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Parameters estimation for mixed generalized inverted exponential distributions with type-II progressive hybrid censoring

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Abstract

The type-II progressive hybrid censoring scheme, which is a mixture of type-II progressive and hybrid censoring schemes, has become substantially fashionable due to its flexibility of allowing for random removals of the remaining survival units at each failure time and terminating the experiment at a pre-specified time. In the literature, this censoring scheme has been used to analyze lifetime data for general population distributions such as exponential distributions and Weibull distributions. However, we seldom focus on parameter estimations for the mixture distribution, which is an important class of models in reliability analysis. This paper aims to investigate the estimation problem of mixed generalized inverted exponential distribution (MGIED) under the type-II progressive hybrid censoring scheme. The maximum likelihood estimators (MLEs) of the model are obtained via EM algorithm. Some simulations are implemented and a case of analysis is provided to illustrate the proposed method.

Keywords: Type-II progressive hybrid censoring, Mixed generalized inverted exponential distribution (MGIED), EM algorithm

Mathematics Subject Classification (2010): 62D05;62F10; 62N05

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1. Introduction

The type-II progressive censoring scheme has increasingly become attention-paying for analyzing lifetime data in the literature of life testing in the past ten years, see Balakrishnan & Aggarwala (2000), Balakrishnan *et al.* (2004), Balakrishnan *et al.* (2007), Basak *et al.* (2009), Chen & Lio (2010), Hashemi *et al.* (2010), Wang (2010), Raqab & Madi (2011), Ismail (2012), Balakrishnan & Kundu (2013), etc. This class of censoring schemes considers random removals of the remaining survival units at each failure time and has higher efficiency than the traditional type-I and type-II censoring schemes.

However, the type-II progressive censoring scheme has one defect that the test may have a quite large length. Kundu & Joarder (2006) proposed a new censoring scheme by combining hybrid censoring scheme with the progressive censoring scheme, which is defined as the type-II progressive hybrid censoring scheme, where the experiment is terminated at a pre-specified time. Some inferential results of this new censoring scheme have been developed by Childs *et al.* (2008). In addition, Lin *et al.* (2012) discussed inference for Weibull distributions (WD) with progressive hybrid censoring. Refer to Balakrishnan & Kundub (2013) for a review about hybrid censoring, including progressive hybrid censoring. Cho *et al.* (2015) proposed a generalized progressive hybrid censoring scheme for exponential distribution.

Mixture models have been paid much attention to in many fields of applied science, such as medicine research, cluster analysis and reliability analysis in the last three decades. For example, for mixed exponential distribution (MED) under the complete sample, conventional type-I censoring and type-II censoring have been widely studied. However, we may be subjected to more complex censoring mechanisms like progressive censoring or progressive hybrid censoring. For instance, Solimana (2006) developed Bayesian estimators for the finite mixture of Rayleigh distribution with progressively censored data. Afify (2011) researched the MLEs of mixed Rayleigh distribution for progressively censored data. Tian *et al.* (2014) investigated the estimation problem of the mixed generalized exponential distribution (MGED) based on progressively type-II censored data. In this paper, we introduce a more flexible mixture distribution which is composed of generalized inverted exponential distributions (GIEDs) and consider its estimation under progressive type-II hybrid censored samples.

GIED was introduced by Abouammoh & Alshingiti (2009) and many of its good distributional properties and reliability characteristics were also derived. They observed that the hazard rate functions of GIED can be increasing, or decreasing but not constant depending on the value of the shape parameter. They also said that in many situations, GIED can provide a better fit than gamma, generalized exponential, Weibull and inverted exponential distributions. Due to the convenient structure of this distribution, GIED has been used in many application fields, for example in accelerated life testing, queue theory, modeling wind speeds etc. Recently, Dey & Pradhan (2014) studied the estimation problem of GIED under the hybrid censoring scheme. Krishna & Kumar (2014) studied reliability estimation of GIED under progressive type II censored samples.

The MGIED with K components has its pdf (probability density function) and cdf (cumulative distribution function) respectively as follows

$$\begin{aligned} f(x;p,\alpha,\lambda) &= \sum_{k=1}^{K} p_k \frac{\alpha_k\lambda_k}{x^2} e^{-\frac{\lambda_k}{x}}(1-e^{-\frac{\lambda_k}{x}})^{\alpha_k-1}, \\ F(x;p,\alpha,\lambda) &= 1-\sum_{k=1}^{K} p_k(1-e^{-\frac{\lambda_k}{x}})^{\alpha_k},\ x\geq 0, \end{aligned} \tag{1}$$

where $p = (p_1, \cdots, p_{K-1}), \lambda = (\lambda_1, \cdots, \lambda_K); 0 < p_k < 1, k = 1, \cdots, K-1,\ p_K = 1 - \sum_{k=1}^{K-1} p_k; \lambda_k > 0, k = 1, 2, \cdots, K$.

The remaining of the article is arranged as follows. In section 2, we develop the MLEs of MGIED under the type-II progressive hybrid censoring. In section 3, we derive the closed form of the estimators via EM algorithm. In section 4, some simulation studies are conducted to illustrate the performance of the proposed algorithm. In section 5, a failure time data set is analyzed for further illustrated purpose. In the last section, we come to some conclusions about this paper.

2. The likelihood function

2.1. The censoring scheme. The type-II progressive hybrid censoring scheme in the life testing can de depicted as follows: suppose we put n identical items to test under the same external conditions and denote the lifetimes of all n units by $X_1, X_2, \cdots, X_n$. The integer $m < n$ is fixed in the beginning of the experiment and $R_1, R_2, \cdots, R_m$ are pre-specified integers. The time point T is a specified constant before the experiment as well. Denote the time of first failure as $X_{1:m:n}$, when R_1 of the remaining surviving units are randomly removed. Then, R_2 of the remaining surviving units are randomly removed at the second failure time $X_{2:m:n}$. This process continues till, at the time of the m-th failure $X_{m:m:n}$ or time point T, all the remaining surviving units are removed and then the test is terminated. And if the m-th failure occurs ahead of the time point T, the experiment stops at the time point $X_{m:m:n}$. To the contrary, if the m-th failure occurs after the time point T and only J failures occur before the time point T, with $1 \le J < m$, then, at the time point T, all the remaining R_J^* units are removed and the experiment is terminated. Clearly, we have $R_J^* = n - J - (R_1 + R_2 + \cdots + R_J)$. Denote the two cases above as case I and case II respectively. And this censoring scheme is so-called type-II progressive hybrid censoring scheme. Particularly, under type-II progressive hybrid censoring scheme, we only have one case of the following two types of observations:

(2) Case I: $\{X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}\},\ X_{m:m:n} < T$.

(3) Case II: $\{X_{1:m:n}, X_{2:m:n}, \cdots, X_{J:m:n}\},\ X_{J:m:n} < T < X_{J+1:m:n}$.

It is clear that for case II, $X_{J+1:m:n}, \cdots, X_{m:m:n}$ are not observed and the experimental time T is assumed to be bounded in practical applications.

2.2. The likelihood function. In this subsection, we try to consider how to obtain the MLEs of model parameters p and λ. Based on the observations (2) and (3), we give the likelihood function as follows:

Case I:

$$\begin{aligned} & L(p, \lambda) \\ = & \prod_{i=1}^{m} [f(x_{i:m:n}; p, \lambda) \cdot (1 - F(x_{i:m:n}; p, \lambda))^{R_i}] \\ = & \prod_{i=1}^{m} [\sum_{k=1}^{K} p_k \frac{\alpha_k \lambda_k}{x_{i:m:n}^2} e^{-\frac{\lambda_k}{x_{i:m:n}}} (1 - e^{-\frac{\lambda_k}{x_{i:m:n}}})^{\alpha_k - 1} \cdot (\sum_{k=1}^{K} p_k (1 - e^{-\frac{\lambda_k}{x_{i:m:n}}})^{\alpha_k})^{R_i}]. \end{aligned}$$

We obtain log-likelihood (LL) function as follows

$$(4) \qquad \begin{aligned} l(p, \lambda) = & \sum_{i=1}^{m} [\ln(\sum_{k=1}^{K} p_k \frac{\alpha_k \lambda_k}{x_{i:m:n}^2} e^{-\frac{\lambda_k}{x_{i:m:n}}} (1 - e^{-\frac{\lambda_k}{x_{i:m:n}}})^{\alpha_k - 1}) \\ & + R_i \cdot \ln(\sum_{k=1}^{K} p_k (1 - e^{-\frac{\lambda_k}{x_{i:m:n}}})^{\alpha_k})]. \end{aligned}$$

