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On the Approximation of Highly Oscillatory Integral Equations Via Radial Kernels

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Abstract

In this work we used radial kernels for computing more generalized fast oscillatory integral equations. The proposed method is based on radial kernels. The present method is efficient for computing oscillatory integral equations with large oscillation parameters. The proposed method is very robust and capable of handling fast oscillatory integral equations.

1. INTRODUCTION

The approximation of integral equations of first kind containing highly oscillatory Bessel kernels

$$\int_a^x J_\nu(\sigma(x-t))y(t)\,dt = h(x),\ \ x\in[a,T], \tag{1}$$

where $y(x)$ is a function to be determined for values of x in the domain $[a,T]$. Such type of problems received the attention of many researchers in [2, 4, 10, 11]. Here $h(x)$ is a smooth function and σ is a large oscillation parameter. The big difficulty of this type of Volterra integral equation (1) is the large oscillation parameter σ. Due to this parameter the kernel function $J_0(\sigma(x-t))$ become highly oscillatory. Consequently standard numerical methods cannot be used immediately to approximate these types of integral equations. It is shown in the work of (see for example [10], Sec.1.8.1) that for the case when $h(x)\in C^1[a,T]$ the analytic solution of (1) may be given as

$$y(x) = \frac{1}{\sigma}\left(\frac{d^2}{dx^2}+\sigma^2\right)^2 \int_a^x (x-t)\,J_1(\sigma(x-t))h(t)\,dt,\ \ x\in[a,T]. \tag{2}$$

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Although this analytical solution involve Bessel integral with high oscillation parameter and cannot be evaluated in the present closed form. We need to use some reliable and robust numerical schemes to get approximate solution. For the evaluation of the integral equations of the type

$$\int_a^x K(x,t)\,y(t)\,dt = h(x), \quad x \in [a,T], \tag{3}$$

robust numerical techniques have been developed by the authors in (see [1, 3, 4, 6]). However these approaches cannot be applied to the integral equations of the type (1), due to the kernel $J_\nu(\sigma x)$ contain highly oscillatory large parameter σ. The evaluation of the integral containing $J_\nu(\sigma x)$ using standard quadrature technique is extremely difficult where the computation cost exponentially increases with increase in σ (see [5, 7]). In the present work a kernel based numerical scheme is constructed to approximate the integral equation of type (1). In other words we constructed a numerical scheme using radial kernels to approximate the integral

$$I = \int_a^b f(x)\,J(\sigma x)dx, \tag{4}$$

where σ is parameter of oscillatory function $J_\nu(\sigma x)$. The usual numerical methods for evaluating such integrals face difficulties. Some efficient methods are available in the literature (for example see [7, 9, 16, 17, 18]) to approximate the given type of integrals very efficiently. In the proposed method the problem of integral computation is converted into a system of ODEs without any boundary conditions which satisfies some differential conditions. The resultant system of ODEs is approximate with the proposed kernel based method.

2. PRELIMINARIES

2.1. Lemma 2.0.1 [12]

Let $\{x_i\,,\,f_i\}_{i=1}^N$ be N data points in $X=\{x_1,....,x_N\}\subset\Omega=[a,b]$, let $h=\sup_{x\in\Omega}\min_{x_j\in X}\|x-x_j\|$ be the fill distance in Ω, the kernel-based interpolant $s\in C^\beta(\Omega)$ for the function $f\in C^\beta(\Omega)$ for all points in X. Then the error estimate

$$\|f-s\|_{L_2(\Omega)} \le 2C\,h^\beta\,|f|_{W_2{}^\beta},$$

Where the $W_2{}^\beta$ is the Sobolev space, $\beta \le N$, and $C=\max\{C_1\sqrt{b-a}\,,\,C_2\}$, where C_1 and C_2 depends on β and N. Further if $l\in N$ and $|\alpha|\le l$, then

$$|D^\alpha f(x)-D^\alpha s(x)| \le C_l\,h^{l-|\alpha|}\,|f|_{W_2{}^\beta}.$$

2.2. Lemma 2.0.2 [7]

Given a vector $u(x)=(u_1(x),u_2(x),.....,u_m(x))^t$ which satisfies (10) and $q(x)$ be monotonic over $[a,b]$, then $W(x)=u(q(x))$ satisfies the equation

$$W'(x)=B(x)W(x), \tag{5}$$

With $B(x)$ is of order $m \times m$ containing functions which are non-rapid oscillatory.

