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Research Article

Some notes on the fine spectrum of quintet band matrix operator over c_0 and c

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Abstract

In this work, we determine the fine spectrum of quintet band matrix operator $G(r,s,t,u,v)$ over c_0 and c. The quintet band matrix $G(r,s,t,u,v)$ is the general form of the matrices $D(r,0,s,0,t)$, Δ^4, $Q(r,s,t,u)$, Δ^3, $D(r,0,0,s)$, $B(r,s,t)$, Δ^2, $B(r,s)$, Δ, right shift and Zweier matrices, where Δ^4, $Q(r,s,t,u)$, Δ^3, $B(r,s,t)$, Δ^2, $B(r,s)$ and Δ are called fourth order difference, quadruple band, third order difference, triple band, second order difference, double band(generalized difference) and difference matrix, respectively.

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1. Introduction

w, which is the set of all sequences with each real (or complex) term is a vector space under the operations of point-wise addition and multiplication with scalar. Each vector subspace of w is called a sequence space. The well known spaces such that the spaces of all bounded, null, convergent and absolutely p-summable sequences are symbolized by ℓ_∞, c_0, c and ℓ_p, respectively, where $p \in [1,\infty)$. A BK-space is a Banach sequence space whose each of the maps $p_i : X \longrightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$.

The sequence spaces ℓ_∞, c_0 and c are known to be BK-spaces with their norm defined by $\|x\|_\infty = \sup_{j\in\mathbb{N}} |x_j|$ and ℓ_p is known to be a BK-space with its norm defined by

$$\|x\|_p = \left(\sum_{j=0}^{\infty} |x_j|^p \right)^{\frac{1}{p}}$$

where $1 \leq p < \infty$.

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Given a sequence $x = (x_k)$ and an infinite matrix $A = (a_{nk})$ with complex entires, the A-transform of x is defined by

$$(Ax)_k = \sum_{j=0}^{\infty} a_{kj} x_j$$

and is considered to be convergent for all $k \in \mathbb{N}$ [30].

Given two sequence spaces X and Y and an infinite matrix $A = (a_{nk})$ with complex entires, the matrix domain of A on the sequence space X is defined by

$$X_A = \left\{ x = (x_j) \in w : Ax \in X \right\}$$

and the class of all matrices provided $Ax \in Y, \forall x \in X$ is denoted by $(X : Y)$.

Let two Banach spaces X and Y and a bounded linear operator $T : X \longrightarrow Y$ be given. Then, the sets $D(T)$, $R(T)$ and $B(X, Y)$ are called the domain of T, the range of T and the set of all bounded linear operators from X into Y, respectively. Also, we use a notation of the form $B(X) = B(X, X)$.

Let an arbitrary Banach space X be given, X^* be continuous dual of X and $T \in B(X)$. Then, T^*, which is the adjoint of T, is defined on the X^* as follows:

$$(T^* f)(x) = f(Tx)$$

for all $f \in X^*$ and $x \in X$.

Let I be the identity operator on $D(T)$. Then, the perturbed operator on $D(T)$ is defined by the equality

$$T_\alpha = T - \alpha I$$

where $\alpha \in \mathbb{C}$ [19].

If T_α has an inverse, it is denoted by T_α^{-1} and is called the resolvent operator of T because it is used to solve the equation $T_\alpha x = y$, namely $x = T_\alpha^{-1} y$ [19].

Spectral theory deals with the properties of the operators T_α and T_α^{-1} depending on the complex number α [19].

For a given linear operator $T : D(T) \subset X \longrightarrow X$, where $X \neq \{\theta\}$ is a normed space, a complex number α is called a regular value of T if the following conditions are met.

(i) T_α^{-1} exists
(ii) T_α^{-1} is bounded
(iii) The domain of T_α^{-1} is dense in X.

The resolvent set of T consists of all regular values α of T and is denoted by $\rho(T, X)$ [19].

The set defined by $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ is called spectrum of T. A complex number α is called a spectral value of T in case of $\alpha \in \sigma(T, X)$ [19]. The set $\sigma(T, X)$ can be divided into three disjoint sets as follows:

The set of all α values in which T_α^{-1} does not exists is called the point spectrum(or discrete spectrum) of T and is denoted by $\sigma_p(T, X)$. Each of the α elements belonging to the $\sigma_p(T, X)$ is called an eigenvalue of T [19].

The set of all α values in which T_α^{-1} satisfies (i) and (iii) but does not satisfy (ii) is called the continuous spectrum of T and is denoted by $\sigma_c(T, X)$ [19].

The set of all α values in which T_α^{-1} satisfies (i) (and may be bounded or not) but does not satisfy (iii) is called the residual spectrum of T and is denoted by $\sigma_r(T, X)$ [19].

For a given Banach space X and a perturbed operator T_α, from the Goldberg [17], some possibilities of T_α can be classified as follows:

(I) $R(T_\alpha) = X$,
(II) $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$,
(III) $\overline{R(T_\alpha)} \neq X$,
(1) T_α^{-1} exists and is continuous,

(2) T_α^{-1} exists but is discontinuous,
(3) T_α has no inverse.

Considering these possibilities together, one can obtain nine different states labelled by $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2$ and III_3. If $T_\alpha \in III_1$, it is understood as T_α satisfies (III) and (1). In case of $T_\alpha \in I_1$ or $T_\alpha \in II_1$ this means that $\alpha \in \rho(T,X)$. In case of an operator belongs to state II_2, it is written $\alpha \in II_2\sigma(T,X)$.

Spectral theory is one of the important topics of functional analysis, quantum mechanics,...which is related to certain inverse operators. These types of operators are used in the problem of solving equations for instance systems of linear algebraic equations, differential equations, integral equations,.... Considering the wide range of uses of this theory, many authors have conducted research on the spectra of certain difference matrix operators on some known sequence spaces. For example, the difference matrix operator Δ on the sequence spaces c_0, c, ℓ_1, ℓ_p, bv and bv_p in [2–4, 6, 20], the double band matrix operator $B(r,s)$ on the sequence spaces c_0, c, ℓ_1, ℓ_p, bv and bv_p in [5, 9, 16], the second order difference matrix operator Δ^2 on the sequence space c_0 in [13], the triple band matrix operator $B(r,s,t)$ on the sequence spaces c_0, c, ℓ_1, ℓ_p, bv and bv_p in [8, 14, 15], the matrix operator $D(r,0,0,s)$ on the sequence spaces c_0, c, ℓ_p, bv_0 and bv_p in [22–25], the quadruple band matrix operator $Q(r,s,t,u)$ on the sequence spaces c_0, c, ℓ_1, ℓ_p, bv and bv_p in [10, 11] and the matrix operator $D(r,0,s,0,t)$ on the sequence spaces c_0 and c in [28]. Also, some authors have examined the spectral property of special operators defined on some of known sequence spaces. For example, Cesáro and p-Cesáro operators in [1, 12, 18], Rhaly operators in [31–34], weighed mean operators in [26] and factorable operators in [27].

2. Fine Spectrum Of Quintet Band Matrix Operator On c_0 And c

In this section, we work on the fine spectrum of the quintet band matrix operator $G(r,s,t,u,v)$ defined on the sequence spaces c_0 and c.

Let's start with some lemmas that are used in the next.

Lemma 2.1 ([29]). *Let an infinite matrix $A = (a_{nk})$ be given. Then, $A = (a_{nk}) \in (c : c)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty \tag{2.1}$$

$$\lim_{n\to\infty} a_{nk} = \mu_k \quad \textit{for all} \quad k \in \mathbb{N} \tag{2.2}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \mu \tag{2.3}$$

Lemma 2.2 ([29]). *Let an infinite matrix $A = (a_{nk})$ be given. Then, $A = (a_{nk}) \in (c_0 : c_0)$ if and only if* (2.1) *and* (2.2) *hold with* $\mu_k = 0$, $\forall k \in \mathbb{N}$.

Lemma 2.3 ([21]). *Given a BK-space X and an infinite matrix $A = (a_{nk})$. Then, for all $A = (a_{nk}) \in (X : X)$, there exists a $T \in B(X)$ such that $T(x) = Ax$, that is $(X : X) \subset B(X)$.*

For given $r,s,t,u,v \in \mathbb{C} \setminus \{0\}$, the quintet band matrix $G = G(r,s,t,u,v) = (g_{nk}(r,s,t,u,v))$ is defined by

$$g_{nk}(r,s,t,u) = \begin{cases} r & , \quad k = n \\ s & , \quad k = n-1 \\ t & , \quad k = n-2 \\ u & , \quad k = n-3 \\ v & , \quad k = n-4 \\ 0 & , \quad \textit{otherwise} \end{cases}$$

for all $n, k \in \mathbb{N}$. Here, it is clearly seen that the equalities $G(r,0,t,0,v) = D(r,0,s,0,t)$, $G(1,-4,6,-4,1) = \Delta^4$, $G(r,s,t,u,0) = Q(r,s,t,u)$, $G(1,-3,3,-1,0) = \Delta^3$, $G(r,0,0,u,0) = D(r,0,0,s)$, $G(r,s,t,0,0) = B(r,s,t)$, $G(1,-2,1,0,0) = \Delta^2$, $G(r,s,0,0,0) = B(r,s)$ and $G(1,-1,0,0,0) = \Delta$ are satisfied, where Δ^4, $Q(r,s,t,u)$, Δ^3, $B(r,s,t)$, Δ^2, $B(r,s)$ and Δ are called fourth order difference, quadruple band, third order difference, triple band, second order difference, double band and difference matrix, respectively. Because of this, our findings obtained from the quintet band matrix $G(r,s,t,u,v)$ are more general and more comprehensive than the findings obtained from the matrices defined above.

