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Certain Classes of Analytic Functions Associated with Conic Domains

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Abstract – In this paper, we define new subclasses of k-uniformly Janowski starlike and k-uniformly Janowski convex functions associated with m-symmetric points. The integral representations, convolution properties and sufficient conditions for the functions belong to this class are investigated.

Keywords – *Subordination, convolution, m-symmetric points.*

1. Introduction

Let A denote the class of functions f (z) of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $E = \{z : z \in C \text{ and } |z| < 1\}$. Furthermore S, represents class of all functions in A which are univalent in E. Sakaguchi [6] introduced a class S_s^* of functions starlike with respect to symmetric points, it consists of functions $f(z) \in S$ satisfying the inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad (z \in E). \tag{1.2}$$

Following him, many authors studied this class and its subclasses see [7, 8, 9].

Das and Singh [16] in 1977 extend the results of Sakaguchi to other class in E, namely convex functions with respect to symmetric points. Let C_s denote the class of convex functions with respect to symmetric points and satisfying the condition

$$\mathrm{Re}\left(\frac{(zf'(z))'}{f'(z) - f'(-z)}\right) > 0, \quad (z \in E).$$

It is also well known [16] that $f \in C_s$ if and only if $zf'(z) \in S_s^*$.

Chand and Singh [1] introduced a class S_s^m of functions starlike with respect to m-symmetric points, which consists of functions $f(z) \in S$, satisfying the inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f_m(z)}\right) > 0, \quad (z \in E), \tag{1.3}$$

where

$$f_m(z) = \frac{1}{m}\sum_{\mu=0}^{m-1} \varepsilon^{-\mu} f(\varepsilon^{\mu} z), \quad (\varepsilon^{\mu} = 1, m \in N). \tag{1.4}$$

From equation (1.4) we can write

$$f_m(z) = \frac{1}{m}\sum_{\mu=0}^{m-1} \varepsilon^{-\mu} f(\varepsilon^{\mu} z) = \frac{1}{m}\sum_{\mu=0}^{m-1} \varepsilon^{-\mu}\left(\varepsilon^{\mu} z + \sum_{n=2}^{\infty} a_n (\varepsilon^{\mu} z)^n\right)$$

$$= z + \sum_{n=2}^{\infty} b_n a_n z^n \tag{1.5}$$

where

$$b_n = \frac{1}{m}\sum_{\mu=0}^{m-1} \varepsilon^{(n-1)\mu} = \begin{cases} 1, & \mathrm{n} = lm+1, \\ 0, & \mathrm{n} \neq \mathrm{lm}+1, \end{cases} \tag{1.6}$$

where $l, m \in N;\ \mathrm{n} \geq 2, \varepsilon^{\mathrm{m}} = 1$.

Note that the accompanying characters follow directly from the above definition [10].

$$f_m(\varepsilon^{\mu} z) = \varepsilon^{\mu} f_m(z), \tag{1.7}$$

$$f_m'(\varepsilon^{\mu} z) = f_m(z) = \frac{1}{m}\sum_{\mu=0}^{m-1} f'(\varepsilon^{\mu} z), \quad (z \in \mathrm{E}). \tag{1.8}$$

Definition 1. For $f(z) \in A$ given by (1.1) and $g(z) \in A$ of the form

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n = (g * f)(z), \quad (z \in \mathrm{E}).$$

For two functions F(z) and G(z) analytic in E, we say that F(z) is subordinate to G(z) denoted by $F \prec G$ or $F(z) \prec G(z)$, if there exists an analytic function $w(z)$ with $|w(z)| < 1$ such that $F(z) = G(w(z))$. Furthermore if the function G(z) is univalent in E then we have the following equivalence [13,14,15]

$$F(z) \prec G(z) \Leftrightarrow F(0) = G(0) \text{ and } F(E) \subseteq G(E).$$

Definition 2. A function $p(z)$ is said to be in the class P[A,B], if it is analytic in E with $p(0) = 1$ and

$$p(z) \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1.$$

Geometrically, if a function p belongs to P [A,B], then it maps the open unit disc E onto the disk characterized by the domain

$$\Omega[A,B] = \left\{ w : \left| w - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \right\}.$$

The class P [A,B], is connected with the class P of functions with positive real part by the relation

$$p(z) \in P, \text{ if and only if } \frac{(A+1)p(z)-(A-1)}{(B+1)p(z)-(B-1)} \in P[A,B].$$