2.3. Lemma 2.0.3 [7]

Suppose the vectors $W = \left(w_1(x), w_2(x), ..., w_k(x)\right)^t$ and $Z = \left(z_1(x), z_2(x), ...z_l(x)\right)^t$ satisfies the following equations

$$W'(x) = B_1(x)W(x), \tag{6}$$

And

$$Z'(x) = B_2(x)Z(x), \tag{7}$$

respectively, where matrices B_1 and B_2 are matrices of order $k \times k$, and $l \times l$ of non-highly oscillatory functions. Then the vector $u = \left\{w_j z_i \middle| j = 1,..., k, i = 1, ..., l\right\}$ satisfies the equation

$$u'(x) = A(x)u(x), \tag{8}$$

with $m = kl$ and $A(x)$ is matrix of order $m \times m$ of non-highly oscillatory functions.

3. LOCALIZED KERNEL BASED METHOD

Consider a more generalized class of rapidly oscillatory integrals like

$$I = \int_a^b f^t(x)u(x)\,dx \equiv \int_a^b \langle f, u\rangle(x)\,dx, \tag{9}$$

With $f(x) = \left(f_i(x), i = 1, ..., m\right)^t$, is a vector of non-rapidly oscillatory functions, and $u(x) = \left(u_i(x), i = 1, ..., m\right)^t$, is vector of linearly independent rapidly oscillatory functions. It is shown in the work [7] that $\left\{u_i\right\}_{i=1}^m$ hold the ODE system

$$u'(x) = A(x)u(x), \tag{10}$$

and eventually the matrix $A(x)$ of order $m \times m$ becomes a matrix of non-oscillatory functions. The work [7] leads to approximate I in (9) by the derivative of given known function. It is assume to find $p(x) = \left(p_1(x), ..., p_m(x)\right)^t$, Such that

$$\langle p, u\rangle' = \langle f, u\rangle. \tag{11}$$

Thus to approximate the integral I by

$$I \approx \int_a^b \langle p, u\rangle'(x)\,dx = p^t(b)u(b) - p^t(a)u(a). \tag{12}$$

Expanding (11) and using (10) we get

$$\langle p, u\rangle' = \langle p', u\rangle + \langle p, u'\rangle = \langle p', u\rangle + \langle p, Au\rangle = \langle p' + A^t p, u\rangle \approx \langle f, u\rangle. \tag{13}$$

By the assumption that $\{u_i\}_{i=1}^{m}$ are linearly independent which implies that p must approximate solution of the ODEs system

$$Lp = p' + A^t p = f, \tag{14}$$

where the function f and the matrix A are non- rapidly oscillatory. It was investigated in the work [8], that the system (14) may have a solution which is not oscillatory at all. This non-oscillatory solution of the PDEs system can be approximated accurately by collocation methods using some suitable basis functions. In the present work we extended the idea [7] to approximate the linear differential operator L and construct a sparse differentiation matrix corresponding to (14). To construct local interpolant (see for example [13, 14]), at each center $x_i \in \Omega_i \subset \Omega$, we define

$$v^i(x_k) = \sum_{x_j \in \Omega_i} c_j^i \phi^i\left(\left\|x_k - x_j\right\|\right),\ x_k, x_j \in \Omega_i, \tag{15}$$

where c_j^i are the expansion coefficients, $\left\|x_k - x_j\right\|$, denotes the norm of the difference of centers x_k and x_j, $\phi(r)$ a radial kernel, with the radial distance $r \geq 0$, and $\Omega_i \subset \Omega$ is a local sub-domain corresponding to each center x_i and contains n nearest centers around the center x_i. For each node x_i, we get the $n \times n$ linear systems

$$v^i = B^i c^i, i = 1, ..., N, \tag{16}$$

where the matrix B^i has the elements $b_{k,j}^i = \phi\left(\left\|x_k - x_j\right\|\right), x_k, x_j \in \Omega_i$. Next to approximate the linear differential operator $Lv(x)$, apply L to (15) we get

$$Lv^i(x_i) = \sum_{x_j \in \Omega_i} c_j^i L\phi^i\left(\left\|x_i - x_j\right\|\right). \tag{17}$$