If the Lemmas 2.1 and 2.2 are applied to the quintet band matrix, we write

$$\sup_{n\in\mathbb{N}} \sum_k |g_{nk}(r,s,t,u,v)| = |r| + |s| + |t| + |u| + |v| < \infty,$$

$$\lim_{n\to\infty} g_{nk}(r,s,t,u,v) = 0 \quad \text{for all} \quad k \in \mathbb{N},$$

$$\lim_{n\to\infty} \sum_k g_{nk}(r,s,t,u,v) = r + s + t + u + v$$

These result in $G(r,s,t,u,v) \in (c:c)$ and $G(r,s,t,u,v) \in (c_0:c_0)$.

Also, considering Lemma 2.3, the following Corollaries can be given.

Corollary 2.4. *The operator defined by $G(r,s,t,u,v) : c \longrightarrow c$ is bounded and linear provided $\|G(r,s,t,u,v)\|_{(c:c)} = |r| + |s| + |t| + |u| + |v|$.*

Corollary 2.5. *The operator defined by $G(r,s,t,u,v) : c_0 \longrightarrow c_0$ is bounded and linear provided $\|G(r,s,t,u,v)\|_{(c_0:c_0)} = |r| + |s| + |t| + |u| + |v|$.*

Let $\alpha \in \mathbb{C}$ and $r,s,t,u,v \in \mathbb{C} \setminus \{0\}$ be given. Then, according to the fundamental theorem of algebra, the fourth degree equation

$$(r-\alpha)z^4 + sz^3 + tz^2 + uz + v = 0 \tag{2.4}$$

has four roots so that $z_1 = -\frac{s}{4(r-\alpha)} + \frac{1}{2}(-a-b)$, $z_2 = -\frac{s}{4(r-\alpha)} + \frac{1}{2}(-a+b)$, $z_3 = -\frac{s}{4(r-\alpha)} + \frac{1}{2}(a-c)$ and $z_4 = -\frac{s}{4(r-\alpha)} + \frac{1}{2}(a+c)$ where

$$a = \sqrt{\frac{s^2}{4(r-\alpha)^2} - \frac{2t}{3(r-\alpha)} + \frac{d}{3(r-\alpha)\sqrt[3]{2}} + \frac{(t^2 - 3su + 12(r-\alpha)v)\sqrt[3]{2}}{3(r-\alpha)d}},$$

$$b = \sqrt{\frac{s^2}{2(r-\alpha)^2} - \frac{4t}{3(r-\alpha)} - \frac{d}{3(r-\alpha)\sqrt[3]{2}} - \frac{(t^2 - 3su + 12(r-\alpha)v)\sqrt[3]{2}}{3(r-\alpha)d} - \frac{-\frac{s^3}{(r-\alpha)^3} + \frac{4ts}{(r-\alpha)^2} - \frac{8u}{(r-\alpha)}}{4a}},$$

$$c = \sqrt{\frac{s^2}{2(r-\alpha)^2} - \frac{4t}{3(r-\alpha)} - \frac{d}{3(r-\alpha)\sqrt[3]{2}} - \frac{(t^2 - 3su + 12(r-\alpha)v)\sqrt[3]{2}}{3(r-\alpha)d} + \frac{-\frac{s^3}{(r-\alpha)^3} + \frac{4ts}{(r-\alpha)^2} - \frac{8u}{(r-\alpha)}}{4a}},$$

$$d = \sqrt[3]{\xi + \sqrt{\xi^2 - 4\left(t^2 - 3su + 12(r-\alpha)v\right)^3}},$$

and

$$\xi = 2t^3 - 9sut - 72(r-\alpha)vt + 27(r-\alpha)u^2 + 27s^2v$$

Moreover, using a simple calculation, the following equalities can be obtained.

$$z_1 + z_2 + z_3 + z_4 = -\frac{s}{r-\alpha},$$

$$z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4 = \frac{t}{r-\alpha},$$

$$z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4 = -\frac{u}{r-\alpha}$$

and

$$z_1 z_2 z_3 z_4 = \frac{v}{r-\alpha}$$

where $r \neq \alpha$.

Before moving on the main results, we would like to draw attention to the following remark. Herein and throughout the rest of the study, unless otherwise stated, we suppose that φ_1, φ_2, φ_3 and φ_4 are random four roots of the equation (2.4) and $max\{|\varphi_1|, |\varphi_2|, |\varphi_3|\} \leq |\varphi_4|$. If other possibilities of the roots of the above equation are chosen, the same results can be obtained using a similar method.

Theorem 2.6. *Let the set S be defined as follows*

$$S = \{\alpha \in \mathbb{C} : 1 \leq |\varphi_4|\}.$$

Then, $\sigma(G(r,s,t,u,v), c_0) = S$.

Proof. What is required for the proof is to show that $(G(r,s,t,u,v) - \alpha I)^{-1}$ exists and belongs to $B(c_0)$ for $\alpha \notin S$ and $G(r,s,t,u,v) - \alpha I$ has not an inverse for $\alpha \in S$.

When $\alpha \notin S$ is taken, it is clearly seen that it must be $\alpha \neq r$. Because of this reason, $G(r,s,t,u,v) - \alpha I$ is a triangle, that is $G(r,s,t,u,v) - \alpha I$ uniquely has an inverse such that

$$(G(r,s,t,u,v) - \alpha I)^{-1} = \begin{bmatrix} a_1 & 0 & 0 & 0 & \dots \\ a_2 & a_1 & 0 & 0 & \dots \\ a_3 & a_2 & a_1 & 0 & \dots \\ a_4 & a_3 & a_2 & a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where

$$a_1 = \frac{1}{r-\alpha}$$

$$a_2 = \frac{\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4}{r-\alpha} = \frac{-\frac{s}{r-\alpha}}{r-\alpha} = -\frac{s}{(r-\alpha)^2}$$

$$\begin{aligned} a_3 &= \frac{\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 + \varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_1\varphi_4 + \varphi_2\varphi_3 + \varphi_2\varphi_4 + \varphi_3\varphi_4}{r-\alpha} \\ &= \frac{1}{r-\alpha}\left[-\frac{s}{r-\alpha}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) - \frac{t}{r-\alpha}\right] \\ &= \frac{1}{r-\alpha}\left[\frac{s^2}{(r-\alpha)^2} - \frac{t}{r-\alpha}\right] \\ &= \frac{s^2 - t(r-\alpha)}{(r-\alpha)^3} \end{aligned}$$

$$\begin{aligned}
a_4 &= \frac{1}{r-\alpha}\Big[\varphi_1^3+\varphi_2^3+\varphi_3^3+\varphi_4^3+\varphi_1\varphi_2^2+\varphi_1\varphi_3^2+\varphi_1\varphi_4^2+\varphi_2\varphi_1^2+\varphi_2\varphi_3^2+\varphi_2\varphi_4^2+\varphi_3\varphi_1^2\\
&+ \varphi_3\varphi_2^2+\varphi_3\varphi_4^2+\varphi_4\varphi_1^2+\varphi_4\varphi_2^2+\varphi_4\varphi_3^2+\varphi_1\varphi_2\varphi_3+\varphi_1\varphi_2\varphi_4+\varphi_1\varphi_3\varphi_4+\varphi_2\varphi_3\varphi_4\Big]\\
&= \frac{1}{r-\alpha}\Big[-\frac{s}{r-\alpha}\Big[(\varphi_1+\varphi_2+\varphi_3+\varphi_4)^2-(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_1\varphi_4+\varphi_2\varphi_3+\varphi_2\varphi_4+\varphi_3\varphi_4)\Big]\\
&- \frac{t}{r-\alpha}(\varphi_1+\varphi_2+\varphi_3+\varphi_4)-\frac{u}{r-\alpha}\Big]\\
&= \frac{1}{r-\alpha}\Big[-\frac{s^3}{(r-\alpha)^3}+\frac{2st}{(r-\alpha)^2}-\frac{u}{r-\alpha}\Big]\\
&= \frac{-s^3+2st(r-\alpha)-u(r-\alpha)^2}{(r-\alpha)^4}
\end{aligned}$$