Case II:

$$\begin{aligned} L(p,\lambda) &= \prod_{i=1}^{J}[f(x_{i:m:n};p,\lambda)\cdot(1-F(x_{i:m:n};p,\lambda))^{R_i}]\cdot(1-F(T;p,\lambda))^{R_J^*} \\ &= \prod_{i=1}^{J}[\sum_{k=1}^{K}p_k\frac{\alpha_k\lambda_k}{x_{i:m:n}^2}e^{-\frac{\lambda_k}{x_{i:m:n}}}(1-e^{-\frac{\lambda_k}{x_{i:m:n}}})^{\alpha_k-1}\cdot(\sum_{k=1}^{K}p_k(1-e^{-\frac{\lambda_k}{x_{i:m:n}}})^{\alpha_k})^{R_i}] \\ &\cdot(\sum_{k=1}^{K}p_k(1-e^{-\frac{\lambda_k}{T}})^{\alpha_k})^{R_J^*}. \end{aligned}$$

The related LL function

$$\begin{aligned} (5)\qquad l(p,\lambda) &= \sum_{i=1}^{J}[\ln(\sum_{k=1}^{K}p_k\frac{\alpha_k\lambda_k}{x_{i:m:n}^2}e^{-\frac{\lambda_k}{x_{i:m:n}}}(1-e^{-\frac{\lambda_k}{x_{i:m:n}}})^{\alpha_k-1}) \\ &+R_i\cdot\ln(\sum_{k=1}^{K}p_k(1-e^{-\frac{\lambda_k}{x_{i:m:n}}})^{\alpha_k})]+R_J^*\ln(\sum_{k=1}^{K}p_k(1-e^{-\frac{\lambda_k}{T}})^{\alpha_k}). \end{aligned}$$

Note that LL functions in (4) and (5) can be combined as follows:

$$\begin{aligned} (6)\qquad l(p,\lambda) &= \sum_{i=1}^{D}[\ln(\sum_{k=1}^{K}p_k\frac{\alpha_k\lambda_k}{x_{i:m:n}^2}e^{-\frac{\lambda_k}{x_{i:m:n}}}(1-e^{-\frac{\lambda_k}{x_{i:m:n}}})^{\alpha_k-1}) \\ &+R_i\cdot\ln(\sum_{k=1}^{K}p_k(1-e^{-\frac{\lambda_k}{x_{i:m:n}}})^{\alpha_k})]+R_D^*\ln(\sum_{k=1}^{K}p_k(1-e^{-\frac{\lambda_k}{T}})^{\alpha_k}), \end{aligned}$$

where $R_D^*=0, D=m$ for case I and $R_D^*=R_J^*, D=J$ for case II.

The MLEs of MGIED model (1) are too difficult to be solved in closed forms because of the complexity of the above LL function. We will utilize the popular EM algorithm to address this problem.

3. The proposed estimation

Suppose $X_1, X_2, \cdots, X_n$ are n identical independent samples from model(1), and denote

$$f_{kj}=\frac{\alpha_k\lambda_k}{x_j^2}e^{-\frac{\lambda_k}{x_j}}(1-e^{-\frac{\lambda_k}{x_j}})^{\alpha_k-1},\ s_{kj}=(1-e^{-\frac{\lambda_k}{x_j}})^{\alpha_k},\ k=1,\cdots,m,$$

$$f_j=\sum_{k=1}^{K}p_kf_{kj},\ s_j=\sum_{k=1}^{K}p_ks_{kj}\ ,j=1,\cdots,n.$$

An indicator vector of X_j is introduced by $I_j=(I_{j1},\cdots,I_{jK})$, where I_{jk} is a dichotomous variable which only takes the value 1 if X_j comes from the (k)th component, and 0 otherwise. Denote $I=(I_1,\cdots,I_n)$ as an indicator matrix composed of n indicator vectors of all life variables $X_1,\cdots,X_n$. Additionally, it is easy to know that the random vector $I_j=(I_1,\cdots,I_{jK})$ follows the multinomial distribution. But since I_j is not observable, we can deem it as the missing data. $I_j^{(1)}=(I_{j1}^{(1)},\cdots,I_{jK}^{(1)})$ and $I_j^{(2)}=(I_{j1}^{(2)},\cdots,I_{jK}^{(2)})$ are denoted as the indicator vectors of the complete data and the right-censored data respectively in the remaining parts of the paper.

For the complete observation data X_j, the joint pdf of X_j and $I_j^{(1)}$ is

$$g(x_j,I_j^{(1)}|p,\lambda)=\prod_{k=1}^{K}[p_kf_{kj}]^{I_{jk}^{(1)}}.$$

The conditional pdf of $I_j^{(1)}$ on X_j is

$$P(I_{jk}^{(1)} = 1|x_j, p, \lambda) = \frac{p_k f_{kj}}{f_j}, k = 1, 2, \cdots, K$$

For the censored data X_j, the joint pdf of X_j and $I_j^{(2)}$ is

$$g(x_j, I_j^{(2)}|p, \lambda) = \prod_{k=1}^{K} [p_k s_{kj}]^{I_{jk}^{(2)}}.$$

The conditional pdf of $I_j^{(2)}$ on X_j is

$$P(I_{jk}^{(2)} = 1|x_j, p, \lambda) = \frac{p_k s_{kj}}{s_j}, k = 1, 2, \cdots, K$$

Assume the life variable follows MGIED model (1). We conduct the life-testing experiment with type-II progressive hybrid censoring as described in section 2. Then in this type-II progressive hybrid censoring experiment, we can observe the complete failure times of D units as $X_{1:m:n}, X_{2:m:n}, \cdots, X_{D:m:n}$. Then, the observed data can be simply denoted as $X = (X_{1:m:n}, X_{2:m:n}, \cdots, X_{D:m:n})$. We denote $Z = \{Z_{ij}, j = 1, 2, \cdots, R_D; i = 1, 2, \cdots, D\} \cup \{Z_{T,j}, j = 1, 2, \cdots, R_D^*\}$, where $\{Z_{ij}, j = 1, 2, \cdots, R_D; i = 1, 2, \cdots, D\}$ represents the j-th censored variables at the failure time $X_{i:m:n}$, and $\{Z_{T,j}, j = 1, 2, \cdots, R_D^*\}$ denotes the j-th censored variables at the censored time T. Then, all $Z = \{Z_{ij}, j = 1, 2, \cdots, R_D; i = 1, 2, \cdots, D\} \cup \{Z_{T,j}, j = 1, 2, \cdots, R_D^*\}$ are unobservable. So we can denote all the missing data as (I, Z) and the complete data as $W = (X, I, Z)$. In the following paper, we will obtain the MLEs of all unknown parameters via EM algorithm. For introduction of the EM algorithm, readers can refer to Dempster *et al.* (1977) etc. EM algorithm is a most popular method in the fields of statistics and data mining which is often used to recognize mixture models and address missing data.

The joint pdf of the complete data W is given by

$$\begin{aligned}
f(p, \alpha, \lambda|W) \quad \propto \quad & \prod_{i=1}^{D} [(\prod_{k=1}^{K} (p_k \frac{\alpha_k \lambda_k}{x_{i:m:n}^2} e^{-\frac{\lambda_k}{x_{i:m:n}}} (1 - e^{-\frac{\lambda_k}{x_{i:m:n}}})^{\alpha_k - 1})^{I_{ik}^{(1)}}) \\
& \cdot (\prod_{j=1}^{R_i} \prod_{k=1}^{K} (p_k \frac{\alpha_k \lambda_k}{z_{ij}^2} e^{-\frac{\lambda_k}{z_{ij}}} (1 - e^{-\frac{\lambda_k}{z_{ij}}})^{\alpha_k - 1})^{I_{ijk}^{(2)}})] \\
& \cdot \prod_{j=1}^{R_D^*} \prod_{k=1}^{K} (p_k \frac{\alpha_k \lambda_k}{z_{T,j}^2} e^{-\frac{\lambda_k}{z_{T,j}}} (1 - e^{-\frac{\lambda_k}{z_{T,j}}})^{\alpha_k - 1})^{I_{jk}^{(2)}}.
\end{aligned}$$

Then, LL function of the complete data is

$$\begin{aligned}
\ln f(p, \alpha, \lambda|W) \quad = \quad & \sum_{i=1}^{D} \sum_{k=1}^{K} \{I_{ik}^{(1)} [\ln(p_k \alpha_k \lambda_k) - 2\ln(x_{i:m:n}) - \frac{\lambda_k}{x_{i:m:n}} \\
& + (\alpha_k - 1) \cdot \ln(1 - e^{-\frac{\lambda_k}{x_{i:m:n}}})] \\
& + \sum_{j=1}^{R_i} I_{ijk}^{(2)} [\ln(p_k \alpha_k \lambda_k) - 2\ln(z_{ij}) - \frac{\lambda_k}{z_{ij}} + (\alpha_k - 1) \cdot \ln(1 - e^{-\frac{\lambda_k}{z_{ij}}})]\} \\
& + \sum_{j=1}^{R_D^*} \sum_{k=1}^{K} I_{jk}^{(2)} [\ln(p_k \alpha_k \lambda_k) - 2\ln(z_{T,j}) - \frac{\lambda_k}{z_{T,j}} \\
& + (\alpha_k - 1) \cdot \ln(1 - e^{-\frac{\lambda_k}{z_{T,j}}})].
\end{aligned}$$

