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A Note on the Integral Representation of Some Relative Growth Indicators of Entire Algebroidal Functions

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Abstaract — Let p and q be any two positive integers. In this paper the concept of two relative growth indicators namely *relative* (p, q)*-th type* and *relative* (p, q)*-th weak type* of entire functions with respect to entire algebroidal functions have been introduced from the view point of their integral representations. Here we also investigate the equivalence of the computational definitions with their respective integral representations.

Keywords — *Entire function, entire algebroidal function, growth, order (lower order), relative order (relative lower order), growth indicator.*

1 Introduction

The *order* and *lower order* of an entire function f which is generally used in computational purposes are classical in complex analysis. Bernal [1] and [2], introduced the *relative order* (respectively *relative lower order)* between two entire functions to avoid comparing growth just with $\exp z$. Extending the notion of *relative order* (respectively *relative lower order)* Ruiz et al. [8] introduced the *relative* (p, q)*-th order* (respectively *relative lower* (p, q)*-th order)* where p and q are any two positive integers. Now to compare the growth of entire functions having the same *relative* (p, q)*-th order* or *relative lower* (p, q)*-th order*, we would like to introduce the definition of *relative* (p, q)*-th type* and *relative* (p, q)*-th weak type* of entire functions with respect to entire algebroidal functions and establish their respective integral representations. We also investigate the equivalence of the computational definitions and their corresponding integral representations of the relative growth indicators as stated above in case of entire algebroidal functions.

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Let F and G be two k-valued function defined by the following irreducible equation

$$f_kF^k + f_{k-1}F^{k-1} + f_{k-2}F^{k-2} + ... + f_0 = 0$$

$$g_kG^k + g_{k-1}G^{k-1} + g_{k-2}G^{k-2} + ... + g_0 = 0$$

where $f_k \neq 0$, $g_k \neq 0$ where f_i $(i = 0, 1, 2, \ldots, k-1)$ and g_i $(i = 0, 1, 2, \ldots, k-1)$ are entire functions having no common zeros. If at least one of the f_i $(i = 0, 1, 2, \ldots, k)$ is transcendental then F is called a k-valued algebroidal function. Further, if $f_k \equiv 1$ then F is called a k-valued entire algebroidal function and similar for G.

Let us consider the definition of *relative* (p, q)*-th order* $\rho_G^{(p,q)}(f_i)$ (respectively *relative* (p, q)*-th lower order* $\lambda_G^{(p,q)}(f_i)$) of an entire functions f_i with respect to an entire algebroidal function G, in the light of *index-pair* which is as follows:

Definition 1.1. [8] Let G be any entire algebroidal function as defined above with index-pair (m, p) .Also let f_i's$(i = 0, 1, 2, \ldots, k-1)$ be entire functions with index-pair (m, q) where p, q, m are positive integers such that $m \geq \max(p, q)$. Then the relative (p, q)-th order of f_i with respect to G is defined as

$$\rho_G^{(p,q)}(f_i) = \limsup_{r\to\infty} \frac{\log^{[p]} M_G^{-1} M_{f_i}(r)}{\log^{[q]} r}.$$

Analogously, the relative (p, q)-th lower order of f_i with respect to $G$is defined by:

$$\lambda_G^{(p,q)}(f_i) = \liminf_{r\to\infty} \frac{\log^{[p]} M_G^{-1} M_{f_i}(r)}{\log^{[q]} r}.$$

In order to refine the above growth scale, now we intend to introduce the definition of another growth indicator, called *relative* (p, q) *-th type* of entire algebroidal function with respect to another entire algebroidal function in the light of their *index-pair* which is as follows:

Definition 1.2. Let $f_i's$ $(0 \leq i \leq k-1)$ be entire functions with index-pair (m_1, q) and G be any entire algebroidal function with index-pair (m_2, p) where $m_1 = m_2 = m$ and p, q, m are all positive integers such that $m \geq \max\{p, q\}$. The relative (p, q) th type of entire functions f_i with respect to the entire algebroidal function G having finite positive relative (p, q) th order $\rho_G^{(p,q)}(f_i)$ $\left(0 < \rho_G^{(p,q)}(f_i) < \infty\right)$ is defined as :

$$\sigma_G^{(p,q)}(f_i) = \inf\left\{\begin{array}{c} \phi > 0 : M_{f_i}(r) < M_G\left[\exp^{[p-1]}\left(\phi(\log^{[q-1]} r)^{\rho_G^{(p,q)}(F)}\right)\right] \\ \text{for all } r > r_0(\phi) > 0 \end{array}\right\}$$
$$= \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}}.$$