This class was presented by Janowski [2] and explored by a few creators. Kanas and Wisniowska [4,3] presented and examined the class k-ST of k-starlike functions and the relating class k-UCV of k-uniformly convex functions. These were characterized subject to the conic region k, Ω_k, $k \geq 0$, as

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

This domain represents the right half plane, a parabola, a hyperbola and an ellipse for k = 0, k = 1, 0 < k < 1 and k > 1 respectively. The external functions for these conic regions are

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z}, & k = 0, \\ 1 + \dfrac{2}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, & k = 1, \\ 1 + \dfrac{2}{1-k^2}\sinh^2\left\{\left(\dfrac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right\}, & 0 < k < 1, \\ 1 + \dfrac{2}{k^2-1}\sin\left(\dfrac{\pi}{2R(t)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{t}}} \dfrac{d(x)}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\right) + \dfrac{1}{k^2-1}, & k > 1, \end{cases} \tag{1.9}$$

where

$$u(z)=\frac{z-\sqrt{t}}{1-\sqrt{tx}}, \qquad (\mathrm{z}\in \mathrm{E}),$$

and $t\in(0,1)$ and z is chosen such that $k=\cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$. Here $R(t)$ is Legendre's complete elliptic integral of first kind and $R'(t)$ is the complementary integral of $R(t)$.

Following are the definitions of classes k - ST and k - UCV .

Definition 3. A function $f(z)\in$ A is said to be in the class k - ST , if and only if

$$\frac{\mathrm{zf}'(\mathrm{z})}{f(z)}\prec \mathrm{p_k}(\mathrm{z}), \quad (\mathrm{z}\in \mathrm{E}, \mathrm{k}\geq 0).$$

Definition 4. A function $f(z)\in$ A is said to be in the class k - UCV , if and only if

$$\frac{(\mathrm{zf}'(\mathrm{z}))'}{f'(z)}\prec \mathrm{p_k}(\mathrm{z}), \quad (\mathrm{z}\in \mathrm{E}, \mathrm{k}\geq 0).$$

The classes k - ST and k - UCV were further generalized by Shams et al, [11], to the $K\mathrm{D}(\mathrm{k},\beta)$ and $S\mathrm{D}(\mathrm{k},\beta)$, respectively subject to the conic domain $G(\mathrm{k},\beta)$, $k\geq 0$ and $0\leq\beta<1$ which is

$$G(k,\beta)=\{w:\mathrm{Re}w>k|w-1|+\beta\}.$$

Now using the concepts of Janowski functions and the conic regions, we defne the following

$$\mathrm{p}(\mathrm{z})\prec\frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \qquad k\geq 0,$$

where $p_k(z)$ is defined by (1.9) and $-1\leq \mathrm{B}<\mathrm{A}\leq 1$.

Geometrically, the function $\mathrm{p}(\mathrm{z})\in k-[\mathrm{A},\mathrm{B}]$, takes all values from the domain $\Omega_k[A,B]$, $-1\leq B<A\leq 1, \mathrm{k}\geq 0$ which is define as

$$\Omega_k[A,B]=\left\{w:\ \mathrm{Re}\left(\frac{(\mathrm{B}-1)w(z)-(A-1)}{(\mathrm{B}+1)w(z)-(A+1)}\right)>k\left|\frac{(B-1)w(z)-(A-1)}{(B+1)w(z)-(A+1)}-1\right|\right\},$$

or equivalently

$$\begin{aligned}\Omega_k[A,B]=\Big\{u+iv:&\left[(B^2-1)(u^2+v^2)-2(AB-1)u+(A^2-1)\right]^2\\ &>k^2\left[\left(-2(B+1)(u^2+v^2)+2(A+B+2)u-2(A+1)\right)^2+4(A-B)^2v^2\right]\Big\}.\end{aligned}$$

The domain $\Omega_k[A,B]$ retains the conic domain Ω_k inside the circular region defined by $\Omega[A,B]$. The impact of $\Omega[A,B]$ on the conic domain Ω_k changes the original shape of the conic regions. The ends of hyperbola and parabola gets closer to one another but never meet anywhere and the ellipse gets the oval shape. When $A \to 1, B \to -1$ the radiuses of the circular disk define by $\Omega[A,B]$ tends to infinity, consequently the arm of the hyperbola and parabolas expand to the oval terns into ellipse. We see that $\Omega_k[1,-1] = \Omega_k$, the conic domain define by Kanas and Wisniowska [3].