There are two steps in each iteration for the EM algorithm: the Expectation step and the Maximization step. For given initial values $p^{(0)}, \alpha^{(0)}, \lambda^{(0)}$ of unknown parameters p, α, λ, we can obtain the MLE via the following two steps:

E step: Given the $(h-1)$-th iteration values $\Theta^{(h-1)} = \{p^{(h-1)}, \alpha^{(h-1)}, \lambda^{(h-1)}\}$, the Q function of the h-th iteration is given by

$$
\begin{aligned}
& Q(\Theta|\Theta^{(h-1)}, W) = E[\ln f(\Theta|W)|\Theta^{(h-1)}, W] \\
= & \sum_{i=1}^{D}\sum_{k=1}^{K}\{[\ln(p_k\alpha_k\lambda_k) - 2\ln(x_{i:m:n}) - \frac{\lambda_k}{x_{i:m:n}} + (\alpha_k - 1)\cdot\ln(1 - e^{-\frac{\lambda_k}{x_{i:m:n}}})] \\
& \cdot E(I_{ik}^{(1)}|\Theta^{(h-1)}) + \sum_{j=1}^{R_i}[ln(p_k\alpha_k\lambda_k)\cdot E(I_{ijk}^{(2)}|\Theta^{(h-1)}) - 2E(I_{ijk}^{(2)}\ln(z_{ij})|\Theta^{(h-1)}) \\
& -\lambda_k E(I_{ijk}^{(2)} z_{ij}^{-1}|\Theta^{(h-1)}) + (\alpha_k - 1)\cdot E(I_{ijk}^{(2)}\ln(1 - e^{-\frac{\lambda_k}{z_{ij}}})|\Theta^{(h-1)})]\} \\
& + \sum_{j=1}^{R_D^*}\sum_{k=1}^{K}[ln(p_k\alpha_k\lambda_k)\cdot E(I_{jk}^{(2)}|\Theta^{(h-1)}) - 2E(I_{jk}^{(2)}\ln(z_{T,j})|\Theta^{(h-1)}) \\
& -\lambda_k E(I_{jk}^{(2)} z_{T,j}^{-1}|\Theta^{(h-1)}) + (\alpha_k - 1)\cdot E(I_{jk}^{(2)}\ln(1 - e^{-\frac{\lambda_k}{z_{T,j}}})|\Theta^{(h-1)})]
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{i=1}^{D}\sum_{k=1}^{K}\{[\ln(p_k\alpha_k\lambda_k) - 2\ln(x_{i:m:n}) - \frac{\lambda_k}{x_{i:m:n}} + (\alpha_k - 1)\cdot\ln(1 - e^{-\frac{\lambda_k}{x_{i:m:n}}})] \\
& \cdot E(E(I_{ik}^{(1)}|\Theta^{(h-1)}, Z)) + \sum_{j=1}^{R_i}[ln(p_k\alpha_k\lambda_k)\cdot E(E(I_{ijk}^{(2)}|\Theta^{(h-1)}, Z)) \\
& -2E(E(\ln(z_{ij})\cdot I_{ijk}^{(2)}|\Theta^{(h-1)}, Z)) - \lambda_k E(E(z_{ij}^{-1}\cdot I_{ijk}^{(2)}|\Theta^{(h-1)}, Z)) \\
& +(\alpha_k - 1)\cdot E(E(\ln(1 - e^{-\frac{\lambda_k}{z_{ij}}})\cdot I_{ijk}^{(2)}|\Theta^{(h-1)}, Z))]\} \\
& + \sum_{j=1}^{R_D^*}\sum_{k=1}^{K}[ln(p_k\alpha_k\lambda_k)\cdot E(E(I_{jk}^{(2)}|\Theta^{(h-1)}, Z)) - 2E(E(\ln(z_{T,j})\cdot I_{jk}^{(2)}|\Theta^{(h-1)}, Z)) \\
& -\lambda_k E(E(z_{T,j}^{-1}\cdot I_{jk}^{(2)}|\Theta^{(h-1)}, Z)) + (\alpha_k - 1)\cdot E(E(\ln(1 - e^{-\frac{\lambda_k}{z_{T,j}}})\cdot I_{jk}^{(2)}|\Theta^{(h-1)}, Z))] \\
= & \sum_{i=1}^{D}\sum_{k=1}^{K}\{[\ln(p_k\alpha_k\lambda_k) - 2\ln(x_{i:m:n}) - \frac{\lambda_k}{x_{i:m:n}} + (\alpha_k - 1)\cdot\ln(1 - e^{-\frac{\lambda_k}{x_{i:m:n}}})] \\
& \cdot a_{ki}^{(h-1)}(x_{i:m:n}) + \sum_{j=1}^{R_i}[ln(p_k\alpha_k\lambda_k)\cdot E(b_{ki}^{(h-1)}(z_{ij})) - 2E(b_{ki}^{(h-1)}(z_{ij})\cdot\ln(z_{ij})) \\
& -\lambda_k E(b_{ki}^{(h-1)}(z_{ij})\cdot z_{ij}^{-1}) + (\alpha_k - 1)\cdot E(b_{ki}^{(h-1)}(z_{ij})\cdot\ln(1 - e^{-\frac{\lambda_k}{z_{ij}}})]\} \\
& + \sum_{j=1}^{R_D^*}\sum_{k=1}^{K}[ln(p_k\alpha_k\lambda_k)\cdot E(b_{ki}^{(h-1)}(z_{T,j})) - 2E(b_{ki}^{(h-1)}(z_{T,j})\cdot\ln(z_{T,j})) \\
& -\lambda_k E(b_{ki}^{(h-1)}(z_{T,j})\cdot z_{T,j}^{-1}) + (\alpha_k - 1)\cdot E(b_{ki}^{(h-1)}(z_{T,j})\cdot\ln(1 - e^{-\frac{\lambda_k}{z_{T,j}}}))],
\end{aligned}
$$

where,

$$a_{ki}^{(h-1)}(x) = \frac{p_k^{(h-1)}\cdot f_{ki}^{(h-1)}(x)}{f_i^{(h-1)}(x)},\ b_{ki}^{(h-1)}(x) = \frac{p_k^{(h-1)}\cdot s_{ki}^{(h-1)}(x)}{s_i^{(h-1)}(x)};$$

$$f_{ki}^{(h-1)}(x) = \frac{\alpha_k^{(h-1)}\lambda_k^{(h-1)}}{x^2} e^{-\lambda_k^{(h-1)}x^{-1}}(1 - e^{-\lambda_k^{(h-1)}x^{-1}})^{\alpha_k^{(h-1)}-1},$$

$$f_i^{(h-1)}(x) = \sum_{k=1}^{K} p_k^{(h-1)} \cdot f_{ki}^{(h-1)}(x);$$

$$s_{ki}^{(h-1)}(x) = (1 - e^{-\lambda_k^{(h-1)}x^{-1}})^{\alpha_k^{(h-1)}},$$

$$s_i^{(h-1)}(x) = \sum_{k=1}^{K} p_k^{(h-1)} \cdot s_{ki}^{(h-1)}(x),$$

for $k = 1, 2, \cdots, K,\ i = 1, 2, \cdots, D$.