$$\begin{aligned}
a_5 &= \frac{1}{r-\alpha}\Big[\varphi_1^4+\varphi_2^4+\varphi_3^4+\varphi_4^4+\varphi_1^3(\varphi_2+\varphi_3+\varphi_4)+\varphi_2^3(\varphi_1+\varphi_3+\varphi_4)+\varphi_3^3(\varphi_1+\varphi_2+\varphi_4)\\
&+ \varphi_4^3(\varphi_1+\varphi_2+\varphi_3)+\varphi_1^2(\varphi_2\varphi_3+\varphi_2\varphi_4+\varphi_3\varphi_4)+\varphi_2^2(\varphi_1\varphi_3+\varphi_1\varphi_4+\varphi_3\varphi_4)\\
&+ \varphi_3^2(\varphi_1\varphi_2+\varphi_1\varphi_4+\varphi_2\varphi_4)+\varphi_4^2(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)+\varphi_1^2\varphi_2^2+\varphi_1^2\varphi_3^2+\varphi_1^2\varphi_4^2+\varphi_2^2\varphi_3^2\\
&+ \varphi_2^2\varphi_4^2+\varphi_3^2\varphi_4^2+\varphi_1\varphi_2\varphi_3\varphi_4\\
&= -\frac{s}{(r-\alpha)^2}\Big[(\varphi_1+\varphi_2+\varphi_3+\varphi_4)^3-3\Big[(\varphi_1+\varphi_2+\varphi_3+\varphi_4)(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_1\varphi_4\\
&+ \varphi_2\varphi_3+\varphi_2\varphi_4+\varphi_3\varphi_4)-3(\varphi_1\varphi_2\varphi_3+\varphi_1\varphi_2\varphi_4+\varphi_1\varphi_3\varphi_4+\varphi_2\varphi_3\varphi_4)\Big]-6(\varphi_1\varphi_2\varphi_3\\
&+ \varphi_1\varphi_2\varphi_4+\varphi_1\varphi_3\varphi_4+\varphi_2\varphi_3\varphi_4)\Big]+\frac{1}{r-\alpha}\Big[(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_1\varphi_4+\varphi_2\varphi_3+\varphi_2\varphi_4+\varphi_3\varphi_4)^2\\
&+ \frac{u}{r-\alpha}(\varphi_1+\varphi_2+\varphi_3+\varphi_4)-\varphi_1\varphi_2\varphi_3\varphi_4\Big]\\
&= -\frac{s}{(r-\alpha)^2}\Big[\Big(-\frac{s}{r-\alpha}\Big)^3-3\Big(-\frac{s}{r-\alpha}\frac{t}{r-\alpha}+3\frac{u}{r-\alpha}\Big)+6\frac{u}{r-\alpha}\Big]\\
&+ \frac{1}{r-\alpha}\Big[\Big(\frac{t}{r-\alpha}\Big)^2-\frac{u}{r-\alpha}\frac{s}{r-\alpha}-\frac{v}{r-\alpha}\Big]\\
&= \frac{s^4-3(r-\alpha)s^2t+(r-\alpha)^2(t^2+2su)-(r-\alpha)^3v}{(r-\alpha)^5}
\end{aligned}$$

$$\vdots$$

and for all $n \geq 1$, according to behavior of the roots φ_1, φ_2, φ_3 and φ_4 of equation (2.4), a_n can be defined as follows:

Case 1: if $\varphi_1 \neq \varphi_2 \neq \varphi_3 \neq \varphi_4$, we write

$$\begin{aligned}
a_n &= \frac{1}{r-\alpha}\Big[\frac{\varphi_1^{n+2}}{(\varphi_1-\varphi_2)(\varphi_1-\varphi_3)(\varphi_1-\varphi_4)}+\frac{\varphi_2^{n+2}}{(\varphi_2-\varphi_1)(\varphi_2-\varphi_3)(\varphi_2-\varphi_4)}\\
&+ \frac{\varphi_3^{n+2}}{(\varphi_3-\varphi_1)(\varphi_3-\varphi_2)(\varphi_3-\varphi_4)}+\frac{\varphi_4^{n+2}}{(\varphi_4-\varphi_1)(\varphi_4-\varphi_2)(\varphi_4-\varphi_3)}\Big]
\end{aligned}$$

According to the suppositions $\alpha \notin S$ and $max\{|\varphi_1|, |\varphi_2|, |\varphi_3|\} \leq |\varphi_4|$, the inequalities $|\varphi_1| < 1$, $|\varphi_2| < 1$, $|\varphi_3| < 1$ and $|\varphi_4| < 1$ are satisfied, which gives us the result

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{r-\alpha}\Bigg[\frac{\varphi_1^{n+2}}{(\varphi_1-\varphi_2)(\varphi_1-\varphi_3)(\varphi_1-\varphi_4)} + \frac{\varphi_2^{n+2}}{(\varphi_2-\varphi_1)(\varphi_2-\varphi_3)(\varphi_2-\varphi_4)} \\ &+ \frac{\varphi_3^{n+2}}{(\varphi_3-\varphi_1)(\varphi_3-\varphi_2)(\varphi_3-\varphi_4)} + \frac{\varphi_4^{n+2}}{(\varphi_4-\varphi_1)(\varphi_4-\varphi_2)(\varphi_4-\varphi_3)}\Bigg] \\ &= 0\end{aligned}$$

<u>Case 2:</u> if $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi$, we write

$$a_n = \frac{\varphi^{n-1} n(n+1)(n+2)}{6(r-\alpha)}$$

According to the suppositions above, the inequality $|\varphi| < 1$ is satisfied, which gives us the results

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\varphi^{n-1} n(n+1)(n+2)}{6(r-\alpha)} = 0$$

and $(a_n) \in \ell_1$.

<u>Case 3:</u> if $\varphi = \varphi_i = \varphi_j \neq \varphi_l = \varphi_4$ where $i, j, l \in \{1, 2, 3\}$ we write

$$a_n = \frac{1}{(r-\alpha)(\varphi-\varphi_4)^3}\Big[n(\varphi-\varphi_4)(\varphi^{n+1}+\varphi_4^{n+1}) - 2\varphi\varphi_4(\varphi^n - \varphi_4^n)\Big]$$

According to the suppositions above, the inequalities $|\varphi| < 1$ and $|\varphi_4| < 1$ are satisfied, which gives us the results

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(r-\alpha)(\varphi-\varphi_4)^3}\Big[n(\varphi-\varphi_4)(\varphi^{n+1}+\varphi_4^{n+1}) - 2\varphi\varphi_4(\varphi^n - \varphi_4^n)\Big] = 0$$

and $(a_n) \in \ell_1$.

<u>Case 4:</u> if $\varphi = \varphi_1 = \varphi_2 = \varphi_3 \neq \varphi_4$, we write

$$\begin{aligned}a_n &= \frac{1}{2(r-\alpha)(\varphi-\varphi_4)^3}\Big[\varphi^n\varphi_4^2(n+1)(n+2) - 2\varphi^{n+1}\varphi_4 n(n+2) \\ &+ \varphi^{n+2}n(n+1) - 2\varphi_4^{n+2}\Big]\end{aligned}$$

According to the suppositions above, the inequalities $|\varphi| < 1$ and $|\varphi_4| < 1$ are satisfied, which gives us the results

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{2(r-\alpha)(\varphi-\varphi_4)^3}\Big[\varphi^n\varphi_4^2(n+1)(n+2) - 2\varphi^{n+1}\varphi_4 n(n+2) \\ &+ \varphi^{n+2}n(n+1) - 2\varphi_4^{n+2}\Big] = 0\end{aligned}$$

and $(a_n) \in \ell_1$.

<u>Case 5:</u> if $\varphi_i = \varphi_j = \varphi_4 \neq \varphi_l$ where $i, j, l \in \{1, 2, 3\}$ we write

$$\begin{aligned}a_n &= \frac{1}{2(r-\alpha)(\varphi_4-\varphi_l)^3}\Big[\varphi_4^n\varphi_l^2(n+1)(n+2) - 2\varphi_4^{n+1}\varphi_l n(n+2) \\ &+ \varphi_4^{n+2}n(n+1) - 2\varphi_l^{n+2}\Big]\end{aligned}$$

According to the suppositions above, the inequalities $|\varphi| < 1$ and $|\varphi_4| < 1$ are satisfied, which gives us the results

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{2(r-\alpha)(\varphi_4-\varphi_l)^3}\Big[\varphi_4^n\varphi_l^2(n+1)(n+2) - 2\varphi_4^{n+1}\varphi_l n(n+2)\\ &+ \varphi_4^{n+2}n(n+1) - 2\varphi_l^{n+2}\Big] = 0\end{aligned}$$

and $(a_n) \in \ell_1$.