Definition 4. A function $f(z) \in \mathrm{A}$ is said to be in the class $k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$, $-1 \le B < A \le 1$, $k \ge 0$, if and only if

$$\mathrm{Re}\left(\frac{(\mathrm{B}-1)\frac{zf'(z)}{f_m(z)} - (A-1)}{(\mathrm{B}+1)\frac{zf'(z)}{f_m(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{zf'(z)}{f_m(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f_m(z)} - (A+1)} - 1\right|,$$

or equivalently

$$\frac{\mathrm{z}\mathrm{f}'(\mathrm{z})}{f_m(z)} \in k - \mathrm{P}[A,B], \tag{1.10}$$

where $f_m(z)$ is defined by (1.4).

Special Cases:

i). $k\text{-}\mathrm{ST}_s^{(1)}[\mathrm{A,B}] = k\text{-}\mathrm{ST}[\mathrm{A,B}]$, we have the well known class presented and studied in [5].
ii). $0\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}] = \mathrm{S}_s^{(m)}[\mathrm{A,B}]$, see [10].
iii). $k\text{-}\mathrm{ST}_s^{(1)}[1,-1] = k\text{-}\mathrm{ST}$. For this we refer to [4].
iv). $k\text{-}\mathrm{ST}_s^{(1)}[1-2\beta,-1] = \mathrm{SD}[\mathrm{k},\beta,]$, we have the well known class presented and studied in [11].
v). $0\text{-}\mathrm{ST}_1^{(1)}[\mathrm{A,B}] = \mathrm{S}^*[\mathrm{A,B}]$, we have the well known class presented and studied in [2].

Definition 4. A function $f(z) \in \mathrm{A}$ is said to be in the class $k\text{-}\mathrm{UCV}_s^{(m)}[\mathrm{A,B}]$, $k \ge 0$, $-1 \le B < A \le 1$, if and only if

$$\mathrm{Re}\left(\frac{(\mathrm{B}-1)\frac{(zf'(z))'}{f_m'(z)} - (A-1)}{(\mathrm{B}+1)\frac{(zf'(z))'}{f_m'(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{(zf'(z))'}{f_m'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f_m'(z)} - (A+1)} - 1\right|,$$

or equivalently

$$\frac{(zf'(z))'}{f_m'(z)} \in k - \mathrm{P}[A,B], \qquad (1.11)$$

where $f_m(z)$ is defined by (1.4).

Special Cases:

i). $k\text{ - UCV}_s^{(1)}[\mathrm{A,B}] = k\text{ - UCV}[\mathrm{A,B}],$ we have the class introduced and studied in [5].
ii). $k\text{ - UCV}_s^{(1)}[1,\text{-}1] = k\text{ - UCV},$ and this is well known class introduced and studied in [3].
iv). $k\text{ - UCV}_s^{(1)}[1\text{ - }2\beta,\text{-}1] = K\mathrm{D}[\mathrm{k},\beta,],$ see [11].
v). $0\text{ - UCV}_s^{(1)}[\mathrm{A,B}] = C[\mathrm{A,B}],$ we have the well known class introduced and studied in [2].

It is easy to see that:

$$\mathrm{f} \in k\text{ - UCV}_s^{(m)}[\mathrm{A,B}] \Leftrightarrow zf' \in k\text{ - ST}_s^{(m)}[\mathrm{A,B}].$$

2. Main Results

Integral representation. First we give two meaningful conclusions about the classes $k\text{ - ST}_s^{(m)}[\mathrm{A,B}]$ and $k\text{ - UCV}_s^{(m)}[\mathrm{A,B}]$.

Theorem 1. Let $f(z) \in k\text{ - ST}_s^{(m)}[\mathrm{A,B}]$. Then $f_m(z) \in k\text{ - ST}[\mathrm{A,B}] \subseteq k\text{ - ST} \subseteq \mathrm{S}.$

Proof. For $f(z) \in k\text{ - ST}_s^{(m)}[\mathrm{A,B}],$ we can obtain

$$\frac{\mathrm{zf}'(\mathrm{z})}{f_m(z)} \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \qquad (z \in \mathrm{E}). \qquad (2.1)$$