The above definition can alternatively defined in the following manner:

Definition 1.3. Let $f_i's$ $(0 \leq i \leq k-1)$ be entire functions having finite positive *relative* (p,q) *-th order* $\rho_G^{(p,q)}(f_i)$ $\left(0 < \rho_G^{(p,q)}(F) < \infty\right)$with respect to an entire algebroidal function G defined as earlier where p and q are any two positive integers. Then the *relative* (p,q) *-th type* $\sigma_G^{(p,q)}(f_i)$ of entire functions f_i with respect to the entire algebroidal function G is defined as: The integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \; (r_0 > 0)$$

converges for $t > \sigma_G^{(p,q)}(f_i)$and diverges for $t < \sigma_G^{(p,q)}(f_i)$.

Analogously, to determine the relative growth of two entire functions having same non zero finite *relative* (p,q) *-th lower order* with respect to an entire algebroidal function, one can introduce the definition of *relative* (p,q) *-th weak type* of entire functionf_i with respect to an entire algebroidal function G of finite positive *relative* (p,q) *-th lower order* $\lambda_G^{(p,q)}(f_i)$ in the following way:

Definition 1.4. Let $f_i's$ $(i = 0,1,2,\ldots,k-1)$ be entire functions having finite positive relative (p,q) th lower order $\lambda_G^{(p,q)}(f_i)$ $\left(a < \lambda_G^{(p,q)}(f_i) < \infty\right)$ with respect to an entire algebroidal function G where p and q are any two positive integers. Then the *relative* (p,q) *-th weak type* of entire functions f_i with respect to the entire algebroidal function G is defined as :

$$\tau_G^{(p,q)}(f_i) = \liminf_{r \to \infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}}.$$

The above definition can also be alternatively defined as:

Definition 1.5. Let $f_i's$ $(i = 0,1,2,\ldots,k-1)$ be entire functions having finite positive *relative* (p,q) *-th lower order* $\lambda_G^{(p,q)}(f_i)$ $\left(a < \lambda_G^{(p,q)}(f_i) < \infty\right)$ where p and q are any two positive integers. Then the *relative* (p,q) *-th weak type* $\tau_G^{(p,q)}(f_i)$ of entire functions f_i with respect to the entire algebroidal function G is defined as: The integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \; (r_0 > 0)$$

converges for $t > \tau_G^{(p,q)}(f_i)$ and diverges for $t < \tau_G^{(p,q)}(f_i)$.

Next we introduce the following two relative growth indicators which will also enable us for subsequent study.

Definition 1.6. Let f_i's be entire functions having finite positive relative (p,q) th order $\rho_G^{(p,q)}(f_i)$ $\left(a < \rho_G^{(p,q)}(f_i) < \infty\right)$ with respect to an entire algebroidal function

G where p and q are any two positive integers. Then the *relative* (p,q)*-th lower type* of entire functions f_i with respect to an entire algebroidal function G is defined as :

$$\overline{\sigma}_G^{(p,q)}(f_i) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} .$$

The above definition can alternatively be defined in the following manner:

Definition 1.7. Let f_i's be entire functions having finite positive relative $(p,q)-$ th order $\rho_G^{(p,q)}(f_i)$ $\left(a < \rho_G^{(p,q)}(f_i) < \infty\right)$ with respect to an entire algebroidal function G where p and q are any two positive integers. Then the *relative* (p,q) *-th lower type* $\overline{\sigma}_G^{(p,q)}(f_i)$ of entire function f_i with respect to tan entire algebroidal function G is defined as: The integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \; (r_0 > 0)$$

converges for $t > \overline{\sigma}_G^{(p,q)}(f_i)$ and diverges for $t < \overline{\sigma}_G^{(p,q)}(f_i)$.