In the above Q function, denote the conditional pdf(s) of all censored data Z_{ij} for $i = 1, 2, \cdots, D,\ j = 1, 2, \cdots, R_i$ and $Z_{T,j}$ for $j = 1, 2, \cdots, R_D^*$, respectively, as follows

$$p_i(z) = \frac{z^{-2} \cdot \sum_{k=1}^{K} p_k^{(h-1)}\alpha_k^{(h-1)}\lambda_k^{(h-1)} \cdot \exp(-\lambda_k^{(h-1)}z^{-1})(1 - \exp(-\lambda_k^{(h-1)}z^{-1}))^{\alpha_k^{(h-1)}-1}}{\sum_{k=1}^{K} p_k^{(t-1)}(1 - \exp(-\lambda_k^{(h-1)}x_{i:m:n}^{-1}))^{\alpha_k^{(h-1)}}},$$

$$z \in [x_{i:m:n}, +\infty);$$

$$p_T(z) = \frac{z^{-2} \cdot \sum_{k=1}^{K} p_k^{(h-1)}\alpha_k^{(h-1)}\lambda_k^{(h-1)} \cdot \exp(-\lambda_k^{(h-1)}z^{-1})(1 - \exp(-\lambda_k^{(h-1)}z^{-1}))^{\alpha_k^{(h-1)}-1}}{\sum_{k=1}^{K} p_k^{(t-1)}(1 - \exp(-\lambda_k^{(h-1)}T^{-1}))^{\alpha_k^{(h-1)}}},$$

$$z \in [T, +\infty).$$

Then, we have

$$\begin{aligned} & Q(\Theta|\Theta^{(h-1)}, W) \\ = \quad & \sum_{i=1}^{D}\sum_{k=1}^{K}\{[\ln(p_k\alpha_k\lambda_k) - 2\ln(x_{i:m:n}) - \frac{\lambda_k}{x_{i:m:n}} + (\alpha_k - 1) \cdot \ln(1 - e^{-\frac{\lambda_k}{x_{i:m:n}}})] \cdot \triangle 1_{ki}^{(h-1)} \\ & + R_i[ln(p_k\alpha_k\lambda_k) \cdot \triangle 2_{ki}^{(h-1)} - 2 \cdot \triangle 3_{ki}^{(h-1)} - \lambda_k \triangle 4_{ki}^{(h-1)} + (\alpha_k - 1) \cdot \triangle 5_{ki}^{(h-1)}]\} \\ & + R_D^* \sum_{k=1}^{K}[ln(p_k\alpha_k\lambda_k) \cdot \triangle 6_{T,k}^{(h-1)} - 2 \cdot \triangle 7_{T,k}^{(h-1)} - \lambda_k \triangle 8_{T,k}^{(h-1)} + (\alpha_k - 1) \cdot \triangle 9_{T,k}^{(h-1)}], \end{aligned}$$

where

$$\triangle 1_{ki}^{(h-1)} = a_{ki}^{(h-1)}(x_{i:m:n}), \quad \triangle 2_{ki}^{(h-1)} = \int_{x_{i:m:n}}^{\infty} b_{ki}^{(h-1)}(x) \cdot p_i(x)dx,$$

$$\triangle 3_{ki}^{(h-1)} = \int_{x_{i:m:n}}^{\infty} \ln(x) \cdot b_{ki}^{(h-1)}(x) \cdot p_i(x)dx, \quad \triangle 4_{ki}^{(h-1)} = \int_{x_{i:m:n}}^{\infty} x^{-1} \cdot b_{ki}^{(h-1)}(x) \cdot p_i(x)dx,$$

$$\triangle 5_{ki}^{(h-1)} = \int_{x_{i:m:n}}^{\infty} \ln(1 - e^{-\frac{\lambda_k}{x}}) \cdot b_{ki}^{(h-1)}(x) \cdot p_i(x)dx, \quad \triangle 6_{T,k}^{(h-1)} = \int_{T}^{\infty} b_{ki}^{(h-1)}(x) \cdot p_T(x)dx,$$

$$\triangle 7_{T,k}^{(h-1)} = \int_{T}^{\infty} \ln(x) \cdot b_{ki}^{(h-1)}(x) \cdot p_T(x)dx, \quad \triangle 8_{T,k}^{(h-1)} = \int_{T}^{\infty} x^{-1} \cdot b_{ki}^{(h-1)}(x) \cdot p_T(x)dx,$$

$$\triangle 9_{T,k}^{(h-1)} = \int_{T}^{\infty} \ln(1 - e^{-\frac{\lambda_k}{x}}) \cdot b_{ki}^{(h-1)}(x) \cdot p_T(x)dx,$$

where $\triangle 5_{ki}^{(h-1)}$ and $\triangle 9_{T,k}^{(h-1)}$ can be simply approximated as

$$\tilde{\triangle} 5_{ki}^{(h-1)} \approx \int_{x_{i:m:n}}^{\infty} \ln(1 - e^{-\frac{\lambda_k^{(h-1)}}{x}}) \cdot b_{ki}^{(h-1)}(x) \cdot p_i(x)dx,$$

$$\tilde{\triangle} 9_{T,k}^{(h-1)} \approx \int_{T}^{\infty} \ln(1 - e^{-\frac{\lambda_k^{(h-1)}}{x}}) \cdot b_{ki}^{(h-1)}(x) \cdot p_T(x)dx.$$

M step: We maximize numerically the Q function in E-step with respect to unknown parameters p, α, λ to update estimates of the parameters, denoted by $p^{(h)}, \alpha^{(h)}, \lambda^{(h)}$. We set

$$
\begin{aligned}
(7) \qquad \frac{\partial Q}{\partial \alpha_k} &= \frac{1}{\alpha_k}[\sum_{i=1}^{D}(\triangle 1_{ki}^{(h-1)} + R_i \cdot \triangle 2_{ki}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,k}^{(h-1)}] \\
&\quad +[\sum_{i=1}^{D}(\ln(1-\exp(-\lambda_k x_{i:m:n}^{-1})) \cdot \triangle 1_{ki}^{(h-1)} + R_i \cdot \tilde{\triangle} 5_{ki}^{(h-1)}) \\
&\quad +R_D^* \cdot \tilde{\triangle} 9_{T,k}^{(h-1)}] \\
&= 0, \quad k = 1, \cdots, K. \\
(8) \qquad \frac{\partial Q}{\partial \lambda_k} &= \frac{1}{\lambda_k}[\sum_{i=1}^{D}(\triangle 1_{ki}^{(h-1)} + R_i \cdot \triangle 2_{ki}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,k}^{(h-1)}] \\
&\quad -[\sum_{i=1}^{D}(x_{i:m:n}^{-1} \frac{1-\alpha_k \exp(-\lambda_k x_{i:m:n}^{-1})}{1-\exp(-\lambda_k x_{i:m:n}^{-1})} \cdot \triangle 1_{ki}^{(h-1)} \\
&\quad +R_i \cdot \triangle 4_{ki}^{(h-1)}) + R_D^* \cdot \triangle 8_{T,k}^{(h-1)}] \\
&= 0, k = 1, \cdots, K.
\end{aligned}
$$

$$
\begin{aligned}
(9) \qquad \frac{\partial Q}{\partial p_k} &= \frac{1}{p_k}[\sum_{i=1}^{D}(\triangle 1_{ki}^{(h-1)} + R_i \cdot \triangle 2_{ki}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,k}^{(h-1)}] \\
&\quad - \frac{1}{1-\sum_{l=1}^{K-1} p_l} \cdot [\sum_{i=1}^{D}(\triangle 1_{Ki}^{(h-1)} + R_i \cdot \triangle 2_{Ki}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,K}^{(h-1)}] \\
&= 0, k = 1, \cdots, K-1.
\end{aligned}
$$

From (7), we obtain

$$
(10) \qquad \hat{\alpha}_k^{(h)} = -\frac{\sum_{i=1}^{D}(\triangle 1_{ki}^{(h-1)} + R_i \cdot \triangle 2_{ki}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,k}^{(h-1)}}{\sum_{i=1}^{D}[\ln(1-\exp(-\lambda_k^{(h-1)} x_{i:m:n}^{-1})) \cdot \triangle 1_{ki}^{(h-1)} + R_i \cdot \tilde{\triangle} 5_{ki}^{(h-1)}] + R_D^* \cdot \tilde{\triangle} 9_{T,k}^{(h-1)}}, \quad k = 1, \cdots, K.
$$

From (8), we obtain

$$
(11) \qquad \hat{\lambda}_k^{(h)} = \frac{\sum_{i=1}^{D}(\triangle 1_{ki}^{(h-1)} + R_i \cdot \triangle 2_{ki}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,k}^{(h-1)}}{\sum_{i=1}^{D}[x_{i:m:n}^{-1} \frac{1-\hat{\alpha}_k^{(h)} \exp(-\lambda_k^{(h-1)} x_{i:m:n}^{-1})}{1-\exp(-\lambda_k^{(h-1)} x_{i:m:n}^{-1})} \cdot \triangle 1_{ki}^{(h-1)} + R_i \cdot \triangle 4_{ki}^{(h-1)}] + R_D^* \cdot \triangle 8_{T,k}^{(h-1)}}, \quad k = 1, \cdots, K.
$$

From (9), we have

$$
\begin{aligned}
& p_k[\sum_{i=1}^{D}(\triangle 1_{Ki}^{(h-1)} + R_i \cdot \triangle 2_{Ki}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,K}^{(h-1)}] \\
& + \sum_{l=1}^{K-1} p_l[\sum_{i=1}^{D}(\triangle 1_{ki}^{(h-1)} + R_i \cdot \triangle 2_{ki}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,k}^{(h-1)}] \\
& = \sum_{i=1}^{D}(\triangle 1_{ki}^{(h-1)} + R_i \cdot \triangle 2_{ki}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,k}^{(h-1)}. \quad k = 1, \cdots, K-1.
\end{aligned}
$$