Substituting z by $\varepsilon^{\mu} z$ respectively $(\mu = 0,1,2,...m-1)$, we have

$$\frac{\varepsilon^{\mu} z\mathrm{f}'(\varepsilon^{\mu} z)}{f_m(\varepsilon^{\mu} z)} \prec \frac{(A+1)p_k(\varepsilon^{\mu} z)-(A-1)}{(B+1)p_k(\varepsilon^{\mu} z)-(B-1)} \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad (z \in \mathrm{E}). \quad (2.2)$$

By definition of $f_m(z)$ and $\varepsilon = \exp\left(\frac{2\pi i}{m}\right)$, we know that $\varepsilon^{-\mu} f_m(\varepsilon^{\mu} z) = f_m(z)$. Then equation (2.2), becomes

$$\frac{z\mathrm{f}'(\varepsilon^{\mu} z)}{f_m(z)} \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad (z \in \mathrm{E}). \qquad (2.3)$$

Let $(\mu = 0,1,2,...m-1)$ in (2.3), respectively and sum them to get

$$\frac{z\mathrm{f}_\mathrm{m}'(z)}{f_m(z)} \prec \frac{1}{m}\sum_{\mu=0}^{m-1}\frac{z\mathrm{f}'(\varepsilon^{\mu} z)}{f_m(z)} \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad (z \in \mathrm{E}).$$

Thus $f_m(z) \in k\text{-ST}[A,B] \subseteq S$.

Putting k = 0 in Theorem 1, we can obtain Corollary 1, below which is comparable to the result obtained by Kwon and Sim [10].

Corollary 1. Let $f(z) \in \mathrm{ST}_s^{(m)}[A,B]$. Then $f_m(z) \in \mathrm{ST}[A,B] \subseteq k\text{-ST} \subseteq S$.

Theorem 2. Let $f(z) \in k\text{-UCV}_s^{(m)}[A,B]$. Then $f_m(z) \in k\text{-UCV}[A,B] \subseteq S$.

Proof. The proof of Theorem 2 is similar to that of Theorem 1 so the details are omitted.

Now we give the integral representations of the functions belonging to the classes $k\text{-ST}_s^{(m)}[A,B]$ and $k\text{-UCV}_s^{(m)}[A,B]$.

Theorem 3. Let $f(z) \in k\text{-ST}_s^{(m)}[A,B]$. Then

$$f_m(z) = z \cdot \left\{ \exp(A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\varepsilon^\mu z} \frac{(p_k(w(t))-1)}{t(B+1)p_k(w(t))-(B-1)} dt \right\}, \tag{2.4}$$

where $w(z)$ analytic function E, with $w(0)=0$ and $|w(z)|<1$.

Proof. Let $f(z) \in k\text{-ST}_s^{(m)}[A,B]$, from definition of the subordination, we can have

$$\frac{zf'(z)}{f_m(z)} \prec \frac{(A+1)p_k(w(z))-(A-1)}{(B+1)p_k(w(z))-(B-1)}, \quad (z \in E), \tag{2.5}$$

where $w(z)$ analytic function E, with $w(0)=0$ and $|w(z)|<1$. Substituting z by $\varepsilon^\mu z$ respectivelty $(\mu = 0,1,2,...m-1)$, we have

$$\frac{zf'(\varepsilon^\mu z)}{\varepsilon^{-\mu} f_m(\varepsilon^\mu z)} = \frac{(A+1)p_k(w(\varepsilon^\mu z))-(A-1)}{(B+1)p_k(w(\varepsilon^\mu z))-(B-1)}, \quad (z \in E). \tag{2.6}$$

For $(\mu = 0,1,2,...m-1)$, $z \in E$. Using the equalities (1.7) and (1.8) we have

$$\frac{zf_m'(z)}{f_m(z)} = \frac{1}{m} \sum_{\mu=0}^{m-1} \frac{(A+1)p_k(w(\varepsilon^\mu z))-(A-1)}{(B+1)p_k(w(\varepsilon^\mu z))-(B-1)}, \quad (z \in E). \tag{2.7}$$

or equivalently

$$\frac{f_m'(z)}{f_m(z)} - \frac{1}{z} = \frac{1}{m} \sum_{\mu=0}^{m-1} \frac{(A-B)(p_k(w(\varepsilon^\mu z))-1)}{z((B+1)p_k(w(\varepsilon^\mu z))-(B-1))}, \quad (z \in E). \tag{2.8}$$

