}{2}\right) &\leq \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}\left(f\circ g^{-1}\right)(x)dx\\ &\leq \frac{2}{s+1}\left[\left(f\circ g^{-1}\right)(g(a))+\left(f\circ g^{-1}\right)(g(b))\right].\end{aligned} \tag{3.2}$$

Where

$$\begin{aligned}\left(f\circ g^{-1}\right)\left(\frac{g(a)+g(b)}{2}\right) &= f\left(\frac{2ab}{a+b}\right),\\ \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}\left(f\circ g^{-1}\right)(x)dx &= \frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\end{aligned}$$

and

$$\left(f\circ g^{-1}\right)(g(a))+\left(f\circ g^{-1}\right)(g(b))=f(a)+f(b). \tag{3.3}$$

If the equation in 3.2 is substituted into the inequality in 3.3, the desired result is obtained.

Theorem 3.3. *Let $a<b$, $s\in(0,1]$ and $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an harmonically (s,P)-function. If f is harmonically symmetric with respect to $\frac{2ab}{a+b}$(i.e. $f(x)=f\left(\left[a^{-1}+b^{-1}-x^{-1}\right]^{-1}\right)$ for all $x\in[a,b]$), then the following inequalities hold:*

$$2^{s-2}f\left(\frac{a+b}{2}\right)\leq f(x)\leq 2^{1-s}\left[f(a)+f(b)\right]$$

for all $x\in I$.

Proof. Let $x \in [a,b]$ be arbitrary point. Since $t^s + (1-t)^s \leq 2^{1-s}$ for all $t \in [0,1]$, we get

$$\begin{aligned} f(x) &= f\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right) \\ &\leq \left(\left(\frac{x-a}{b-a}\right)^s + \left(\frac{b-x}{b-a}\right)^s\right)[f(a)+f(b)] \\ &\leq 2^{1-s}[f(a)+f(b)] \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}x + \frac{1}{2}[a+b-x]\right) \\ &\leq 2^{1-s}[f(x)+f(a+b-x)] \\ &= 2^{2-s}f(x). \end{aligned}$$

This completes the proof. □

Remark 3.4. *We can give a different proof of the Theorem 3.1 as follows:*
Let $g : I \subset (0,\infty) \to \mathbb{R}$, $g(x) = x^{-1}$ and $g(I) = \{g(x) : x \in I\}$. If $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an harmonically (s,P)-function then $f \circ g^{-1}$ is an (s,P)-function on $g(I)$. Thus , by Theorem 1.8, we have

$$2^{s-2}\left(f \circ g^{-1}\right)\left(\frac{g(a)+g(b)}{2}\right) \leq \left(f \circ g^{-1}\right)(x) \leq 2^{1-s}\left[\left(f \circ g^{-1}\right)(g(a)) + \left(f \circ g^{-1}\right)(g(b))\right].$$

If we take $g(x) = g^{-1}(x) = x^{-1}$ in the last inequalities, then we obtain the desired result.

4. Some new inequalities for harmonically (s,P)-functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value is harmonically (s,P)-function. İşcan [10] used the following lemma:

Lemma 4.1. *Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx = \frac{b-a}{2}\int_0^1 \frac{(1-2t)}{A_t^2} f'\left(\frac{ab}{A_t}\right)dt,$$

where $A_t = tb + (1-t)a$.

Theorem 4.2. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$ and assume that $f' \in L[a,b]$ and $s \in (0,1]$. If $|f'|$ is harmonically (s,P)-function on interval $[a,b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq (b-a)A\left(\left|f'(a)\right|, \left|f'(b)\right|\right)I_s(a,b) \tag{4.1}$$

where $I_s(a,b) = \int_0^1 \frac{|1-2t|}{A_t^2}\left(t^s + (1-t)^s\right)dt$ and $A(.,.)$ is the arithmetic mean.

Proof. Using Lemma 4.1 and the inequality

$$\left|f'\left(\frac{ab}{A_t}\right)\right| \leq \left(t^s + (1-t)^s\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right],$$

we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{b-a}{2}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]\int_0^1 \frac{|1-2t|}{A_t^2}\left(t^s + (1-t)^s\right)dt.$$

This completes the proof of the theorem. □

Theorem 4.3. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$ and assume that $f' \in L[a,b]$ and $s \in (0,1]$. If $|f'|^q$, $q > 1$, is harmonically (s,P)-function on interval $[a,b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{b-a}{2b^2}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{2}{s+1}\right)^{1/q} A^{1/q}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)F_{s,q}^{1/q}(a,b), \tag{4.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $F_{s,q}(a,b) = {}_2F_1(2q,1;s+2;1-\frac{a}{b}) + {}_2F_1(2q,s+1;s+2;1-\frac{a}{b})$ and A is the arithmetic mean.