From the above equations, we know the h-th iteration value in the M-step with respect to parameters $p_l, \cdots, p_{K-1}$ are the solutions of the linear equation group denoted by $AP = b$, where P, A, b are given respectively by

$$P = (p_1, p_2, \cdots, p_{K-1})^T, A_{K-1} = (a_{ls}),$$

$$a_{ls} = \begin{cases} \sum_{i=1}^{D}[(\triangle 1_{li}^{(h-1)} + \triangle 1_{Ki}^{(h-1)}) + R_i(\triangle 2_{li}^{(h-1)} + \triangle 2_{Ki}^{(h-1)})] + R_D^*(\triangle 6_{T,l}^{(h-1)} + \triangle 6_{T,K}^{(h-1)}), & l = s \\ \sum_{i=1}^{D}(\triangle 1_{li}^{(h-1)} + R_i \cdot \triangle 2_{li}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,l}^{(h-1)}, & l \neq s \end{cases}$$

$$b = (\sum_{i=1}^{D}(\triangle 1_{1i}^{(h-1)} + R_i \cdot \triangle 2_{1i}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,1}^{(h-1)}, \cdots, \sum_{i=1}^{D}(\triangle 1_{K-1,i}^{(h-1)} + R_i \cdot \triangle 2_{K-1,i}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,K-1}^{(h-1)})^T.$$

Because the conditions $\sum_{i=1}^{D}(\triangle 1_{li}^{(h-1)} + R_i \cdot \triangle 2_{li}^{(h-1)}) + R_D^* \cdot \triangle 6_{T,l}^{(h-1)} > 0,\ l = 1, \cdots, K$ holds, we can prove that $rank(A) = K - 1$, namely, A is a reversible matrix. Thus, the only solution of parameter vector P of the t-th iteration in the M-step is given by

$$\hat{p}^{(h)} = (\hat{p}_1^{(h)}, \hat{p}_2^{(h)}, \cdots, \hat{p}_{m-1}^{(h)})^T = A^{-1}b. \tag{12}$$

From the above (10), (11) and (12), we can update $(p^{(h)}, \alpha^{(h)}, \lambda^{(h)})$ by repeating E-step and M-step till the total error of all estimated parameters approach the supposed constraints. As recommended in the literature, in practical applications, it is helpful to run the EM algorithm several times using different starting values to obtain more stable estimates (see Muthen & Shedden (1999), Nityasuddhi & Bohning (2003) and Yao (2013), etc).

4. Simulations

In this section, we carry out some simulation studies to illustrate the finite sample performance of the proposed method under different sample sizes and different censoring schemes. Suppose $X_i, i = 1, 2, \cdots, n$ are n identical independently distributed samples generated from MGIED model (1), we only consider the situation including two mixture components under progressively type-II hybrid censored data. To compare results of different censoring schemes, the true values of parameters are taken as $p_1 = 0.3, \alpha_1 = 1.5, \alpha_2 = 0.8, \lambda_1 = 1.2, \lambda_2 = 3$. The initial values are set as $\hat{p}_1^{(0)} = 0.5, \hat{\alpha}_1^{(0)} = 1.8, \hat{\alpha}_2^{(0)} = 1, \hat{\lambda}_1^{(0)} = 1, \hat{\lambda}_2^{(0)} = 2.7$.

The following three different censoring schemes with three different T values: $T_1 = X_{[\frac{m}{2}]:m:n}$, $T_2 = X_{[\frac{4m}{5}]:m:n}$, $T_3 = X_{m:m:n} + 2$, respectively, are taken into account, where $[x]$ denotes the integral part of a positive number x.

Scheme 1: $R_1 = n - m, R_2 = \cdots = R_m = 0$.

Scheme 2: $R_1 = 0, R_2 = n - m, R_3 = \cdots = R_m = 0$.

Scheme 3: $R_1 = \cdots = R_{m-1} = 0, R_m = n - m$.

To generate a type-II progressive hybrid censored sample from MGIED model (1), we make use of the algorithm suggested in Balakrishnan & Aggarwala (2000) and also used in Kundu & Joarder (2006), which involves the following steps:

(1) Generate m independent and identically distributed (iid) random numbers $U_1, U_2, \cdots, U_m$ from the standard uniform distribution $U[0, 1]$.

(2) Set $Z_i = -\log(1 - U_i)$, so that Z_i's are iid standard exponential distribution variates.

(3) Given n, m and the censoring scheme $R = (R_1, R_2, \cdots, R_m)$, let $Y_1 = Z_1/m$ and for $i = 1, \cdots, m$

$$Y_i = Y_{i-1} + \frac{Z_i}{(n - \sum_{j=1}^{i-1} R_j - i + 1)}.$$

Then, we can obtain a progressive type-II censored sample $(Y_1, Y_2, \cdots, Y_m)$ which comes from standard exponential distribution with censoring scheme $R = (R_1, R_2, \cdots, R_m)$.

(4) Set $W_i = 1 - \exp(-Y_i)$ so that W_i's form a progressive type-II censored sample from uniform distribution $U[0,1]$.

(5) Set $X_{i:m:n} = F^{-1}(W_i)$ so that X_i's form a type-II progressive censored sample from MGIED model (1), where $F(x)$ is its cdf.

(6) If $X_{m:m:n} \leq T$, then the corresponding progressive type-II hybrid censored sample of MGIED is the progressive type-II censored sample $\{(X_{1:m:n}, R_1), \cdots, (X_{m:m:n}, R_m)\}$ and $D = m, R_D^* = 0$ in this case. If $X_{m:m:n} > T$, we can find J such that $X_{J:m:n} < T < X_{J+1:m:n}$, then the corresponding progressive type-II hybrid censored sample is $\{(X_{1:m:n}, R_1), \cdots, (X_{J:m:n}, R_J)\}$ and $R_D^* = R_J^*, D = J$ in this case, where R_J^* is the same as defined before.

Table 1: Bias(s) and RMSE(s) of estimators for Scheme 1

			Bias					**RMSE**				
T	n	m	p_1	α_1	α_2	λ_1	λ_2	p_1	α_1	α_2	λ_1	λ_2
	40	20	−0.137	−0.160	0.179	0.084	−0.485	0.148	0.569	0.233	0.619	0.802
		30	−0.051	−0.045	0.125	−0.004	−0.073	0.082	0.256	0.171	0.315	0.483
		35	−0.018	0.024	0.097	0.004	0.262	0.059	0.216	0.147	0.294	0.608
	60	30	−0.039	−0.059	0.237	−0.057	−0.659	0.049	0.223	0.252	0.290	0.743
T_1		40	−0.015	−0.047	0.174	−0.063	−0.449	0.032	0.188	0.193	0.217	0.516
		50	0.001	−0.068	0.101	−0.031	−0.017	0.030	0.192	0.124	0.208	0.262
	80	50	−0.019	−0.048	0.188	−0.085	−0.551	0.030	0.171	0.200	0.209	0.597
		60	−0.004	−0.052	0.139	−0.063	−0.298	0.025	0.164	0.156	0.174	0.364
		70	0.009	−0.063	0.086	−0.038	0.080	0.028	0.171	0.109	0.171	0.242
	40	20	−0.092	−0.270	0.121	0.167	−0.496	0.113	0.395	0.200	0.599	0.725
		30	−0.015	−0.181	0.049	0.079	−0.204	0.063	0.286	0.149	0.336	0.410
		35	0.018	−0.149	0.005	0.084	0.024	0.056	0.248	0.129	0.324	0.350
	60	30	−0.026	−0.145	0.174	−0.008	−0.678	0.042	0.240	0.203	0.285	0.749
T_2		40	0.001	−0.153	0.097	−0.016	−0.497	0.031	0.220	0.139	0.211	0.543
		50	0.021	−0.188	0.025	0.006	−0.148	0.039	0.243	0.097	0.201	0.243
	80	50	−0.005	−0.151	0.121	−0.045	−0.589	0.026	0.207	0.149	0.184	0.621
		60	0.013	−0.166	0.055	−0.025	−0.373	0.030	0.213	0.098	0.162	0.408
		70	0.029	−0.188	0.009	−0.004	−0.074	0.042	0.230	0.079	0.160	0.170
	40	20	−0.080	−0.339	0.152	0.178	−0.549	0.103	0.438	0.226	0.570	0.733
		30	−0.007	−0.263	0.077	0.073	−0.321	0.061	0.346	0.173	0.337	0.442
		35	0.025	−0.237	0.028	0.064	−0.168	0.059	0.309	0.144	0.306	0.342
	60	30	−0.022	−0.184	0.186	0.010	−0.692	0.040	0.259	0.217	0.279	0.755
T_3		40	0.004	−0.197	0.102	−0.023	−0.551	0.032	0.248	0.147	0.207	0.586
		50	0.025	−0.234	0.028	−0.007	−0.282	0.042	0.276	0.111	0.194	0.331
	80	50	−0.002	−0.192	0.124	−0.049	−0.626	0.026	0.234	0.158	0.182	0.653
		60	0.016	−0.210	0.060	−0.034	−0.456	0.033	0.245	0.109	0.159	0.480
		70	0.001	−0.233	0.006	−0.017	−0.237	0.045	0.264	0.084	0.155	0.275