Integrating equality (2.8) , we have

$$\log\frac{f_m(z)}{z}=(A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^z\frac{\left(p_k\left(w\left(\varepsilon^\mu\zeta\right)\right)-1\right)}{\zeta\left((B+1)p_k\left(w\left(\varepsilon^\mu\zeta\right)\right)-(B-1)\right)}d\zeta$$
$$=(A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z}\frac{\left(p_k(w(t))-1\right)}{t\left((B+1)p_k(w(t))-(B-1)\right)}dt.$$

Therefore arranging equality (2.9) for $f_m(z)$ we can obtain

$$f_m(z)=z\cdot\left\{\exp(A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z}\frac{\left(p_k(w(t))-1\right)}{t\left((B+1)p_k(w(t))-(B-1)\right)}dt\right\},$$

and so the proof of Theorem 3 is complete.

Putting m = 1, in Theorem 3, we can obtain Corollary 2.

Corollary 2. Let $f(z)\in k$ - ST[A,B]. Then

$$f(z)=z\cdot\left\{\exp(A-B)\int_0^z\frac{\left(p_k(w(t))-1\right)}{t(B+1)p_k(w(t))-(B-1)}dt\right\}, \tag{2.11}$$

where $\mathrm{w}(\mathrm{z})$ analytic function E, with $\mathrm{w}(0)=0$ and $|w(\mathrm{z})|<1$.

Putting k = 0, in Theorem 3, we can obtain Corollary 3, below which is comparable to the result obtained by Kwon and Sim [10].

Corollary 3. Let $f(z)\in \mathrm{ST}_{\mathrm{s}}^{(\mathrm{m})}[\mathrm{A,B}]$. Then

$$f_m(z)=z\cdot\left\{\exp(A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z}\frac{w(t)}{t(1+Bw(t))}dt\right\}, \tag{2.12}$$

where $\mathrm{w}(\mathrm{z})$ analytic function E, with $\mathrm{w}(0)=0$ and $|w(\mathrm{z})|<1$.

Putting m = 1, A = 1 and B = -1 in Theorem 3, we can obtain Corollary 4.

Corollary 4. Let $f(z)\in k$ - ST. Then

$$f(z)=z\cdot\left\{\exp\int_0^z\left(p_k(w(t))-1\right)dt\right\}, \tag{2.13}$$

where $\mathrm{w}(\mathrm{z})$ analytic function E, with $\mathrm{w}(0)=0$ and $|w(\mathrm{z})|<1$.

Theorem 4. Let $f(z) \in k\text{ - UCV}_{\mathrm{s}}^{(\mathrm{m})}[\mathrm{A,B}]$. Then

$$f_m(z) = \int_0^z \exp\left((A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z} \frac{(p_k(w(t))-1)}{t(B+1)p_k(w(t))-(B-1)}dt \right)d\zeta. \qquad (2.14)$$

where $\mathrm{w}(\mathrm{z})$ analytic function E, with $\mathrm{w}(0)=0$ and $|w(\mathrm{z})|<1$.

Proof. The proof of Theorem 4 is similar to that of Theorem 3 so the details are omitted. ▪

Theorem 4. Let $f(z) \in k\text{ - ST}_{\mathrm{s}}^{(\mathrm{m})}[\mathrm{A,B}]$. Then

$$f(z) = \int_0^z \exp\left((A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z} \frac{(p_k(w(t))-1)}{t(B+1)p_k(w(t))-(B-1)}dt \right)$$
$$\times\left(\frac{((A+1)p_k(w(\zeta))-(A-1))}{(B+1)p_k(w(\zeta))-(B-1)} \right)d\zeta. \qquad (2.15)$$

where $\mathrm{w}(\mathrm{z})$ analytic function E, with $\mathrm{w}(0)=0$ and $|w(\mathrm{z})|<1$.

Proof. Let $f(z) \in k\text{ - ST}_{\mathrm{s}}^{(\mathrm{m})}[\mathrm{A,B}]$. Then from equalities (2.4) and (2.5) we have

$$f'(z) = \left(\frac{f_m(z)}{z}\right)\left(\frac{((A+1)p_k(w(\zeta))-(A-1))}{(B+1)p_k(w(\zeta))-(B-1)} \right)$$
$$= \exp\left((A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z} \frac{(p_k(w(t))-1)}{t(B+1)p_k(w(t))-(B-1)}dt \right)$$
$$\times\left(\frac{((A+1)p_k(w(\zeta))-(A-1))}{(B+1)p_k(w(\zeta))-(B-1)} \right)d\zeta, \qquad (2.16)$$

Integrating the equality (2.16) , we have

$$f(z) = \int_0^z \exp\left((A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z} \frac{(p_k(w(t))-1)}{t(B+1)p_k(w(t))-(B-1)}dt \right)$$
$$\times\left(\frac{((A+1)p_k(w(\zeta))-(A-1))}{(B+1)p_k(w(\zeta))-(B-1)} \right)d\zeta.$$

and so the proof of Theorem 5 is completed.