Case 6: if $\varphi = \varphi_i = \varphi_j \neq \varphi_l \neq \varphi_4$ where $i, j, l \in \{1, 2, 3\}$ we write

$$\begin{aligned}a_n &= \frac{1}{(r-\alpha)(\varphi-\varphi_4)^2}\Big[\varphi^{n+1}\Big((n-1)\frac{\varphi-\varphi_4}{\varphi-\varphi_l} + \frac{\varphi^2 - 2\varphi(\varphi_4+\varphi_l) + 3\varphi_4\varphi_l}{(\varphi-\varphi_l)^2}\Big)\\ &+ \frac{\varphi_4^{n+2}}{\varphi_4-\varphi_l} - \frac{(\varphi-\varphi_4)^2}{(\varphi-\varphi_l)^2(\varphi_4-\varphi_l)}\varphi_l^{n+2}\Big]\end{aligned}$$

According to the suppositions above, the inequalities $|\varphi| < 1$, $|\varphi_l| < 1$ and $|\varphi_4| < 1$ are satisfied, which gives us the results

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{(r-\alpha)(\varphi-\varphi_4)^2}\Big[\varphi^{n+1}\Big((n-1)\frac{\varphi-\varphi_4}{\varphi-\varphi_l} + \frac{\varphi^2 - 2\varphi(\varphi_4+\varphi_l) + 3\varphi_4\varphi_l}{(\varphi-\varphi_l)^2}\Big)\\ &+ \frac{\varphi_4^{n+2}}{\varphi_4-\varphi_l} - \frac{(\varphi-\varphi_4)^2}{(\varphi-\varphi_l)^2(\varphi_4-\varphi_l)}\varphi_l^{n+2}\Big] = 0\end{aligned}$$

and $(a_n) \in \ell_1$.

Case 7: if $\varphi_i = \varphi_4 \neq \varphi_l \neq \varphi_j$ where $i, j, l \in \{1, 2, 3\}$ we write

$$\begin{aligned}a_n &= \frac{1}{(r-\alpha)(\varphi_4-\varphi_j)^2}\Big[\varphi_4^{n+1}\Big((n-1)\frac{\varphi_4-\varphi_j}{\varphi_4-\varphi_l} + \frac{\varphi_4^2 - 2\varphi_4(\varphi_j+\varphi_l) + 3\varphi_j\varphi_l}{(\varphi_4-\varphi_l)^2}\Big)\\ &+ \frac{\varphi_j^{n+2}}{\varphi_j-\varphi_l} - \frac{(\varphi_4-\varphi_j)^2}{(\varphi_4-\varphi_l)^2(\varphi_j-\varphi_l)}\varphi_l^{n+2}\Big]\end{aligned}$$

According to the suppositions above, the inequalities $|\varphi_j| < 1$, $|\varphi_l| < 1$ and $|\varphi_4| < 1$ are satisfied, which gives us the results

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{(r-\alpha)(\varphi_4-\varphi_j)^2}\Big[\varphi_4^{n+1}\Big((n-1)\frac{\varphi_4-\varphi_j}{\varphi_4-\varphi_l} + \frac{\varphi_4^2 - 2\varphi_4(\varphi_j+\varphi_l) + 3\varphi_j\varphi_l}{(\varphi_4-\varphi_l)^2}\Big)\\ &+ \frac{\varphi_j^{n+2}}{\varphi_j-\varphi_l} - \frac{(\varphi_4-\varphi_j)^2}{(\varphi_4-\varphi_l)^2(\varphi_j-\varphi_l)}\varphi_l^{n+2}\Big] = 0\end{aligned}$$

and $(a_n) \in \ell_1$.

According to the results above, $a_n \to 0 (n \to \infty)$ and $(a_n) \in \ell_1$ are provided when $\alpha \notin S$.

Now, considering that the inequalities $|\varphi_1| < 1$, $|\varphi_2| < 1$, $|\varphi_3| < 1$ and $|\varphi_4| < 1$ are satisfied, we can write

$$\begin{aligned}
\|(G(r,s,t,u,v)-\alpha I)^{-1}\|_{(c_0:c_0)} &= \sup_{n\in\mathbb{N}}\sum_{k=1}^{n}|a_k| = \sum_{k=1}^{\infty}|a_k| \\
&\leq \frac{1}{|r-\alpha||(\varphi_1-\varphi_2)(\varphi_1-\varphi_3)(\varphi_1-\varphi_4)|}\sum_{k=1}^{\infty}|\varphi_1|^{k+2} \\
&+ \frac{1}{|r-\alpha||(\varphi_2-\varphi_1)(\varphi_2-\varphi_3)(\varphi_2-\varphi_4)|}\sum_{k=1}^{\infty}|\varphi_2|^{k+2} \\
&+ \frac{1}{|r-\alpha||(\varphi_3-\varphi_1)(\varphi_3-\varphi_2)(\varphi_3-\varphi_4)|}\sum_{k=1}^{\infty}|\varphi_3|^{k+2} \\
&+ \frac{1}{|r-\alpha||(\varphi_4-\varphi_1)(\varphi_4-\varphi_2)(\varphi_4-\varphi_3)|}\sum_{k=1}^{\infty}|\varphi_4|^{k+2} \\
&< \infty.
\end{aligned}$$

This result shows us that $\sigma(G(r,s,t,u,v),c_0) \subset S$.

Now, let's take $\alpha \in S$. In case of $\alpha = r$, we obtain $G(r,s,t,u,v) - \alpha I = G(0,s,t,u,v)$. In that case, since $G(0,s,t,u,v)$ does not have a dense range, it is not invertible.

When $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi$, we have

$$a_n = \frac{\varphi^{n-1}n(n+1)(n+2)}{6(r-\alpha)}$$

for all $n \geq 1$. As per the assumption $\alpha \in S$, the inequality $|\varphi| \geq 1$ is satisfied. Because of this, $a_n \nrightarrow 0$, that is $(G(r,s,t,u,v) - \alpha I)^{-1}$ is not in $B(c_0)$.

When $\varphi = \varphi_i = \varphi_j \neq \varphi_l = \varphi_4$ where $i,j,l \in \{1,2,3\}$, we have

$$a_n = \frac{1}{(r-\alpha)(\varphi-\varphi_4)^3}\left[n(\varphi-\varphi_4)(\varphi^{n+1}+\varphi_4^{n+1}) - 2\varphi\varphi_4(\varphi^n - \varphi_4^n)\right]$$

for all $n \geq 1$. As per the assumption $\alpha \in S$, the inequality $|\varphi_4| \geq 1$ is satisfied. Because of this, $a_n \nrightarrow 0$, that is $(G(r,s,t,u,v) - \alpha I)^{-1}$ is not in $B(c_0)$.

When $\varphi = \varphi_1 = \varphi_2 = \varphi_3 \neq \varphi_4$ we have

$$\begin{aligned}
a_n &= \frac{1}{2(r-\alpha)(\varphi-\varphi_4)^3}\Big[\varphi^n\varphi_4^2(n+1)(n+2) - 2\varphi^{n+1}\varphi_4 n(n+2) \\
&+ \varphi^{n+2}n(n+1) - 2\varphi_4^{n+2}\Big]
\end{aligned}$$

for all $n \geq 1$. As per the assumption $\alpha \in S$, the inequality $|\varphi_4| \geq 1$ is satisfied. Because of this, $a_n \nrightarrow 0$, that is $(G(r,s,t,u,v) - \alpha I)^{-1}$ is not in $B(c_0)$.

When $\varphi_i = \varphi_j = \varphi_4 \neq \varphi_l$ where $i,j,l \in \{1,2,3\}$ we have

$$\begin{aligned}
a_n &= \frac{1}{2(r-\alpha)(\varphi_4-\varphi_l)^3}\Big[\varphi_4^n\varphi_l^2(n+1)(n+2) - 2\varphi_4^{n+1}\varphi_l n(n+2) \\
&+ \varphi_4^{n+2}n(n+1) - 2\varphi_l^{n+2}\Big]
\end{aligned}$$

for all $n \geq 1$. As per the assumption $\alpha \in S$, the inequality $|\varphi_4| \geq 1$ is satisfied. Because of this, $a_n \nrightarrow 0$, that is $(G(r,s,t,u,v) - \alpha I)^{-1}$ is not in $B(c_0)$.

When $\varphi = \varphi_i = \varphi_j \neq \varphi_l \neq \varphi_4$ where $i, j, l \in \{1, 2, 3\}$ we write

$$\begin{aligned} a_n &= \frac{1}{(r-\alpha)(\varphi-\varphi_4)^2}\Bigg[\varphi^{n+1}\left((n-1)\frac{\varphi-\varphi_4}{\varphi-\varphi_l}+\frac{\varphi^2-2\varphi(\varphi_4+\varphi_l)+3\varphi_4\varphi_l}{(\varphi-\varphi_l)^2}\right) \\ &+ \frac{\varphi_4^{n+2}}{\varphi_4-\varphi_l}-\frac{(\varphi-\varphi_4)^2}{(\varphi-\varphi_l)^2(\varphi_4-\varphi_l)}\varphi_l^{n+2}\Bigg] \end{aligned}$$

for all $n \geq 1$. As per the assumption $\alpha \in S$, the inequality $|\varphi_4| \geq 1$ is satisfied. Because of this, $a_n \nrightarrow 0$, that is $(G(r, s, t, u, v) - \alpha I)^{-1}$ is not in $B(c_0)$.