Definition 1.8. Let f_i's be entire functions having finite positive relative (p,q)*-th lower order* $\lambda_G^{(p,q)}(f_i)$ $\left(a < \lambda_G^{(p,q)}(f_i) < \infty\right)$ and G be an entire algebroidal function . Then the growth indicator $\overline{\tau}_G^{(p,q)}(f_i)$ of an entire function $f_i$with respect to the entire algebroidal function G is defined as :

$$\overline{\tau}_G^{(p,q)}(f_i) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} .$$

The above definition can also be alternatively defined as:

Definition 1.9. Let f_i's be entire functions having finite positive *relative* (p,q) *-th lower order* $\lambda_G^{(p,q)}(f_i)$ $\left(a < \lambda_G^{(p,q)}(f_i) < \infty\right)$with respect to the entire algebroidal function G where p and q are any two positive integers. Then the growth indicator $\overline{\tau}_G^{(p,q)}(f_i)$ of entire function f_i with respect to the entire algebroidal function G is defined as: The integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \; (r_0 > 0)$$

converges for $t > \overline{\tau}_G^{(p,q)}(f_i)$ and diverges for $t < \overline{\tau}_G^{(p,q)}(f_i)$.

Now a question may arise about the equivalence of the definitions of *relative* (p,q) *-th type* and *relative* (p,q) *-th weak type* with their integral representations. In the present paper we would like to establish such equivalence of Definition 1.2 with Definition 1.3 and Definition 1.4 with Definition 1.5 and also investigate some growth properties related to *relative* (p,q) *-th type* and *relative* (p,q) *-th weak type* of entire function with respect to an entire algebroidal function.

2 Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 2.1. Let the integral $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\log^{[q-1]} r\right)^A\right]^{t+1}} dr\ (r_0 > 0)$ converges where $0 < A < \infty$. Then

$$\lim_{r\to\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^t} = 0 .$$

Proof. Since the integral $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\log^{[q-1]} r\right)^A\right]^{t+1}} dr\ (r_0 > 0)$ converges, then

$$\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr < \varepsilon, \text{ if } r_0 > R(\varepsilon) .$$

Therefore,

$$\int\limits_{r_0}^{\exp\left(\log^{[q-1]} r_0\right)^A + r_0} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr < \varepsilon .$$

Since $\log^{[p-2]} M_G^{-1} M_{f_i}(r)$ increases with r, so

$$\int\limits_{r_0}^{\exp\left(\log^{[q-1]} r_0\right)^A + r_0} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr \geq$$

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r_0)}{\left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]^{t+1}} \cdot \left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right] .$$

i.e., for all sufficiently large values of r,

$$\int\limits_{r_0}^{\exp\left(\log^{[q-1]} r_0\right)^A + r_0} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr \geq$$

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r_0)}{\left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]^{t}} ,$$

so that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r_0)}{\left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]^{t}} < \varepsilon \text{ if } r_0 > R(\varepsilon) .$$

$$i.e., \lim_{r\to\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^t} = 0.$$

This proves the lemma. □

3 Theorems

In this section we state the main results of this paper.

Theorem 3.1. Let f_i's $(i = 0, 1, 2, ...k-1)$ be entire functions having finite positive *relative* (p, q) *-th order* $\rho_G^{(p,q)}(f_i)$ $\left(0 < \rho_g^{(p,q)}(f) < \infty\right)$ and *relative* (p, q) *-th type* $\sigma_G^{(p,q)}(f_i)$with respect to an entire algebroidal function G as defined in the introductory section where p and q are any two positive integers. Then Definition 1.2 and Definition 1.3 are equivalent.

Proof. Let us considerf_i's $(i = 0, 1, 2, ...k-1)$ be entire functions and G be an entire algebroidal function such that $\rho_G^{(p,q)}(f_i)$ $\left(0 < \rho_G^{(p,q)}(f_i) < \infty\right)$ exists for any two positive integers p and q.

Case I. $\sigma_G^{(p,q)}(f_i) = \infty$.

Definition 1.2 ⇒ Definition 1.3.