Putting k = 0, in Theorem 5, we can obtain Corollary 5, below which is comparable to the result obtained by Kwon and Sim [10].

Corollary 5. Let $f(z) \in k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$. Then

$$f(z)=\int_0^z \exp\left((A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu z}\frac{w(t)}{t(1+Bw(t))}dt\right)\left(\frac{1+Aw(\zeta)}{1+Bw(\zeta)}\right)d\zeta, \tag{2.17}$$

where $\mathrm{w}(\mathrm{z})$ analytic function E, with $\mathrm{w}(0)=0$ and $|w(\mathrm{z})|<1$.

Theorem 4. Let $f(z) \in k\text{-}\mathrm{USV}_s^{(m)}[\mathrm{A,B}]$. Then

$$f_m(z)=\int_0^z\left\{\frac{1}{\zeta}\int_0^\zeta \exp\left((A-B)\frac{1}{m}\sum_{\mu=0}^{m-1}\int_0^{\varepsilon^\mu \zeta}\frac{(p_k(w(t))-1)}{t(B+1)p_k(w(t))-(B-1)}dt\right)\right.$$
$$\left.\times\left(\frac{((A+1)p_k(w(\zeta))-(A-1))}{(B+1)p_k(w(\zeta))-(B-1)}\right)d\zeta d\xi.\right.$$

where $\mathrm{w}(\mathrm{z})$ analytic function E, with $\mathrm{w}(0)=0$ and $|w(\mathrm{z})|<1$.

Convolution conditions: In this section, we provide the convolutions conditions for the classes $k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$ and $k\text{-}\mathrm{UCV}_s^{(m)}[\mathrm{A,B}]$.

Theorem 5. A function $f(z) \in k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$, if and only if

$$\frac{1}{z}\left\{f(z)*\left(\frac{z}{(1-z)^2}\left((B+1)p_k\left(e^{i\vartheta}\right)-(B-1)\right)-\left((A+1)p_k\left(e^{i\vartheta}\right)-(A-1)\right)h(z)\right)\right\}\neq 0, \tag{2.18}$$

for all $z\in \mathrm{E}$ and $0\le \vartheta<2\pi$, where h(z) is given by (2.24).

Proof. Assume that $f(z) \in k\text{-}\mathrm{ST}_s^{(m)}[\mathrm{A,B}]$, then we have

$$\frac{\mathrm{z}f'(\mathrm{z})}{f_m(z)}\prec\frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad (z\in \mathrm{E}), \tag{2.19}$$

if and only if

$$\frac{\mathrm{z}f'(\mathrm{z})}{f_m(z)}\neq\frac{(A+1)p_k\left(e^{i\vartheta}\right)-(A-1)}{(B+1)p_k\left(e^{i\vartheta}\right)-(B-1)}, \quad (z\in \mathrm{E}). \tag{2.20}$$

for all $z\in \mathrm{E}$ and $0\le \vartheta<2\pi$. The condition (2.20), can be written as

$$\frac{1}{z}\left\{zf'(z)\left(\left((B+1)p_k\left(e^{i\vartheta}\right)-(B-1)\right)-f_m(z)\left((A+1)p_k\left(e^{i\vartheta}\right)-(A-1)\right)h(z)\right)\right\}\neq 0, \tag{2.21}$$

On the other hand it is well known that

$$zf'(z) = f(z) * \frac{z}{(1-z)^2}, \quad (z \in \mathrm{E}). \tag{2.22}$$

And from the definition of $f_m(z)$ we have

$$f_m(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (f * h)(z), \tag{2.23}$$

where

$$h(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{2.24}$$

where b_n is given by (1.6). Substituting (2.22) and (2.23) in (2.21), we can get (2.18). This completes the proof of the Theorem 7.

Putting m = 1, in Theorem 7, we can obtain Corollary 6.