When $\varphi_i = \varphi_4 \neq \varphi_l \neq \varphi_j$ where $i, j, l \in \{1, 2, 3\}$ we write

$$\begin{aligned} a_n &= \frac{1}{(r-\alpha)(\varphi_4-\varphi_j)^2}\Bigg[\varphi_4^{n+1}\left((n-1)\frac{\varphi_4-\varphi_j}{\varphi_4-\varphi_l}+\frac{\varphi_4^2-2\varphi_4(\varphi_j+\varphi_l)+3\varphi_j\varphi_l}{(\varphi_4-\varphi_l)^2}\right) \\ &+ \frac{\varphi_j^{n+2}}{\varphi_j-\varphi_l}-\frac{(\varphi_4-\varphi_j)^2}{(\varphi_4-\varphi_l)^2(\varphi_j-\varphi_l)}\varphi_l^{n+2}\Bigg] \end{aligned}$$

for all $n \geq 1$. As per the assumption $\alpha \in S$, the inequality $|\varphi_4| \geq 1$ is satisfied. Because of this, $a_n \nrightarrow 0$, that is $(G(r, s, t, u, v) - \alpha I)^{-1}$ is not in $B(c_0)$.

So, we may assume that $\alpha \neq r$ and $\varphi_1 \neq \varphi_2 \neq \varphi_3 \neq \varphi_4$.

Since $\alpha \neq r$, $G(r, s, t, u, v) - \alpha I$ is a triangular matrix. In addition, because of $\varphi_1 \neq \varphi_2 \neq \varphi_3 \neq \varphi_4$, from our supposition, $max\{|\varphi_1|, |\varphi_2|, |\varphi_3|\} \leq |\varphi_4|$ can be written, which gives us the result

$$\begin{aligned} a_n &= \frac{1}{r-\alpha}\Bigg[\frac{\varphi_1^{n+2}}{(\varphi_1-\varphi_2)(\varphi_1-\varphi_3)(\varphi_1-\varphi_4)}+\frac{\varphi_2^{n+2}}{(\varphi_2-\varphi_1)(\varphi_2-\varphi_3)(\varphi_2-\varphi_4)} \\ &+ \frac{\varphi_3^{n+2}}{(\varphi_3-\varphi_1)(\varphi_3-\varphi_2)(\varphi_3-\varphi_4)}+\frac{\varphi_4^{n+2}}{(\varphi_4-\varphi_1)(\varphi_4-\varphi_2)(\varphi_4-\varphi_3)}\Bigg] \nrightarrow 0 \end{aligned}$$

that is $\sum_{m=1}^{\infty} |a_m|$ diverges. Therefore, $(G(r, s, t, u, v) - \alpha I)^{-1}$ is not in $B(c_0)$. This shows us that the coverage $S \subset \sigma(G(r, s, t, u, v), c_0)$ is provided, namely $S = \sigma(G(r, s, t, u, v), c_0)$. This completes the proof of theorem. □

Theorem 2.7. $\sigma_p(G(r, s, t, u, v), c_0) = \emptyset$

Proof. Considering $x \neq \theta = (0, 0, 0, ...)$, let us suppose that $G(r, s, t, u, v)x = \alpha x$ in c_0. Let the first non-zero term in the entries of the sequence $x = (x_n)$ be x_{n_0}. Then, if we solve the equation below

$$vx_{n_0-4} + ux_{n_0-3} + tx_{n_0-2} + sx_{n_0-1} + rx_{n_0} = \alpha x_{n_0}$$

we obtain that $\alpha = r$. Moreover, by solving the next equation

$$vx_{n_0-3} + ux_{n_0-2} + tx_{n_0-1} + sx_{n_0} + rx_{n_0+1} = \alpha x_{n_0+1}$$

$x_{n_0} = 0$ is obtained, which contradicts the supposition $x_{n_0} \neq 0$. This completes the proof of theorem. □

As a preliminary to the next theorem, we would like to make the following two remarks and a Lemma. The dual space of c_0 is symbolized with c_0^* which is isometrically isomorphic to the sequence space ℓ_1. If a bounded operator $T : c_0 \longrightarrow c_0$ is defined via the matrix A, then the adjoint operator of T denoted by $T^* : c_0^* \longrightarrow c_0^*$ is defined via the transpose matrix A^t.

Lemma 2.8 ([17])**.** *T has a dense range if and only if T^* is one to one.*

Theorem 2.9. *Let a set S_1 be defined as*

$$S_1 = \{\alpha \in \mathbb{C} : 1 < |\varphi_4|\}.$$

Then, $\sigma_p(G(r, s, t, u, v)^*, c_0^*) = S_1$.

Proof. Let us suppose that $G(r,s,t,u,v)^*x = \alpha x$ in $c_0^* \cong \ell_1$ where $x \neq \theta = (0,0,0,...)$. Let's take a look the following system of linear equations

$$\begin{cases} rx_0 + sx_1 + tx_2 + ux_3 + vx_4 = \alpha x_0 \\ rx_1 + sx_2 + tx_3 + ux_4 + vx_5 = \alpha x_1 \\ rx_2 + sx_3 + tx_4 + ux_5 + vx_6 = \alpha x_2 \\ rx_3 + sx_4 + tx_5 + ux_6 + vx_7 = \alpha x_3 \\ \qquad\qquad\vdots \end{cases}$$

In case of $\alpha = r$ then $x_0 \neq 0$ can be chosen. Because of this, $x = (x_0, 0, 0, 0, ...)$ becomes an eigenvector corresponding to $\alpha = r$.

So now let's assume that $\alpha \neq r$. Then, we obtain

$$\begin{aligned} x_n = & \, vb_{n-2}x_3 - \big[tb_{n-3} + sb_{n-4} + (r-\alpha)b_{n-5}\big]x_2 \\ & - \big[sb_{n-3} + (r-\alpha)b_{n-4}\big]x_1 - (r-\alpha)b_{n-3}x_0 \end{aligned} \tag{2.5}$$

for all $n \geq 4$, where

<u>**i-)**</u> in case of $\varphi_1 \neq \varphi_2 \neq \varphi_3 \neq \varphi_4$

$$\begin{aligned} b_n &= \frac{1}{v}\Bigg[\frac{\frac{1}{\varphi_1^{n+2}}}{(\frac{1}{\varphi_1}-\frac{1}{\varphi_2})(\frac{1}{\varphi_1}-\frac{1}{\varphi_3})(\frac{1}{\varphi_1}-\frac{1}{\varphi_4})} + \frac{\frac{1}{\varphi_2^{n+2}}}{(\frac{1}{\varphi_2}-\frac{1}{\varphi_1})(\frac{1}{\varphi_2}-\frac{1}{\varphi_3})(\frac{1}{\varphi_2}-\frac{1}{\varphi_4})} \\ &+ \frac{\frac{1}{\varphi_3^{n+2}}}{(\frac{1}{\varphi_3}-\frac{1}{\varphi_1})(\frac{1}{\varphi_3}-\frac{1}{\varphi_2})(\frac{1}{\varphi_3}-\frac{1}{\varphi_4})} + \frac{\frac{1}{\varphi_4^{n+2}}}{(\frac{1}{\varphi_4}-\frac{1}{\varphi_1})(\frac{1}{\varphi_4}-\frac{1}{\varphi_2})(\frac{1}{\varphi_4}-\frac{1}{\varphi_3})}\Bigg] \end{aligned}$$

for all $n \geq 1$,

<u>**ii-)**</u> in case of $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi$

$$b_n = \frac{n(n+1)(n+2)}{\varphi^{n-1}6v} \tag{2.6}$$

for all $n \geq 1$,

<u>**iii-)**</u> in case of $\varphi = \varphi_i = \varphi_j = \varphi_k \neq \varphi_l$ where $i,j,l,k \in \{1,2,3,4\}$

$$\begin{aligned} b_n = & \frac{1}{2v(\frac{1}{\varphi}-\frac{1}{\varphi_l})^3}\Bigg[\frac{1}{\varphi^n\varphi_l^2}(n+1)(n+2) - \frac{2}{\varphi^{n+1}\varphi_l}n(n+2) \\ & + \frac{1}{\varphi^{n+2}}n(n+1) - \frac{2}{\varphi_l^{n+2}}\Bigg] \end{aligned} \tag{2.7}$$