As $\sigma_G^{(p,q)}(f_i) = \infty$, from Definition 1.2 we have for arbitrary positive C and for a sequence of values of r tending to infinity that

$$\log^{[p-1]} M_G^{-1} M_{f_i}(r) > C \cdot \left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}$$

$$i.e., \quad \log^{[p-2]} M_G^{-1} M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^C . \tag{1}$$

If possible, let the integral $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right]^{C+1}} dr$ $(r_0 > 0)$ be converge.

Then by Lemma 2.1,

$$\limsup_{r\to\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^C} = 0 .$$

So for all sufficiently large values of r,

$$\log^{[p-2]} M_G^{-1} M_{f_i}(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^C . \tag{2}$$

Therefore from (1) and (2) we arrive at a contradiction.

Hence $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{C+1}} dr \ (r_0 > 0)$ diverges whenever C is finite, which is the Definition 1.3.

Definition 1.3 $\Rightarrow$ Definition 1.2.

Let C be any positive number. Since $\sigma_G^{(p,q)}(f_i) = \infty$, from Definition 1.3, the divergence of the integral $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{C+1}} dr \ (r_0 > 0)$ gives for arbitrary positive ε and for a sequence of values of r tending to infinity

$$\log^{[p-2]} M_G^{-1} M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{C-\varepsilon}$$

$$i.e.,\ \log^{[p-1]} M_G^{-1} M_{f_i}(r) > (C-\varepsilon)\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)},$$

which implies that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} \geq G - \varepsilon\ .$$

Since $C > 0$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} = \infty\ .$$

$\square$

Thus Definition 1.2 follows.

Case II. $0 \leq \sigma_G^{(p,q)}(f_i) < \infty$.

Definition 1.2 $\Rightarrow$ Definition 1.3.

Subcase (A). $0 < \sigma_G^{(p,q)}(f_i) < \infty$.

Let f_i 's$(i = 0, 1, 2, ...k-1)$ be entire functions and G be an entire algebroidal function such that $0 < \sigma_G^{(p,q)}(f_i) < \infty$ exists for any two positive integers p and q. Then according to the Definition 1.2, for arbitrary positive ε and for all sufficiently large values of r, we obtain that

$$\log^{[p-1]} M_G^{-1} M_{f_i}(r) < \left(\sigma_G^{(p,q)}(f_i) + \varepsilon\right)\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}$$

$$i.e.,\ \log^{[p-2]} M_G^{-1} M_{f_i}(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)+\varepsilon}$$

$$i.e.,\ \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^t} < \frac{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)+\varepsilon}}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^t}$$

$$i.e.,\ \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^t} < \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t-\sigma_G^{(p,q)}(f_i)+\varepsilon}}.$$

Therefore $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr\ (r_0 > 0)$ converges for $t > \sigma_G^{(p,q)}(f_i)$.

Again by Definition 1.2, we obtain for a sequence values of r tending to infinity that

$$\log^{[p-1]} M_G^{-1} M_{f_i}(r) > \left(\sigma_G^{(p,q)}(f_i) - \varepsilon\right)\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}$$

$$i.e.,\ \log^{[p-2]} M_G^{-1} M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)-\varepsilon}. \qquad (3)$$

So for $t < \sigma_G^{(p,q)}(f_i)$, we get from (3) that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^t} > \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t-\ \sigma_G^{(p,q)}(f_i)-\varepsilon}}.$$

Therefore $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr\ (r_0 > 0)$ diverges for $t < \sigma_G^{(p,q)}(f_i)$.

Hence $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr\ (r_0 > 0)$ converges for $t > \sigma_G^{(p,q)}(f_i)$ and diverges for $t < \sigma_G^{(p,q)}(f_i)$.

Subcase (B). $\sigma_G^{(p,q)}(f_i) = 0.$

When $\sigma_G^{(p,q)}(f_i) = 0$ for any two positive integers p and q , Definition 1.2 gives for all sufficiently large values of r that

$$\frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} < \varepsilon \ .$$

Then as before we obtain that $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\ \left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right]^{t+1}} dr\,(r_0 > 0)$ converges for $t > 0$ and diverges for $t < 0$.