Corollary 6. A function $f(z) \in k\text{-}\mathrm{ST}[\mathrm{A},\mathrm{B}]$, if and only if

$$\frac{1}{z}\left\{ f(z) * \left(\frac{z}{(1-z)^2}\left(1 + Be^{i\vartheta}\right) - \frac{z}{(1-z)}\left(1 + Ae^{i\vartheta}\right) \right) \right\} \neq 0, \tag{2.25}$$

for all $z \in \mathrm{E}$.

Putting k = 0, in Theorem 7, we can obtain Corollary 7.

Corollary 7. A function $f(z) \in \mathrm{ST}_s^{(m)}[\mathrm{A},\mathrm{B}]$, if and only if

$$\frac{1}{z}\left\{ f(z) * \left(\frac{z}{(1-z)^2}\left(1 + Be^{i\vartheta}\right) - h(z)\left(1 + Ae^{i\vartheta}\right) \right) \right\} \neq 0, \tag{2.26}$$

for all $z \in \mathrm{E}$ and $0 \leq \vartheta < 2\pi$, where h(z) is given by (2.24).

Theorem 8. A function $f(z) \in k\text{-}\mathrm{UCV}_{\mathrm{s}}^{(\mathrm{m})}[\mathrm{A},\mathrm{B}]$, if and only if

$$\frac{1}{z}\left\{ f(z) * \left(\frac{z(B+1)p_k\left(e^{i\vartheta}\right) - (B-1)}{(1-z)^2} - \left((A+1)p_k\left(e^{i\vartheta}\right) - (A-1)\right)h(z) \right) \right\} \neq 0, \tag{2.27}$$

for all $z \in \mathrm{E}$ and $0 \leq \vartheta < 2\pi$, where h(z) is given by (2.24).

Proof. The proof of Theorem 8, is similar to that of Theorem 7, so the details are omitted.

Coefficient inequalities: Finally, we provided the sufficient conditions for the functions belonging to classes $k\text{-}\mathrm{ST}_{\mathrm{s}}^{(\mathrm{m})}[\mathrm{A},\mathrm{B}]$ and $k\text{-}\mathrm{UCV}_{\mathrm{s}}^{(\mathrm{m})}[\mathrm{A},\mathrm{B}]$.

Theorem 9. A function $f(z) \in \mathrm{A}$ is said to be in the class $k\text{-}\mathrm{ST}_{\mathrm{s}}^{(\mathrm{m})}[\mathrm{A},\mathrm{B}]$, if it satisfies the condition

$$\sum_{n=1}^{\infty} 2(k+1)mn + |(mn(B+1)+(B-A))||a_{mn+1}| + \sum_{n=2,\, n\neq lm+1}^{\infty} 2(k+1)n + |n(B+1)||a_n| < |B-A|, \tag{2.28}$$

where $f_m(z)$ is given by (1.5) with $k \geq 0$, $-1 \leq B < A \leq 1$.

Proof. Assume that (2.28) holds, then it suffices to show that

$$k\left|\frac{(B-1)\frac{zf'(z)}{f_m(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f_m(z)}-(A+1)}-1\right| - \mathrm{Re}\left(\frac{(\mathrm{B}-1)\frac{zf'(z)}{f_m(z)}-(A-1)}{(\mathrm{B}+1)\frac{zf'(z)}{f_m(z)}-(A+1)}-1\right) < 1 \tag{2.29}$$

we have

$$\begin{aligned}
&k\left|\frac{(B-1)\frac{zf'(z)}{f_m(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f_m(z)}-(A+1)}-1\right| - \mathrm{Re}\left(\frac{(\mathrm{B}-1)\frac{zf'(z)}{f_m(z)}-(A-1)}{(\mathrm{B}+1)\frac{zf'(z)}{f_m(z)}-(A+1)}-1\right) \\
&< (k+1)\left|\frac{(B-1)zf'(z)-(A-1)f_m(z)}{(B+1)zf'(z)-(A+1)f_m(z)}-1\right| \\
&= 2(k+1)\left|\frac{f_m(z)-zf'(z)}{(B+1)zf'(z)-(A+1)f_m(z)}-1\right| \\
&\leq 2(k+1)\frac{\sum_{n=2}^{\infty}|b_n-n||a_n|}{|B-A|-\sum_{n=2}^{\infty}|n(B+1)-(A+1)b_n||a_n|}.
\end{aligned}$$

The last expression is bounded by 1, if

$$\sum_{n=2}^{\infty} 2(k+1)(n-b_n) + |(n(B+1)-(A+1)b_n)||a_n| < |B-A|. \tag{2.30}$$

Using (1.6) in (2.30) we have

$$\sum_{n=1}^{\infty} 2(k+1)mn + |(mn(B+1)+(B-A))||a_{mn+1}| + \sum_{n=2,\, n\neq lm+1}^{\infty} 2(k+1)n + |n(B+1)||a_n| < |B-A|,$$

and this completes the proof of Theorem 9.