for all $n \geq 1$,

<u>**iv-)**</u> in case of $\varphi = \varphi_i = \varphi_j \neq \varphi_k = \varphi_l$ where $i,j,l,k \in \{1,2,3\}$

$$b_n = \frac{1}{v(\frac{1}{\varphi}-\frac{1}{\varphi_l})^3}\left[n\left(\frac{1}{\varphi}-\frac{1}{\varphi_l}\right)\left(\frac{1}{\varphi^{n+1}}+\frac{1}{\varphi_l^{n+1}}\right) - \frac{2}{\varphi\varphi_l}\left(\frac{1}{\varphi^n}-\frac{1}{\varphi_l^n}\right)\right] \tag{2.8}$$

for all $n \geq 1$,

<u>**v-)**</u> in case of $\varphi = \varphi_i = \varphi_j \neq \varphi_l \neq \varphi_k$ where $i,j,l,k \in \{1,2,3\}$ we write

$$\begin{aligned} b_n = & \frac{1}{v(\frac{1}{\varphi}-\frac{1}{\varphi_k})^2}\Bigg[\frac{1}{\varphi^{n+1}}\left((n-1)\frac{\frac{1}{\varphi}-\frac{1}{\varphi_k}}{\frac{1}{\varphi}-\frac{1}{\varphi_l}} + \frac{\frac{1}{\varphi^2}-\frac{2}{\varphi}(\frac{1}{\varphi_k}+\frac{1}{\varphi_l})+\frac{3}{\varphi_k\varphi_l}}{(\frac{1}{\varphi}-\frac{1}{\varphi_l})^2}\right) \\ & + \frac{\frac{1}{\varphi_k^{n+2}}}{\frac{1}{\varphi_k}-\frac{1}{\varphi_l}} - \frac{(\frac{1}{\varphi}-\frac{1}{\varphi_k})^2}{(\frac{1}{\varphi}-\frac{1}{\varphi_l})^2(\frac{1}{\varphi_k}-\frac{1}{\varphi_l})}\frac{1}{\varphi_l^{n+2}}\Bigg] \end{aligned} \tag{2.9}$$

for all $n \geq 1$.

If α is a complex number satisfying the condition $1<|\varphi_4|$, in this case, when $x_0=1$, $x_1=\frac{1}{\varphi_4}$, $x_2=\frac{1}{\varphi_4^2}$ and $x_3=\frac{1}{\varphi_4^3}$ are chosen, the following equality is obtained.

$$\begin{aligned} x_n &= \frac{vb_{n-2}}{\varphi_4^3} - \Big[tb_{n-3}+sb_{n-4}+(r-\alpha)b_{n-5}\Big]\frac{1}{\varphi_4^2} - \Big[sb_{n-3}+(r-\alpha)b_{n-4}\Big]\frac{1}{\varphi_4} \\ &\quad -(r-\alpha)b_{n-3} \\ &= \frac{1}{\varphi_4^n}\left[\frac{-\varphi_4^3+(\varphi_1+\varphi_2+\varphi_3)\varphi_4^2-(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)\varphi_4+\varphi_1\varphi_2\varphi_3}{(\varphi_1-\varphi_4)(\varphi_2-\varphi_4)(\varphi_3-\varphi_4)}\right] \\ &= \frac{1}{\varphi_4^n} \end{aligned}$$

for all $n\geq 4$. As a result, since $\frac{1}{|\varphi_4|}<1$, we infer that $x=(x_n)\in\ell_1$. This shows us that $S_1\subset\sigma_p(G(r,s,t,u,v)^*,\ell_1)$.

Let $\alpha\in\mathbb{C}$ that satisfies the inequality $|\varphi_4|\leq 1$. Under this condition, we should prove that $\alpha\notin\sigma_p(G(r,s,t,u,v)^*,\ell_1)$.

The following equality can be written by using the (2.5) relation:

$$\begin{aligned} \frac{x_{n+1}}{x_n} &= \frac{vb_{n-1}x_3-\big[tb_{n-2}+sb_{n-3}+(r-\alpha)b_{n-4}\big]x_2-\big[sb_{n-2}+(r-\alpha)b_{n-3}\big]x_1-(r-\alpha)b_{n-2}x_0}{vb_{n-2}x_3-\big[tb_{n-3}+sb_{n-4}+(r-\alpha)b_{n-5}\big]x_2-\big[sb_{n-3}+(r-\alpha)b_{n-4}\big]x_1-(r-\alpha)b_{n-3}x_0} \\ &= \frac{b_{n-4}}{b_{n-5}}\left[\frac{\frac{vb_{n-1}}{(r-\alpha)b_{n-4}}x_3-\Big(\frac{tb_{n-2}}{(r-\alpha)b_{n-4}}+\frac{sb_{n-3}}{(r-\alpha)b_{n-4}}+1\Big)x_2-\Big(\frac{sb_{n-2}}{(r-\alpha)b_{n-4}}+\frac{b_{n-3}}{b_{n-4}}\Big)x_1-\frac{b_{n-2}}{b_{n-4}}x_0}{\frac{vb_{n-2}}{(r-\alpha)b_{n-5}}x_3-\Big(\frac{tb_{n-3}}{(r-\alpha)b_{n-5}}+\frac{sb_{n-4}}{(r-\alpha)b_{n-5}}+1\Big)x_2-\Big(\frac{sb_{n-3}}{(r-\alpha)b_{n-5}}+\frac{b_{n-4}}{b_{n-5}}\Big)x_1-\frac{b_{n-3}}{b_{n-5}}x_0}\right]. \end{aligned}$$

Here, we would like to remind again that $z_1z_2z_3z_4=\frac{v}{r-\alpha}$, $z_1+z_2+z_3+z_4=-\frac{s}{r-\alpha}$ and $z_1z_2+z_1z_3+z_1z_4+z_2z_3+z_2z_4+z_3z_4=\frac{t}{r-\alpha}$.

So now let's analyze the behaviour of the roots $\varphi_1,\varphi_2,\varphi_3$ and φ_4.

Case 1: $|\varphi_i|<|\varphi_j|<|\varphi_k|<|\varphi_4|\leq 1$, where $i,j,k\in\{1,2,3\}$, that is $\varphi_1\neq\varphi_2\neq\varphi_3\neq\varphi_4$. So, we obtain

$$\lim_{n\to\infty}\frac{b_{n-3}}{b_{n-4}}=\lim_{n\to\infty}\frac{b_{n-4}}{b_{n-5}}=\frac{1}{\varphi_i},\ \lim_{n\to\infty}\frac{b_{n-2}}{b_{n-4}}=\lim_{n\to\infty}\frac{b_{n-3}}{b_{n-5}}=\frac{1}{\varphi_i^2}$$

and

$$\lim_{n\to\infty}\frac{b_{n-1}}{b_{n-4}}=\lim_{n\to\infty}\frac{b_{n-2}}{b_{n-5}}=\frac{1}{\varphi_i^3}$$

If $\frac{v}{(r-\alpha)\varphi_i^3}x_3-\Big(\frac{t}{(r-\alpha)\varphi_i^2}+\frac{s}{(r-\alpha)\varphi_i}+1\Big)x_2-\Big(\frac{s}{(r-\alpha)\varphi_i^2}+\frac{1}{\varphi_i}\Big)x_1-\frac{1}{\varphi_i^2}x_0=0$, so, we get $x_n=\frac{1}{\varphi_4^n}x_0$. Thus, $x=(x_n)\notin\ell_1$ since $|\varphi_4|\leq 1$ is satisfied. If the above equality does not hold, we get

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right|=\frac{1}{|\varphi_i|}>1.$$

Case 2: $|\varphi|=|\varphi_1|=|\varphi_2|=|\varphi_3|=|\varphi_4|\leq 1$. In this situation, $\varphi=\varphi_1=\varphi_2=\varphi_3=\varphi_4$ and by using (2.5), we write

$$\lim_{n\to\infty}\frac{b_{n-3}}{b_{n-4}}=\lim_{n\to\infty}\frac{b_{n-4}}{b_{n-5}}=\frac{1}{\varphi},\ \lim_{n\to\infty}\frac{b_{n-2}}{b_{n-4}}=\lim_{n\to\infty}\frac{b_{n-3}}{b_{n-5}}=\frac{1}{\varphi^2}$$

and

$$\lim_{n\to\infty}\frac{b_{n-1}}{b_{n-4}}=\lim_{n\to\infty}\frac{b_{n-2}}{b_{n-5}}=\frac{1}{\varphi^3}$$

Thus, we have

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right|=\frac{1}{|\varphi|}>1.$$