Table 2: Bias(s) and RMSE(s) of estimators for Scheme 2

			Bias					RMSE				
T	n	m	p_1	α_1	α_2	λ_1	λ_2	p_1	α_1	α_2	λ_1	λ_2
T_1	40	20	−0.126	−0.070	0.195	0.033	−0.331	0.136	0.894	0.238	0.509	0.709
		30	−0.047	−0.018	0.128	−0.013	0.068	0.077	0.236	0.171	0.304	0.510
		35	−0.016	0.040	0.098	0.002	0.367	0.056	0.211	0.146	0.289	0.696
	60	30	−0.039	−0.072	0.231	−0.049	−0.535	0.049	0.225	0.245	0.282	0.632
		40	−0.016	−0.050	0.169	−0.054	−0.314	0.032	0.191	0.188	0.216	0.419
		50	0.001	−0.069	0.099	−0.027	0.074	0.030	0.193	0.123	0.208	0.285
	80	50	−0.020	−0.052	0.184	−0.077	−0.443	0.031	0.173	0.196	0.207	0.505
		60	−0.004	−0.055	0.137	−0.057	−0.206	0.025	0.167	0.153	0.174	0.306
		70	0.009	−0.064	0.085	−0.035	0.149	0.028	0.172	0.108	0.172	0.283
T_2	40	20	−0.085	−0.217	0.128	0.122	−0.333	0.105	0.351	0.196	0.508	0.608
		30	−0.012	−0.163	0.050	0.074	−0.070	0.060	0.265	0.147	0.327	0.384
		35	0.020	−0.140	0.006	0.084	0.117	0.055	0.239	0.130	0.320	0.391
	60	30	−0.026	−0.147	0.168	0.001	−0.552	0.042	0.242	0.196	0.278	0.635
		40	0.001	−0.155	0.093	−0.007	−0.364	0.031	0.222	0.135	0.211	0.435
		50	0.021	−0.369	0.023	0.010	−0.070	0.039	0.244	0.096	0.202	0.211
	80	50	−0.005	−0.154	0.118	−0.035	−0.483	0.026	0.210	0.145	0.184	0.525
		60	0.012	−0.168	0.053	−0.019	−0.284	0.031	0.216	0.096	0.163	0.335
		70	0.029	−0.189	0.008	0.000	−0.012	0.042	0.231	0.079	0.162	0.161
T_3	40	20	−0.073	−0.293	0.158	0.142	−0.389	0.096	0.395	0.226	0.511	0.607
		30	−0.005	−0.248	0.078	0.069	−0.201	0.058	0.328	0.173	0.330	0.375
		35	0.026	−0.231	0.029	0.063	−0.088	0.058	0.301	0.145	0.303	0.329
	60	30	−0.022	−0.187	0.179	0.000	−0.569	0.040	0.262	0.210	0.275	0.641
		40	0.004	−0.199	0.097	−0.013	−0.428	0.032	0.250	0.142	0.206	0.479
		50	0.025	−0.236	0.026	−0.002	−0.206	0.042	0.277	0.110	0.195	0.274
	80	50	−0.002	−0.195	0.120	−0.039	−0.525	0.026	0.237	0.154	0.181	0.558
		60	0.016	−0.212	0.058	−0.028	−0.374	0.033	0.247	0.107	0.160	0.407
		70	0.003	−0.234	0.004	−0.016	−0.182	0.045	0.265	0.084	0.157	0.233

Table 3: Bias(s) and RMSE(s) of estimators for Scheme 3

T	n	m	Bias p_1	Bias α_1	Bias α_2	Bias λ_1	Bias λ_2	RMSE p_1	RMSE α_1	RMSE α_2	RMSE λ_1	RMSE λ_2
T_1	40	20	−0.092	0.125	0.204	−0.007	0.389	0.101	0.570	0.228	0.380	1.068
		30	−0.023	0.189	0.117	0.028	0.829	0.053	0.968	0.151	0.319	1.554
		35	−0.004	0.158	0.088	0.036	0.891	0.046	0.533	0.135	0.304	1.468
	60	30	−0.033	−0.033	0.185	−0.041	0.215	0.044	0.221	0.198	0.255	0.484
		40	−0.009	−0.017	0.130	−0.035	0.432	0.031	0.195	0.149	0.205	0.616
		50	0.006	−0.046	0.077	−0.009	0.621	0.031	0.188	0.102	0.205	0.769
	80	50	−0.012	−0.025	0.138	−0.052	0.383	0.027	0.174	0.149	0.196	0.493
		60	0.002	−0.032	0.102	−0.034	0.502	0.026	0.171	0.119	0.170	0.602
		70	0.013	−0.052	0.067	−0.020	0.595	0.030	0.173	0.092	0.171	0.670
T_2	40	20	−0.024	0.186	0.104	0.174	0.902	0.053	0.764	0.145	0.443	1.631
		30	0.024	−0.003	0.022	0.160	0.824	0.050	0.233	0.122	0.349	1.080
		35	0.039	−0.077	−0.012	0.130	0.611	0.060	0.022	0.120	0.339	0.793
	60	30	−0.005	−0.079	0.086	0.080	0.624	0.035	0.020	0.116	0.269	0.809
		40	0.021	−0.109	0.030	0.072	0.644	0.039	0.198	0.087	0.228	0.751
		50	0.034	−0.172	−0.011	0.059	0.539	0.048	0.232	0.084	0.219	0.595
	80	50	0.017	−0.017	0.041	0.051	0.627	0.033	0.192	0.076	0.190	0.690
		60	0.029	−0.147	0.000	0.047	0.598	0.042	0.205	0.065	0.172	0.648
		70	0.039	−0.180	−0.020	0.035	0.480	0.050	0.226	0.073	0.167	0.517
T_3	40	20	0.004	0.034	0.093	0.252	0.798	0.047	0.568	0.148	0.465	1.269
		30	0.040	−0.146	0.022	0.173	0.468	0.061	0.244	0.147	0.369	0.627
		35	0.049	−0.199	−0.015	0.117	0.236	0.069	0.272	0.145	0.330	0.405
	60	30	0.013	−0.124	0.060	0.140	0.613	0.039	0.213	0.106	0.305	0.736
		40	0.035	−0.182	0.001	0.093	0.499	0.050	0.238	0.089	0.240	0.561
		50	0.043	−0.242	−0.034	0.049	0.282	0.056	0.279	0.102	0.207	0.339
	80	50	0.031	−0.183	0.010	0.076	0.546	0.043	0.229	0.073	0.195	0.595
		60	0.040	−0.216	−0.021	0.050	0.405	0.051	0.251	0.079	0.167	0.445
		70	0.007	−0.242	−0.046	0.023	0.204	0.057	0.272	0.092	0.158	0.255

We consider different n, m and T for the given four censoring schemes. We repeat the process $s = 500$ times for different sample sizes $n = 40, 60, 80$ respectively. If we denote parameter estimates of the k-th experiment as $\hat{\Theta}^{(k)} = (\hat{p}_1^{(k)}, \hat{\alpha}_1^{(k)}, \hat{\alpha}_2^{(k)}, \hat{\lambda}_1^{(k)}, \hat{\lambda}_2^{(k)})$, $(k = 1, \cdots, s)$, the final means and root mean square errors (RMSEs) of the estimates are given respectively by $\widehat{Mean}(\Theta_j) = \frac{1}{s}\sum_{k=1}^{s} \hat{\Theta}_j^{(k)}$ and $\widehat{RMSE}(\Theta_j) = \sqrt{\frac{1}{s}\sum_{k=1}^{s}(\hat{\Theta}_j^k - \Theta_j)^2}$, where $\hat{\Theta}_j$ is the j-th coordinate of the unknown parameter vector Θ. All the computation results for different (n, m) and T and censoring schemes are shown in Table 1, Table 2 and Table 3, respectively. All computations are conducted via matlab2013. Computer codes for all simulations and additional results can be obtained from the first author upon request.