Proof. Using Lemma 4.1, Hölder's integral inequality and the following inequality

$$\left|f'\left(\frac{ab}{A_t}\right)\right|^q \leq (t^s+(1-t)^s)\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]$$

which is harmonically (s,P)-function of $|f'|^q$, we get

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| &\leq \frac{b-a}{2}\left(\int_0^1|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\frac{1}{A_t^{2q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]\int_0^1\frac{t^s+(1-t)^s}{A_t^{2q}}dt\right)^{\frac{1}{q}} \\ &= \frac{b-a}{2b^2}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{2}{s+1}\right)^{1/q}A^{1/q}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right).F_{s,q}^{1/q}(a,b)\end{aligned}$$

where

$$\int_0^1\frac{t^s+(1-t)^s}{A_t^{2q}}dt=\frac{1}{b^{2q}(s+1)}\left[{}_2F_1(2q,1;s+2;1-\frac{a}{b})+{}_2F_1(2q,s+1;s+2;1-\frac{a}{b})\right].$$

This completes the proof of the theorem. □

Theorem 4.4. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$ and assume that $f'\in L[a,b]$ and $s\in(0,1]$ If $|f'|^q, q\geq 1$, is harmonically (s,P)-function on the interval $[a,b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \leq \frac{b-a}{2^{1-1/q}}A^{1/q}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\left(\frac{1}{ab}-\frac{4}{(b-a)^2}\ln\left(\frac{A}{G}\right)\right)^{1-\frac{1}{q}}I_s^{\frac{1}{q}}(a,b), \tag{4.3}$$

where $I_s(a,b)=\int_0^1\frac{|1-2t|}{A_t^2}(t^s+(1-t)^s)dt$, $A(.,.)$ is the arithmetic mean and $G(.,.)$ is the geometric mean.

Proof. From Lemma 4.1, power mean integral inequality and the property of harmonically (s,P)-function of $|f'|^q$, we obtain

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| &\leq \frac{b-a}{2}\left(\int_0^1\frac{|1-2t|}{A_t^2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{|1-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2^{1-1/q}}A^{1/q}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\left(\frac{1}{ab}-\frac{4}{(b-a)^2}\ln\left(\frac{A}{G}\right)\right)^{1-\frac{1}{q}}I_s^{\frac{1}{q}}(a,b)\end{aligned}$$

This completes the proof of theorem. □

Corollary 4.5. *Under the assumption of Theorem 4.4, If we take $q=1$ in the inequality (4.3), then we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq(b-a)A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)I_s(a,b).$$

This inequality coincides with the inequality (4.1).

5. Applications for special means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers a,b with $b>a$:
1. The arithmetic mean

$$A:=A(a,b)=\frac{a+b}{2},\quad a,b\geq 0,$$

2. The geometric mean

$$G:=G(a,b)=\sqrt{ab},\quad a,b\geq 0$$

3. The harmonic mean

$$H:=H(a,b)=\frac{2ab}{a+b},\quad a,b>0,$$

4. The logarithmic mean

$$L:=L(a,b)=\begin{cases}\frac{b-a}{\ln b-\ln a}, & a\neq b\\ a, & a=b\end{cases};\ a,b>0$$

5. The p-logarithmic mean

$$L_p:=L_p(a,b)=\begin{cases}\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a\neq b, p\in\mathbb{R}\backslash\{-1,0\}\\ a, & a=b\end{cases};\ a,b>0.$$

6.The identric mean

$$I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a,b > 0,$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 5.1. *Let $a,b \in (0,\infty)$ with $a < b$, $s \in (0,1]$ and $n \in (1,\infty)\setminus\{2\}$. Then, the following inequalities are obtained:*

$$2^{s-2}H^n(a,b) \leq G^2(a,b)L_{n-2}^{n-2}(a,b) \leq \frac{4}{s+1}A(a^n,b^n).$$

Proof. The assertion follows from the inequalities (3.1) for the function

$$f(x) = x^n, \;\; x \in [0,\infty).$$

□

Proposition 5.2. *Let $a,b \in (1,\infty)$ with $a < b$ and $s \in (0,1]$. Then, the following inequalities are obtained:*

$$2^{s-2}\left(H + H^{-1}\right) \leq G^2(A+L) \leq \frac{4}{s+1}\left(A + H^{-1}\right).$$

Proof. The assertion follows from the inequalities (3.1) for the function

$$f(x) = x + x^{-1}, \;\; x \in (1,\infty).$$

□

6. Conclusion

In this paper, a new class of convexity, called the harmonic (s-P) function class, is introduced, and some of its fundamental properties are presented. A Hermite-Hadamard type inequality is established for this class, along with several related results. Furthermore, new integral inequalities are derived by employing certain integral identities from the existing literature. In the final section, applications related to special means are discussed. This study provides a foundation for researchers to derive various new inequalities using the proposed class of functions.

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