Thus combining Subcase (A) and Subcase (B), Definition 1.3 follows.

Definition 1.3 $\Rightarrow$ Definition 1.2.

From Definition 3 and for arbitrary positive ε,the integral

$$\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)+\varepsilon+1}} dr\,(r_0 > 0)$$

converges. Then by Lemma 2.1, we get that

$$\limsup_{r\to\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)+\varepsilon}} = 0 \ .$$

So we obtain all sufficiently large values of r that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)+\varepsilon}} < \varepsilon$$

$$i.e.,\ \log^{[p-2]} M_G^{-1} M_{f_i}(r) < \varepsilon \cdot \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)+\varepsilon}$$

$$i.e.,\ \log^{[p-1]} M_G^{-1} M_{f_i}(r) < \log \varepsilon + \left(\sigma_G^{(p,q)}(f_i) + \varepsilon\right)\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}$$

$$i.e.,\ \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} \leq \sigma_G^{(p,q)}(f_i) + \varepsilon \ .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} \leq \sigma_G^{(p,q)}(f_i) \,. \tag{4}$$

On the other hand, the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)-\varepsilon+1}} dr \; (r_0 > 0)$$

implies that there exists a sequence of values of r tending to infinity such that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)-\varepsilon+1}} > \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{1+\varepsilon}}$$

$$\begin{aligned}
&i.e., \;\; \log^{[p-2]} M_G^{-1} M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{\sigma_G^{(p,q)}(f_i)-2\varepsilon} \\
&i.e., \;\; \log^{[p-1]} M_G^{-1} M_{f_i}(r) > \left(\sigma_G^{(p,q)}(f_i) - 2\varepsilon\right)\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right) \\
&i.e., \;\; \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} > \left(\sigma_G^{(p,q)}(f_i) - 2\varepsilon\right) .
\end{aligned}$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} \geq \sigma_G^{(p,q)}(f_i) \;\; . \tag{5}$$

So from (4) and (5) , we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} = \sigma_G^{(p,q)}(f_i) .$$

This proves the theorem.

Remark 3.2. The similar results follows if we consider an entire algebroidal function F and the entire functions $g_i\,(i = 0, 1, 2, ..., k-1)$ instead of G and f_i respectively in Definition 1.2 and Definition 1.3 **.**

Theorem 3.3. Let $f_i's$ $(i = 0, 1, 2, ..., k-1)$ be entire functions having finite positive *relative* (p, q) *-th lower order* $\lambda_G^{(p,q)}(f_i)$ $\left(0 < \lambda_G^{(p,q)}(f_i) < \infty\right)$ and *relative* (p, q) *-th weak type* $\tau_G^{(p,q)}(f_i)$ with respect to an algebroidal functions G where p and q are any two positive integers. Then Definition 1.4 and Definition 1.5 are equivalent.

Proof. Let us consider $f_i's$ be entire function and G be an entire algebroidal function such that $\lambda_G^{(p,q)}(f_i)\left(0<\lambda_G^{(p,q)}(f_i)<\infty\right)$ exists for any two positive integers p and q.

Case I. $\tau_G^{(p,q)}(f_i)=\infty$.

Definition 1.4 $\Rightarrow$ Definition 1.5.

As $\tau_G^{(p,q)}(f_i)=\infty$, from Definition 1.4 we obtain for arbitrary positive C and for all sufficiently large values of r that

$$\log^{[p-1]} M_G^{-1} M_{f_i}(r) > C\cdot\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}$$

$$i.e.,\ \log^{[p-2]} M_G^{-1} M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{C}. \tag{6}$$

Now if possible let the integral $\int_{r_0}^{\infty}\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{C+1}}dr\ (r_0>0)$ be converge.

Then by Lemma 2.1,

$$\liminf_{r\to\infty}\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{C}}=0\ .$$

So for a sequence of values of r tending to infinity we get that

$$\log^{[p-2]} M_G^{-1} M_{f_i}(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{C}. \tag{7}$$

Therefore from (6) and (7), we arrive at a contradiction.
Hence $\int_{r_0}^{\infty}\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{C+1}}dr\ (r_0>0)$ diverges whenever G is finite, which is Definition 1.5.