Putting m = 1, in Theorem 9, we can obtain Corollary 8, below which is comparable to the result obtained by Noor and Malik [5].

Corollary 8. A function $f(z) \in \mathrm{A}$ is said to be in the class $k\text{-ST}[\mathrm{A},\mathrm{B}]$, if it satisfies the condition

$$\sum_{n=2}^{\infty}\{2(k+1)(n-1)+|(n(B+1)+(A+1))|\}|a_n|<|B-A|,$$

where $k \geq 0$, $-1 \leq B < A \leq 1$.

Putting k = 0, in Theorem 9, we can obtain Corollary 9, below which is comparable to the result obtained by Kwon and Sim [10].

Corollary 9. A function $f(z) \in \mathrm{A}$ is said to be in the class $\mathrm{ST}_{\mathrm{s}}^{(\mathrm{m})}[\mathrm{A},\mathrm{B}]$, if it satisfies the condition

$$\sum_{n=1}^{\infty} mn+(mn+1)(B-A)|a_{mn+1}|+\sum_{n=2,\, n\neq lm+1}^{\infty}|n(B+1)||a_n|<|B-A|,$$

where $f_m(z)$ is given by (1.5) with $-1 \leq B < A \leq 1$.

Putting m = 1, A = 1 and B = -1 in Theorem 9, we can obtain Corollary 10, below which is comparable to the result obtained by Kanas and Wisniowska [3].

Corollary 10. A function $f(z) \in \mathrm{A}$ is said to be in the class $\mathrm{k}\text{-ST}$, if it satisfies the condition

$$\sum_{n=2}^{\infty}\{n+k(n-1)\}|a_n|<1, \quad \mathrm{k} \geq 0.$$

Putting m = 1, $\mathrm{A} = 1 - 2\beta, \mathrm{B} = -1$, with $0 \leq \beta < 1$ in Theorem 9, we can obtain Corollary 11, below which is comparable to the result obtained by Shams et-al [11].

Corollary 11. A function $f(z) \in \mathrm{A}$ is said to be in the class $\mathrm{SD}(\mathrm{k},\beta)$, if it satisfies the condition

$$\sum_{n=2}^{\infty}\{n(k+1)-(k+\beta)\}|a_n|<1-\beta,$$

with $0 \leq \beta < 1$, with $\mathrm{k} \geq 0$.

Putting m = 1, $\mathrm{A} = 1 - 2\beta, \mathrm{B} = -1$, with $0 \leq \beta < 1$ and $\mathrm{k} = 0$ in Theorem 9, we can obtain Corollary 12, below which is comparable to the result obtained by Shams et-al [11].

Corollary 12. A function $f(z) \in \mathrm{A}$ is said to be in the class $\mathrm{S}^*(\beta)$, if it satisfies the condition

$$\sum_{n=1}^{\infty}\{n-\beta\}|a_n|<1-\beta,$$

with $0 \leq \beta < 1$.

Theorem 10. A function $f(z) \in \mathrm{A}$ is said to be in the class k - $\mathrm{UCV}_{\mathrm{s}}^{(\mathrm{m})}[\mathrm{A,B}]$, if it satisfies the condition

$$\sum_{\mathrm{n}=2}^{\infty}\left[2(k+1)mn+\left|(mn(B+1)+(B-A))\right|\right](mn+1)\left|a_{mn+1}\right|$$
$$+\sum_{\mathrm{n}=2,\,\mathrm{n}\neq \mathrm{lm}+1}^{\infty}\left(2(k+1)n+n(B+1)\right)\left|na_n\right|<\left|B-A\right|, \tag{2.28}$$

where $f_m(z)$ is given by (1.5) with $k \geq 0$, $-1 \leq B < A \leq 1$.

Proof. The proof of Theorem 10, is similar to that of Theorem 9, so the details are omitted.

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