From Table 1 to Table 3, it can be seen that on the whole, the EM algorithm has good estimation effect for MGIED model (1) under type-II progressive hybrid censored samples. For all given three sampling schemes, we observe that for fixed n and m, as T increases, RMSEs of estimates for most of estimated parameters decline as expected. Similarly, for fixed n and T, as m increases, RMSEs decrease as expected. For fixed m and T, as n increases, overall, RMSEs decrease for most of parameters. Additionally, for fixed n, m and T, overall, there is no more significant estimation difference for three sampling Schemes.

5. Real-world data analysis

In this section, we discuss a group of failure time data of the air conditional system in Linhart & Zucchini (1986). The following data are failure times of the air conditioning system of an airplane: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

This data set has been analyzed by Gupta & Kundu (2001) and Tian *et al.* (2014). Gupta & Kundu (2001) fitted this data using three life distributions and concluded that the Weibull distribution (WD) can provide a better fit than Gamma distribution (GD) and generalized exponential distribution (GED). Tian *et al.* (2014) fitted this data better by using MED with two components. Tian *et al.* (2014) fitted this data by using MGED with two components. They noticed that mixture distributions has better fit effect than single-population lifetime distributions. In this paper, we also provide the estimation results based on MGIED model (1) with two components. MLEs, K-S distances and p-values of Kolmogorov-Smirnov test, LL values for different life models is listed in Table 4.

Table 4: Estimation results of failure time data

Model	MLEs	K-S	p-value	LL
GED(λ, β)	$\hat{\alpha} = 0.813, \hat{\lambda} = 0.015$	0.183	0.237	−152.22
GD(β)	$\hat{\alpha} = 0.813, \hat{\lambda} = 0.014$	0.171	0.310	−152.17
WD(λ, β)	$\hat{\alpha} = 0.855, \hat{\lambda} = 0.018$	0.154	0.432	−151.94
MGED(λ)	$\hat{p_1} = 0.489, \hat{\alpha_1} = 1.126, \hat{\alpha_2} = 1.052, \widehat{\lambda_1} = 0.044, \hat{\lambda_2} = 0.011$	0.139	0.562	−151.22
MED(λ)	$\hat{p_1} = 0.346, \hat{\lambda_1} = 0.065, \hat{\lambda_2} = 0.012$	0.124	0.697	−151.17
MIGED(λ)	$\hat{p_1} = 0.877, \hat{\alpha_1} = 1.117, \hat{\alpha_2} = 0.930, \hat{\lambda_1} = 23.788, \widehat{\lambda_2} = 2.339$	0.116	0.768	−151.08

From Table 4, we can see that three considered mixture distributions fit better than three single-population life distributions in Gupta & Kundu (2001). In addition, it is observed that MIGED fits the best for this failure time data whereas MED fits the best in the second data in terms of likelihood and Kolmogorov-Smirnov goodness-of-fit test. Therefore, it can be said MIGED works better than other considered distributions. The above data can be deemed as samples form MIGED model (1). Figure 1 is given to compare the Kaplan-Meier product distribution with the fitted survival function. The fitted failure rate function is also listed in Figure 1.

Table 5: The progressive type-II hybrid censored samples.

i	1	2	3	4	5	6	7	8	9	10
t_i	1	3	5	7	11	12	16	20	23	71
d_i	2	2	2	2	2	2	2	2	2	2

Next, we estimate MGIED model based on type-II progressive hybrid censoring scheme $R = (2,2,2,2,2,2,2,2,2,2)$ with $n = 30,\ m = 10,\ T = 80$. The censoring scheme and the censored samples are displayed in Table 5. In this case, the MLEs of MGIED with two mixture components based on the type-II progressive hybrid censoring scheme are

$$\hat{p_1} = 0.927,\ \hat{\alpha}_1 = 0.870, \hat{\alpha}_1 = 2.702, \hat{\lambda}_1 = 20.748,\ \hat{\lambda}_2 = 2.809.$$

The fitted survival function and failure rate function under this type-II progressive hybrid censoring data are plotted in Figure 2.

6. Conclusion

Estimation of MGIED under the type-II progressively hybrid censored data is discussed in this paper. The EM algorithm is employed to obtain the closed form of estimators. Some Monte Carlo simulations are implemented to investigate the performance of the proposed estimation procedure. Finally, a real-world case analysis is presented to illustrate a special application.