Definition 1.5 $\Rightarrow$ Definition 1.4.

Let C be any positive number. Since $\tau_G^{(p,q)}(f_i)=\infty$, from Definition 1.5, the divergence of the integral $\int_{r_0}^{\infty}\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{C+1}}dr\ (r_0>0)$ gives for arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[p-2]} M_G^{-1} M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{C-\varepsilon}$$

$$i.e.,\ \log^{[p-1]} M_G^{-1} M_{f_i}(r) > (C-\varepsilon)\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)},$$

which implies that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} \geq C-\varepsilon\ .$$

Since $C>0$ is arbitrary, it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} = \infty\ .$$

Thus Definition 1.4 follows.

Case II. $0\leq\tau_G^{(p,q)}(f_i)<\infty$.

Definition 1.4 $\Rightarrow$ Definition 1.5.

Subcase (C). $0<\tau_G^{(p,q)}(f_i)<\infty$.

Let f_i's$(i=0,1,2,....,k-1)$ be entire functions and G be an entire algebroidal function such that $0<\tau_G^{(p,q)}(f_i)<\infty$ exists for any two positive integers p and q. Then according to Definition 1.4, for a sequence of values of r tending to infinity, we get that

$$\log^{[p-1]} M_G^{-1}M_{f_i}(r) < \left(\tau_G^{(p,q)}(f_i)+\varepsilon\right)\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}$$

$$i.e.,\ \ \log^{[p-2]} M_G^{-1}M_{f_i}(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)+\varepsilon}$$

$$i.e.,\ \ \frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t}} < \frac{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)+\varepsilon}}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t}}$$

$$i.e.,\ \ \frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t}} < \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t-\ \tau_G^{(p,q)}(f_i)+\varepsilon}}\ .$$

Therefore $\int\limits_{r_0}^{\infty}\frac{\log^{[p-2]} M_G^{-1}M_{f_i}(r)}{\left[\exp\ \left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}}dr\ (r_0>0)$ converges for $k>\tau_G^{(p,q)}(f_i)$. □

Again by Definition 1.4, we obtain for all sufficiently large values of r that

$$\log^{[p-1]} M_G^{-1} M_{f_i}(r) > \left(\tau_G^{(p,q)}(f_i) - \varepsilon\right)\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}$$

$$i.e., \ \log^{[p-2]} M_G^{-1} M_{f_i}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)-\varepsilon}. \tag{8}$$

So for $k < \tau_G^{(p,q)}(f_i)$, we get from (8) that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t}} > \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t- \ \tau_G^{(p,q)}(f_i)-\varepsilon}}.$$

Therefore $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \ (r_0 > 0)$ diverges for $t < \tau_G^{(p,q)}(f_i)$.

Hence $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \ (r_0 > 0)$ converges for $t > \tau_G^{(p,q)}(f_i)$ and diverges for $t < \tau_G^{(p,q)}(f_i)$.

Subcase (D). $\tau_G^{(p,q)}(f_i) = 0$.

When $\tau_G^{(p,q)}(f_i) = 0$ for any two positive integers p and q , Definition 1.4 gives for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} < \varepsilon \, .$$

Then as before we obtain that $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \ (r_0 > 0)$ converges for $t > 0$ and diverges for $t < 0$.

Thus combining Subcase(C) and Subcase(D) , Definition 1.5 follows.

Definition 1.5 ⇒ Definition 1.4.

From Definition 1.5 and for arbitrary positive ε, the integral

$$\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)+\varepsilon+1}} dr \ (r_0 > 0)$$

converges. Then by Lemma 2.1, we get that

$$\liminf_{r\to\infty}\frac{\log^{[p-2]}M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)+\varepsilon}}=0\ .$$

So we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-2]}M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)+\varepsilon}}<\varepsilon$$

$$i.e.,\ \log^{[p-2]}M_G^{-1}M_{f_i}(r)<\varepsilon\cdot\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)+\varepsilon}$$

$$i.e.,\ \log^{[p-1]}M_G^{-1}M_{f_i}(r)<\log\varepsilon+\left(\tau_G^{(p,q)}(f_i)+\varepsilon\right)\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log^{[p-1]}M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}}\le\tau_G^{(p,q)}(f_i)+\varepsilon\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}}\le\tau_G^{(p,q)}(f_i)\ . \tag{9}$$