Case 3: $|\varphi_j| < |\varphi_i| = |\varphi_k| = |\varphi_4| \leq 1$, where $i, j, k \in \{1, 2, 3\}$. In this situation, $\varphi_j \neq \varphi_i = \varphi_k = \varphi_4$ and by using (2.7), we write

$$\lim_{n\to\infty} \frac{b_{n-3}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-4}}{b_{n-5}} = \frac{1}{\varphi_j}, \lim_{n\to\infty} \frac{b_{n-2}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-3}}{b_{n-5}} = \frac{1}{\varphi_j^2}$$

and

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-2}}{b_{n-5}} = \frac{1}{\varphi_j^3}$$

Thus, we have

$$\lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{|\varphi_j|} > 1.$$

Case 4: $|\varphi| = |\varphi_i| = |\varphi_j| < |\varphi_k| = |\varphi_4| \leq 1$, where $i, j, k \in \{1, 2, 3\}$. In this situation, $\varphi = \varphi_i = \varphi_j \neq \varphi_k = \varphi_4$ and by using (2.8), we write

$$\lim_{n\to\infty} \frac{b_{n-3}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-4}}{b_{n-5}} = \frac{1}{\varphi}, \lim_{n\to\infty} \frac{b_{n-2}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-3}}{b_{n-5}} = \frac{1}{\varphi^2}$$

and

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-2}}{b_{n-5}} = \frac{1}{\varphi^3}$$

Thus, we have

$$\lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{|\varphi|} > 1.$$

Case 5: $|\varphi| = |\varphi_i| = |\varphi_j| < |\varphi_k| < |\varphi_4| \leq 1$, where $i, j, k \in \{1, 2, 3\}$. In this situation, $\varphi = \varphi_i = \varphi_j \neq \varphi_k \neq \varphi_4$ and by using (2.9), we write

$$\lim_{n\to\infty} \frac{b_{n-3}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-4}}{b_{n-5}} = \frac{1}{\varphi}, \lim_{n\to\infty} \frac{b_{n-2}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-3}}{b_{n-5}} = \frac{1}{\varphi^2}$$

and

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-2}}{b_{n-5}} = \frac{1}{\varphi^3}$$

Thus, we have

$$\lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{|\varphi|} > 1.$$

Case 6: $|\varphi_i| < |\varphi_j| < |\varphi_k| = |\varphi_4| \leq 1$, where $i, j, k \in \{1, 2, 3\}$. In this situation, $\varphi_i \neq \varphi_j \neq \varphi_k = \varphi_4$ and by using (2.9), we write

$$\lim_{n\to\infty} \frac{b_{n-3}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-4}}{b_{n-5}} = \frac{1}{\varphi_i}, \lim_{n\to\infty} \frac{b_{n-2}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-3}}{b_{n-5}} = \frac{1}{\varphi_i^2}$$

and

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-2}}{b_{n-5}} = \frac{1}{\varphi_i^3}$$

Thus, we have

$$\lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{|\varphi_i|} > 1.$$

Case 7: $|\varphi_i| < |\varphi_j| = |\varphi_k| < |\varphi_4| \leq 1$, where $i, j, k \in \{1, 2, 3\}$. In this situation, $\varphi_i \neq \varphi_j = \varphi_k \neq \varphi_4$ and by using (2.9), we write

$$\lim_{n\to\infty} \frac{b_{n-3}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-4}}{b_{n-5}} = \frac{1}{\varphi_i}, \lim_{n\to\infty} \frac{b_{n-2}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-3}}{b_{n-5}} = \frac{1}{\varphi_i^2}$$

and

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_{n-4}} = \lim_{n\to\infty} \frac{b_{n-2}}{b_{n-5}} = \frac{1}{\varphi_i^3}$$

Thus, we have

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{|\varphi_i|} > 1.$$

Case 8: $|\varphi| = |\varphi_1| = |\varphi_2| = |\varphi_3| = |\varphi_4| < 1$. In this situation, $\varphi = \varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$ and by using (2.6), we write

$$\lim_{n\to\infty}\frac{b_{n-3}}{b_{n-4}} = \lim_{n\to\infty}\frac{b_{n-4}}{b_{n-5}} = \frac{1}{\varphi},\ \lim_{n\to\infty}\frac{b_{n-2}}{b_{n-4}} = \lim_{n\to\infty}\frac{b_{n-3}}{b_{n-5}} = \frac{1}{\varphi^2}$$

and

$$\lim_{n\to\infty}\frac{b_{n-1}}{b_{n-4}} = \lim_{n\to\infty}\frac{b_{n-2}}{b_{n-5}} = \frac{1}{\varphi^3}$$

Thus, we have

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{|\varphi|} > 1.$$

Case 9: $|\varphi| = |\varphi_1| = |\varphi_2| = |\varphi_3| = |\varphi_4| = 1$. In this situation, $\varphi = \varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$. Let's assume $\alpha \in \sigma_p(G(r,s,t,u,v)^*, \ell_1)$. This gives us that $\theta \neq x \in \ell_1$. Using the expressions (2.5) and (2.6), the following equality can be written

$$x_n = \frac{(n-1)(n-2)(n-3)}{6\varphi^n}\Big[\frac{n}{n-3}\varphi^3 x_3 - \frac{3n}{n-2}\varphi^2 x_2 + \frac{3n}{n-1}\varphi x_1 - x_0\Big]$$

for all $n \geq 4$.

Since $x = (x_n) \in \ell_1$, that is $\lim_{n\to\infty}|x_n| = 0$, we have two choices as follows. The first of these $x_3 = x_2 = x_1 = x_0 = 0$ gives the result $x = \theta$ and the second of these $x_3 = \frac{x_0}{\varphi^3}, x_2 = \frac{x_0}{\varphi^2}, x_1 = \frac{x_0}{\varphi}$ gives the result $x_n = \frac{x_0}{\varphi^n}$. These cause contradiction. Then, $\alpha \notin \sigma_p(G(r,s,t,u,v)^*, \ell_1)$ is obtained.

Considering d'Alembert test and cases 1, 2, 3, 4, 5, 6, 7, 8 together, we decide that $x = (x_n) \notin \ell_1$. In case 9, $\alpha \in \sigma_p(G(r,s,t,u,v)^*, \ell_1)$ gives rise to a contradiction. This completes the proof of theorem. □

Theorem 2.10. *Let the following set S_2 be defined by*

$$S_2 = \{\alpha \in \mathbb{C} : |\varphi_4| = 1\}.$$

Then, the following two equalities are satisfied.

(I) $\sigma_r(G(r,s,t,u,v), c_0) = S_1$,

(II) $\sigma_c(G(r,s,t,u,v), c_0) = S_2$.

Here, the set S_1 is defined in the Theorem 2.9.

Proof. (I) It is known from Theorem 2.9 that $\sigma_p(G(r,s,t,u,v)^*, c_0^*) = S_1$. Therefore, it is clear that $G(r,s,t,u,v)^* - \alpha I$ is not one to one, $\forall\alpha \in S_1$. When this result and Lemma 2.8 are evaluated together, it is concluded that $G(r,s,t,u,v) - \alpha I$ does not have a dense range, $\forall\alpha \in S_1$.

(II) Since $\sigma(G(r,s,t,u,v), c_0)$ consists of three disjoint parts as follows:

$$\sigma(G(r,s,t,u,v), c_0) = \sigma_p(G(r,s,t,u,v), c_0) \cup \sigma_c(G(r,s,t,u,v), c_0) \cup \sigma_r(G(r,s,t,u,v), c_0)$$

one can conclude that $\sigma_c(G(r,s,t,u,v), c_0) = S_2$. □

Theorem 2.11. *When $\alpha \in \sigma_c(G(r,s,t,u,v), c_0)$, it is satisfied that $\alpha \in II_2\sigma(G(r,s,t,u,v), c_0)$* .

Proof. Let's take $\alpha \in \sigma_c(G(r,s,t,u,v), c_0)$. In this case, it can be deduced from Theorem 2.6 that $(G(r,s,t,u,v) - \alpha I)^{-1}$ is discontinuous and therefore $(G(r,s,t,u,v) - \alpha I)^{-1}$ is unbounded, under the condition $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$. Also from Theorem 2.9, we know that $G(r,s,t,u,v)^* - \alpha I$ is one to one. Thus, by using Lemma 2.8, we conclude that $G(r,s,t,u,v) - \alpha I$ has a dense range.