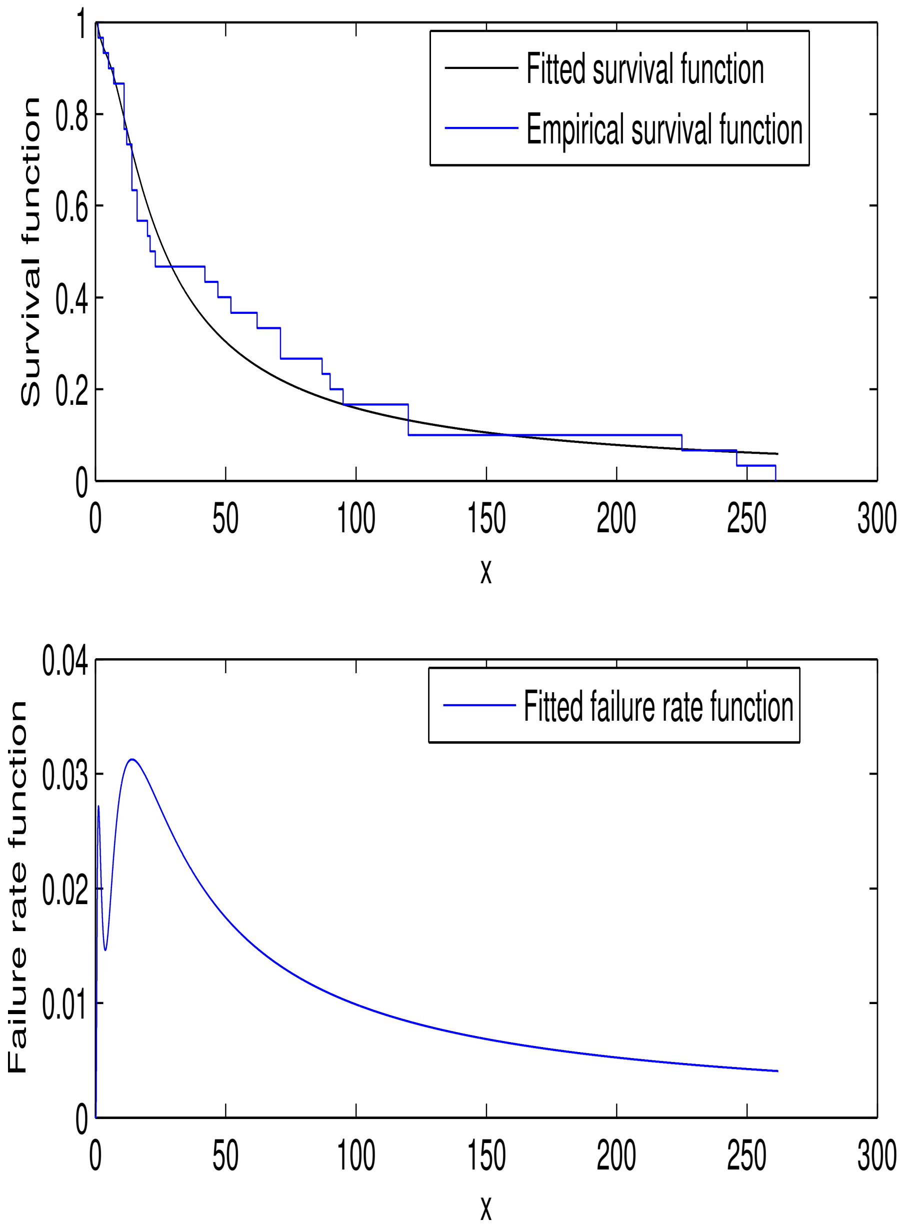

Figure 1. Fitted survival function and failure rate function of complete samples

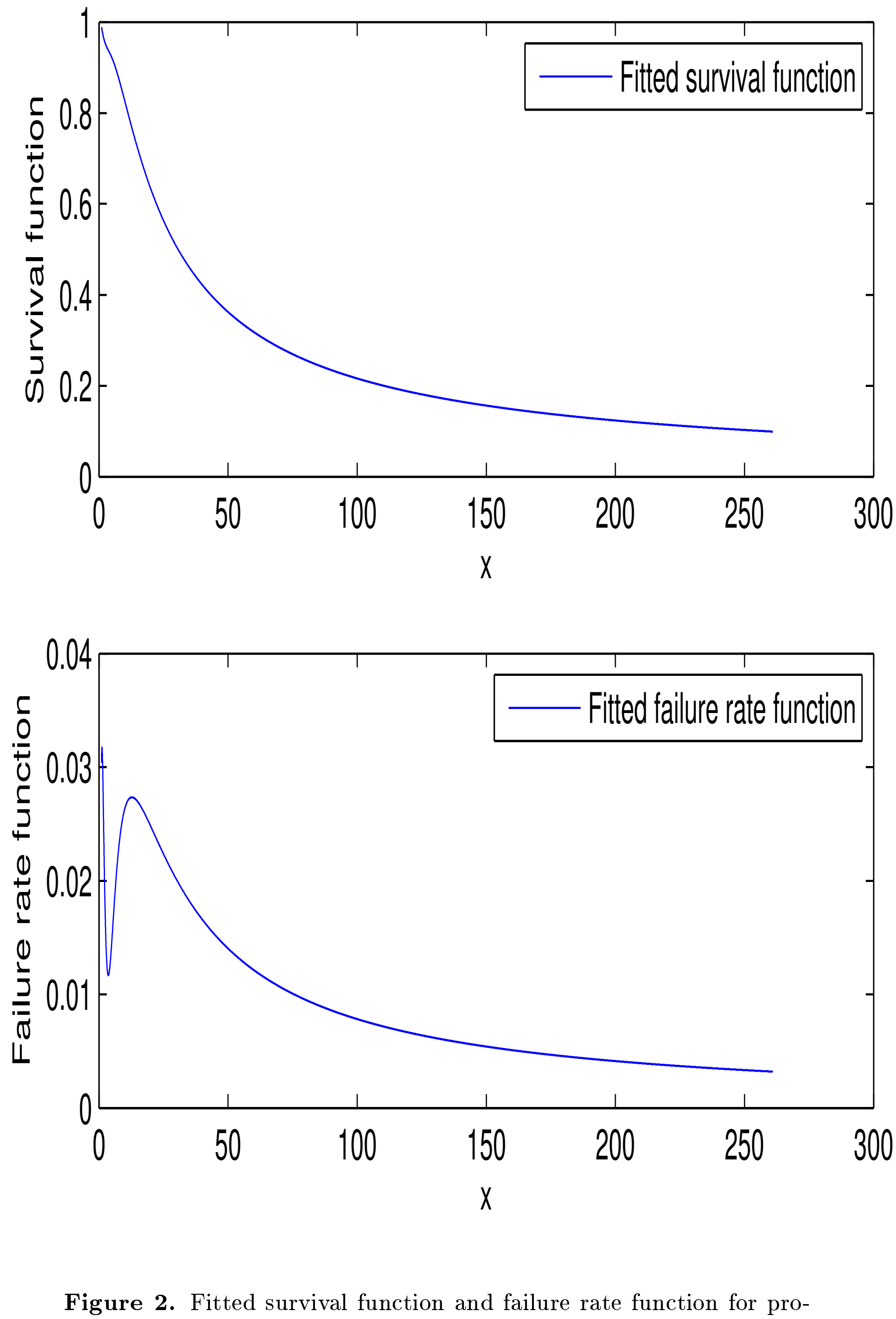

Figure 2. Fitted survival function and failure rate function for progressive type-II hybrid censoring

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References

[1] Abouammoh, A. M. and Alshingiti, A. M. *Reliability estimation of generalized inverted exponential distribution*, Journal of Statistical Computation and Simulation **79** (11), 1301-1315, 2009.

[2] Afify, W. M. *Classical estimation of mixed Rayleigh distribution in type I progressive censored*, Journal of Statistical Theory and Applications **10** (4), 619-632, 2011.

[3] Balakrishnan, N. and Aggarwala, R. *Progressive censoring: Theory, Methods and Applications.* Boston: Birkhauser., 2000.

[4] Balakrishnan, N., Habibi Rad, A. and Arghami, N. R. *Testing exponentiality based on Kullback-Leibler information with progressively type-II censored data*, IEEE Transactions on Reliability **56**, 301-307, 2007.

[5] Balakrishnan, N., Kannan, N., Lin, C. T. and Wu, S .J. *Inference for the extreme value distributions under progressive type-II censoring*, Journal of Statistical Computation and Simulation, *74*, 25-45, 2004.

[6] Balakrishnan, N. and Kundu, D. *Hybrid censoring: Models, inferential results and applications*, Computational Statistics and Data Analysis *57* (1), 166-209, 2013.

[7] Basak, P. and Basak, I., Balakrishnan, N. *Estimation for the three-parameter lognormal distribution based on progressively censored data*, Computational Statistics and Data Analysis *53* (10), 3580-3592, 2009.

[8] Chen, D. G. and Lio, Y. L. *Parameter estimations for generalized exponential distribution under progressive type-I interval censoring*, Computational Statistics and Data Analysis, *54*, 1581-1591, 2010.

[9] Childs, A., Chandrasekar, B. and Balakrishnan, N. *Exact likelihood inference for an exponential parameter under progressive hybrid censoring*, Statistical Models and Methods for Biomedical and Technical Systems 319-330, 2008.

[10] Cho, Y., Sun, H., Lee, K. *Exact likelihood inference for an exponential parameter under generalized progressive hybrid censoring scheme*, Statistical Methodology *23*, 18-34, 2015.

[11] Dempster, A. P., Laird, N. M. and Rubin, D. B. *Maximum likelihood from incomplete data via the EM algorithm*, Journal of Royal Statistical Society Series B *39* (1), 1-38, 1977.

[12] Dey, S., Pradhan, B. *Generalized inverted exponential distribution under hybrid censoring*, Statistical Methodology *18*, 101-114, 2014.

[13] Gupta, R. D. and Kundu, D. *Exponentiated Exponential Family: An Alternative to Gamma and Weibull Distributions*, Computational Statistics and Data Analysis *43* (1), 117-130, 2001.

[14] Hashemi, M., Tavangar, M. and Asadi, M. *Some properties of the residual lifetime of progressively type-II right censored order statistics*, Statistics and Probability Letters *80*, 848-859, 2010.

[15] Ismail, A. A. *Inference in the generalized exponential distribution under partially accelerated tests with progressive Type-II censoring*, Advanced Science Letters *59* (1), 49-56, 2012.

[16] Krishna, H. and Kumar, K. *Reliability estimation in generalized inverted exponential distribution with progressively type II censored sample*, Journal of Statistical Computation and Simulation *83* (6), 1007-1019, 2014.

[17] Kundu, D. and Joarder, A. *Analysis of the type-II progressively hybrid censored data*, Computational Statistical and Data Analysis *50*, 2509-2528, 2006.

[18] Lin, C.T., Chou, C. C. and Huang, Y. L. *Inference for Weibull distribution with progressively hybrid censoring*, Computational Statistics and Data Analysis *56*(3), 451-467, 2012.

[19] Linhart, H., Zucchini, W. *Model selection.* Wiley, Newyork, 1986.

[20] Mokhtari, E. M., Rad, A. H. and Yousefzadeh, F. *Inference for Weibull distribution based on progressively Type-II hybrid censored data*, Journal of Statistical Planning and Inference *141*, 2824-2838, 2011.
[21] Muthen, B., Shedden, K. *Finite mixture modeling with mixture outcomes using the EM algorithm*, Biometrics *55* (2), 463-469, 1999.
[22] Nityasuddhi, D., Bohning, D. *Asymptotic properties of the EM algorithm estimate for normal mixture models with component specific variances*, Computational Statistics & Data Analysis *41* (3-4), 591-601, 2003.
[23] Raqab, M. Z. and Madi, M. T. *Inference for the generalized Rayleigh distribution based on progressively censored data*, Journal of Statistics planning and Inference *141*, 3313-332, 2011.
[24] Solimana, A. A. *Estimators estimators for the finite mixture of Rayleigh model based on progressively censored data*, Communications in Statistics-Theory and Methods *35* (5), 803-820, 2006.
[25] Tian, Y. Z., Tian, M. Z., Zhu, Q. Q. *Estimating a finite mixed exponential distribution under progressively type-II censored data*, Communications in Statistics-Theory and Method *43* (17), 3762-3776, 2014.
[26] Tian, Y. Z., Zhu, Q. Q., Tian, M. Z. *Inference for mixed generalized exponential distribution under progressively type-II censored samples*, Journal of Applied Statistics *41* (3), 660-676, 2014.
[27] Wang, B. X. *Interval estimation for exponential progressive type-II censored step-stress accelerated life-testing*, Journal of Statistical Planning and Inference *140*, 2706-2718, 2010.
[28] Yao, W. X. *A note on EM algorithm for mixture models*, Statistics & Probability Letters *83* (2), 519-526, 2013.