On the other hand, the divergence of the integral

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]}M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)-\varepsilon+1}}dr\ (r_0>0)$$

implies for all sufficiently large values of r that

$$\frac{\log^{[p-2]}M_G^{-1}M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)-\varepsilon+1}}>\frac{1}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{1+\varepsilon}}$$

$$i.e.,\ \log^{[p-2]}M_G^{-1}M_{f_i}(r)>\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{\tau_G^{(p,q)}(f_i)-2\varepsilon}$$

$$i.e.,\ \log^{[p-1]}M_G^{-1}M_{f_i}(r)>\left(\tau_G^{(p,q)}(f_i)-2\varepsilon\right)\left(\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}\right)$$

$$i.e.,\ \frac{\log^{[p-1]}M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}r\right)^{\lambda_G^{(p,q)}(f_i)}}>\left(\tau_G^{(p,q)}(f_i)-2\varepsilon\right)\ .$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} \geq \tau_G^{(p,q)}(f_i)\ . \tag{10}$$

So from (9) and (10) , we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} = \tau_G^{(p,q)}(f_i)\ .$$

This proves the theorem.

Now we state the following two theorems without their proofs as those can easily be carried out with help of Lemma 2.1 and in the line of Theorem 3.1 and Theorem 3.3 respectively.

Theorem 3.4. Let f_i 's be entire functions having finite positive *relative* (p,q)-*th order* $\rho_G^{(p,q)}(f_i)$ $\left(0<\rho_G^{(p,q)}(f_i)<\infty\right)$ and *relative* (p,q) *-th lower type* $\overline{\sigma}_G^{(p,q)}(f_i)$ with respect to an entire algebroidal function G where p and q are any two positive integers. Then Definition 1.6 and Definition 1.7 are equivalent.

Theorem 3.5. Let f_i's be entire functions having finite positive relative (p,q)-*th lower order* $\lambda_G^{(p,q)}(f_i)$ $\left(a<\lambda_G^{(p,q)}(f_i)<\infty\right)$ and the growth indicator $\overline{\tau}_G^{(p,q)}(f_i)$ with respect to an entire algebroidal function G where p and q are any two positive integers. Then Definition 1.8 and Definition 1.9 are equivalent.

Theorem 3.6. Let f_i's be entire functions and G be an entire algebroidal function with $0<\lambda_G^{(p,q)}(f_i)\leq\rho_G^{(p,q)}(f_i)<\infty$ where p and q are any two positive integers. Then

$$(i)\ \ \sigma_G^{(p,q)}(f_i)=\limsup_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}(r)}{\left[\log^{[q-1]} M_{f_i}^{-1}(r)\right]^{\rho_G^{(p,q)}(f_i)}},$$

$$(ii)\ \ \overline{\sigma}_G^{(p,q)}(f_i)=\liminf_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}(r)}{\left[\log^{[q-1]} M_{f_i}^{-1}(r)\right]^{\rho_G^{(p,q)}(f_i)}},$$

$$(iii)\ \ \tau_G^{(p,q)}(f_i)=\liminf_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}(r)}{\left[\log^{[q-1]} M_{f_i}^{-1}(r)\right]^{\lambda_G^{(p,q)}(f_i)}}\mathit{and}$$

$$(iv)\ \ \overline{\tau}_G^{(p,q)}(f_i)=\limsup_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}(r)}{\left[\log^{[q-1]} M_{f_i}^{-1}(r)\right]^{\lambda_G^{(p,q)}(f_i)}}\ .$$

Proof. Taking $M_{f_i}(r)=R$, the theorem follows from the definitions of $\sigma_G^{(p,q)}(f_i)$, $\overline{\sigma}_G^{(p,q)}(f_i)$, $\tau_G^{(p,q)}(f_i)$and $\overline{\tau}_g^{(p,q)}(f)$ respectively.