In the next step, it should be shown that $G(r,s,t,u,v)-\alpha I$ is not surjective. Let's take $y=(1,0,0,...)\in c_0$ provided $(G(r,s,t,u,v)-\alpha I)x=y$. Then, we get $x=(a_n)$, where a_n is defined as in the proof of Theorem 2.6. In conclusion, based on Theorem 2.6, we obtain that $x=(a_n)\notin c_0$ in case of α belongs to the spectrum. This gives us the result $\alpha\in II_2\sigma(G(r,s,t,u,v),c_0)$. This completes the proof of theorem. □

Theorem 2.12. *Let μ_1,μ_2 and μ_3 be random three roots of the equation $sz^3+tz^2+uz+v=0$. Then, the followings hold.*

(I) *If the inequality $|\mu_j|<1$ is provided for every $j\in\{1,2,3\}$, then $r\in III_1\sigma(G(r,s,t,u,v),c_0)$,*
(II) *If the inequality $|\mu_j|\geq 1$ is provided for at least $j\in\{1,2,3\}$, then $r\in III_2\sigma(G(r,s,t,u,v),c_0)$*

Proof. According to Theorem 2.10(I), it is known that $G(r,s,t,u,v)-\alpha I$ is in state III_1 or III_2 in case of $\alpha=r$. Moreover, the left inverse of $G(0,s,t,u,v)$ symbolized by $H=(h_{nk})$ is defined by

$$h_{nk}=\begin{cases}\frac{1}{s}\sum\limits_{j=0}^{n-k+1}\sum\limits_{\nu=0}^{n-k-j+1}\mu_1^{n-k-j-\nu+1}\mu_2^{\nu}\mu_3^{j} & , \quad 1\leq k\leq n+1\\ 0 & , \quad k=0 \text{ or } k>n+1\end{cases}$$

for all $n,k\in\mathbb{N}$. In this case, by using the Lemma 2.2, the followings can be written:

(I) When $|\mu_j|<1$ is provided for all $j\in\{1,2,3\}$, then $H=(h_{nk})\in B(c_0)$, that is $G(0,s,t,u,v)$ has a continuous inverse. For this reason, $r\in III_1\sigma(G(r,s,t,u,v),c_0)$, in case of $|\mu_j|<1$ for all $j\in\{1,2,3\}$.

(II) When $|\mu_j|\geq 1$ is provided for at least $j\in\{1,2,3\}$, then $H=(h_{nk})\notin B(c_0)$ that is $G(0,s,t,u,v)$ does not have a continuous inverse. For this reason, $r\in III_2\sigma(G(r,s,t,u,v),c_0)$ whenever $|\mu_j|\geq 1$ for at least $j\in\{1,2,3\}$. This completes the proof of theorem. □

Theorem 2.13. *Let $\alpha\neq r$ be given. If α belongs to $\sigma_r(G(r,s,t,u,v),c_0)$, then α belongs to $III_2\sigma(G(r,s,t,u,v),c_0)$.*

Proof. Assume that $\alpha\neq r$ and $\alpha\in\sigma_r(G(r,s,t,u,v),c_0)$. Then, the matrix $G(r,s,t,u,v)-\alpha I$ becomes a triangle which yields that it has an inverse. Moreover, it is known that $1<|\varphi_4|$ in case of $\alpha\in\sigma_r(G(r,s,t,u,v),c_0)$. This results in $\lim_{n\to\infty}|a_n|=\infty$ namely $G(r,s,t,u,v)-\alpha I$ has an unbounded inverse, (a_n) used here is as defined in the proof of Theorem 2.6. Also, we infer from Theorem 2.9 and Lemma 2.8 that $G(r,s,t,u,v)^*-\alpha I$ is not one to one, namely $G(r,s,t,u,v)-\alpha I$ does not have a dense range. If these are taken into consideration, we conclude that $\alpha\in III_2\sigma(G(r,s,t,u,v),c_0)$. This completes the proof of theorem. □

Now, let us take a bounded operator such that $L:c\longrightarrow c, L(x)=Dx$. Then, $L^*:c^*\longrightarrow c^*$ acting on $\mathbb{C}\oplus\ell_1$ is defined by

$$L^*=\begin{bmatrix}\chi & 0\\ \eta & D^t\end{bmatrix}$$

where $D=(d_{nk})$ is an infinite matrix, D^t is transpose of D, $\chi=\chi(D)=\lim_{n\to\infty}\sum_k d_{nk}-\sum_k\lim_{n\to\infty}d_{nk}$ and $\eta=(\eta_k)$ is a sequence defined by $\eta_k=\lim_{n\to\infty}d_{nk}$ for all $k\in\mathbb{N}$.

By applying this concept to the operator $G(r,s,t,u,v):c\longrightarrow c$, one can write

$$G(r,s,t,u,v)^*=\begin{bmatrix}r+s+t+u+v & 0\\ 0 & G(r,s,t,u,v)^t\end{bmatrix}$$

for $G(r,s,t,u,v)^*\in B(\ell_1)$.

Theorem 2.14. *Given a set S_1 as defined in Theorem 2.9. Then, $\sigma_p(G(r,s,t,u,v)^*,c^*)=S_1\cup\{r+s+t+u+v\}$.*

Proof. By taking a sequence $x \neq \theta = (0,0,0,...)$, let us suppose that $G(r,s,t,u,v)^*x = \alpha x$ in ℓ_1. Then, by solving the following system of linear equations

$$\begin{cases} (r+s+t+u+v)x_0 = \alpha x_0 \\ rx_1 + sx_2 + tx_3 + ux_4 + vx_5 = \alpha x_1 \\ rx_2 + sx_3 + tx_4 + ux_5 + vx_6 = \alpha x_2 \\ rx_3 + sx_4 + tx_5 + ux_6 + vx_7 = \alpha x_3 \\ \qquad \vdots \end{cases}$$

we obtain

$$\begin{aligned} x_n = {} & vb_{n-2}x_4 - \big[tb_{n-3} + sb_{n-4} + (r-\alpha)b_{n-5}\big]x_3 \\ & - \big[sb_{n-3} + (r-\alpha)b_{n-4}\big]x_2 - (r-\alpha)b_{n-3}x_1 \end{aligned}$$

for all $n \geq 5$. Here, the definition of the sequence (b_n) is as in Theorem 2.9.

By taking $x_0 \neq 0$, one can obtain $\alpha = r+s+t+u$. Because of this, α becomes an eigenvalue according to eigenvector $x = (x_0, 0, 0, ...)$.

By taking $\alpha \neq r+s+t+u+v$, one can obtain $x_0 = 0$. Then, if the method used in Theorem 2.9 is followed similarly, $x = (x_n) \notin \ell_1$ is obtained. This completes the proof of theorem. □

The fine spectrum of the quintet band matrix operator $G(r,s,t,u,v)$ on c can be determined by following similar methods by substituting c instead of c_0 in the relevant Theorems above. Therefore, the next theorem is given without proof.

Theorem 2.15. *Given two sets S and S_2 as defined in Theorem 2.6 and Theorem 2.10, in turn. Then, the following expressions are satisfied.*

(i) $\sigma(G(r,s,t,u,v),c) = S$,

(ii) $\sigma_p(G(r,s,t,u,v),c) = \emptyset$,

(iii) $\sigma_r(G(r,s,t,u,v),c) = \sigma_p(G(r,s,t,u,v)^*, c^*)$,

(iv) $\sigma_c(G(r,s,t,u,v),c) = S_2 \setminus \{r+s+t+u+v\}$,

(v) *In case of $\alpha \notin \sigma(G(r,s,t,u,v),c)$, $G(r,s,t,u,v) - \alpha I \in I_1$ is provided*

(vi) *If the inequality $|\mu_j| < 1$ is provided for all $j \in \{1,2,3\}$, then $r \in III_1\sigma(G(r,s,t,u,v),c)$,*

(vii) *If the inequality $|\mu_j| \geq 1$ is provided for at least $j \in \{1,2,3\}$, then $r \in III_2\sigma(G(r,s,t,u,v),c)$*

(viii) *In case of $\alpha \in \sigma_c(G(r,s,t,u,v),c)$, $\alpha \in II_2\sigma(G(r,s,t,u,v),c)$ is provided*

(ix) *In case of $\alpha \in \sigma_r(G(r,s,t,u,v),c) \setminus \{r\}$, $\alpha \in III_2\sigma(G(r,s,t,u,v),c)$ is provided*

where μ_1, μ_2 and μ_3 are random three roots of the equation $sz^3 + tz^2 + uz + v = 0$.

3. Conclusion

When the quintet band matrix operator $G(r,s,t,u,v)$ defined here is examined, it is easily seen that $G(1,-4,6,-4,1) = \Delta^4$, $G(r,s,t,u,0) = Q(r,s,t,u)$, $G(1,-3,3,-1,0) = \Delta^3$, $G(r,0,0,u,0) = D(r,0,0,s)$, $G(r,s,t,0,0) = B(r,s,t)$, $G(1,-2,1,0,0) = \Delta^2$, $G(r,s,0,0,0) = B(r,s)$ and $G(1,-1,0,0,0) = \Delta$, where Δ^4, $Q(r,s,t,u)$, Δ^3, $B(r,s,t)$, Δ^2, $B(r,s)$ and Δ are named fourth order difference, quadruple band, third order difference, triple band, second order difference, double band and difference matrix, respectively. Because of this, our results obtained from the quintet band matrix operator are more general and more comprehensive than the results on [4], [5], [10], [13], [14] and [22]. Additionally, our study includes the right shift and Zweier matrices [7].

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