In the following theorem we obtain a relationship among $\sigma_G^{(p,q)}(f_i)$, $\overline{\sigma}_G^{(p,q)}(f_i)$, $\overline{\tau}_G^{(p,q)}(f_i)$ and $\tau_G^{(p,q)}(f_i)$. □

Theorem 3.7. Let f_i's be entire functions such that f_i is of regular relative $(p,q)-$ growth with respect to an entire algebroidal function G i.e., $\rho_G^{(p,q)}(f_i) = \lambda_G^{(p,q)}(f_i)$ $\left(0 < \lambda_G^{(p,q)}(f_i) = \rho_G^{(p,q)}(f_i) < \infty\right)$ where p and q are any two positive integers, then the following quantities

$$(i)\ \sigma_G^{(p,q)}(f_i),\quad (ii)\ \tau_G^{(p,q)}(f_i),\quad (iii)\ \overline{\sigma}_G^{(p,q)}(f_i) \text{ and } \quad (iv)\ \overline{\tau}_G^{(p,q)}(f_i)$$

are all equivalent.

From Definition 1.5, it follows that the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \; (r_0 > 0)$$

converges for $t > \tau_G^{(p,q)}(f_i)$and diverges for $t < \tau_G^{(p,q)}(f_i)$.

On the other hand, Definition 1.3 implies that the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \; (r_0 > 0)$$

converges for $t > \sigma_G^{(p,q)}(f_i)$ and diverges for $t < \sigma_G^{(p,q)}(f_i)$.

(i)⇒(ii).

Now it is obvious that all the quantities in the expression

$$\left[\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} - \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}}\right]$$

are of non negative type. So

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} - \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \; (r_0 > 0) \geq 0$$

$$i.e., \int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \geq$$

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}\right)\right]^{t+1}} dr \text{ for } r_0 > 0 \ .$$

$$i.e.,\ \tau_G^{(p,q)}(f_i) \geq \sigma_G^{(p,q)}(f_i) \ . \tag{11}$$

Further f_i 's are of regular relative (p,q) growth with respect to G. Therefore we get that

$$\sigma_G^{(p,q)}(f_i) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}}$$

$$\geq \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} = \tau_G^{(p,q)}(f_i) \ . \tag{12}$$

Hence from (11) and (12), we obtain that

$$\sigma_G^{(p,q)}(f_i) = \tau_G^{(p,q)}(f_i) \ . \tag{13}$$

(ii)⇒(iii).

Since f_i 's are of regular relative (p,q) growth with respect to G i.e., $\rho_G^{(p,q)}(f_i) = \lambda_G^{(p,q)}(f_i)$ we get that

$$\tau_G^{(p,q)}(f_i) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} = \overline{\sigma}_G^{(p,q)}(f_i) \ .$$

(iii)⇒(iv).

In view of (13) and the condition $\rho_G^{(p,q)}(f_i) = \lambda_G^{(p,q)}(f_i)$, it follows that

$$\overline{\sigma}_G^{(p,q)}(f_i) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}}$$

$$i.e., \overline{\sigma}_G^{(p,q)}(f_i) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}}$$

$$i.e.,\ \overline{\sigma}_G^{(p,q)}(f_i) = \tau_G^{(p,q)}(f_i)$$

$$i.e.,\ \overline{\sigma}_G^{(p,q)}(f_i) = \sigma_G^{(p,q)}(f_i)$$

$$i.e.,\ \overline{\sigma}_G^{(p,q)}(f_i) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}}$$

$$i.e.,\ \overline{\sigma}_G^{(p,q)}(f_i) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}}$$

$$i.e.,\ \overline{\sigma}_G^{(p,q)}(f_i) = \overline{\tau}_G^{(p,q)}(f_i)\ .$$

(iv)⇒(i).

As f_i 's are of regular relative (p,q) growth with respect to G i.e., $\rho_G^{(p,q)}(f_i) = \lambda_G^{(p,q)}(f_i)$, we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_i)}} = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_i}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_i)}} = \sigma_G^{(p,q)}(f_i)\,.$$

Thus the theorem follows.

4 Conclusion

The results carried out in this present paper may be viewed from the angle of slowly changing functions as well as for the functions analytic in the unit disc and ploydisc.

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