The expression in (17) may be given by dot product of two vectors,

$$Lv^i(x_i) = w^i . c^i, \tag{18}$$

Where the entries of the vector w^i are given by

$$L\phi^i\left(\left\|x_i - x_j\right\|\right),\ x_j \in \Omega_i. \tag{19}$$

eliminating the coefficients c^i from (16) and (18) we get

$$Lv^i(x_i) = w^i\left(B^i\right)^{-1} v^i = D^i v^i, \tag{20}$$

Where,

$$D^i = w^i\left(B^i\right)^{-1}, \tag{21}$$

is row vector of order $1\times N$ containing n non-zero entries and remaining $N-n$ zeros entries. Consequently for all centers x_i, $i=1,....,N$, the differential operator L can be approximated by a sparse differentiation matrix D of order $N\times N$ given by

$$Lv = Dv. \tag{22}$$

4. GLOBAL KERNEL BASED METHOD

For the given points $x_1,.....,x_N \in \Omega$. The function $p(x)$ may be approximated as linear combination of radial kernels $\phi(r)$, [19]

$$p(x)=\sum_{j=i}^{N} c_j \phi\left(\left\|x-x_j\right\|\right),\ x_j \in \Omega,\ j=1,2,.....,N, \tag{23}$$

let L be the linear operator then from eq (23), we get

$$Lp(x)=\sum_{j=i}^{N} c_j L\phi\left(\left\|x-x_j\right\|\right),\ x_j \in \Omega,\ j=1,2,.....,N. \tag{24}$$

To approximate the system of ODEs (14) we compute (24) for the evaluation points $x \in \{x_1,.....,x_N\} \subset \Omega$ and get

$$Dc = f, \tag{25}$$

where c is $N\times 1$ vector of expansion coefficients and D is $N\times N$ matrix with entries $L\phi\left(\left\|x_i-x_j\right\|\right)_{i,\,j=1}^{N}$, and f is $N\times 1$ vector. The solution of (14) using the global kernel based method is given by

$$p = Hc, \tag{26}$$

where H is $M\times N$ evaluation matrix with the entries $\phi\left(\left\|x_i-x_j\right\|\right), i=1,....,M$, $j=1,....,N$ and the value of c can be obtained from (25).

5. APPLICATION OF THE PROPOSED METHOD

This section is devoted to demonstrate the validity and applicability of the present kernel based method to highly oscillatory integral equations containing the kernel like $J_\nu(\sigma x)$. We consider the integral equation (1) whose solution can be converted in the form containing two highly oscillatory Bessel integrals.

Lemma 5.0.4 ([11])

The exact solution of (1) can be transformed into the following form

$$y(x) = h'(x) + \sigma^2 \int_a^x F(t) J_0(\sigma t)\, dt + \sigma \int_a^x G(t) J_1(\sigma t)\, dt, \tag{27}$$

With $F(t) = h(x-t)$, $G(t) = -h'(x-t)$.
Thus the numerical solution of (1) can be converted into the computation of two highly oscillatory Bessel integrals. To evaluate the integral

$$I_n = \int_a^b f(x) J_\upsilon(\sigma x)\, dx \tag{28}$$

For the present method to approximate the integral the bases are the Bessel functions and can be obtained from the recurrence relations of Bessel function

$$\begin{bmatrix} J_{\nu-1}(\sigma x) \\ J_\nu(\sigma x) \end{bmatrix}' = \begin{bmatrix} \frac{\nu-1}{x} & -\sigma \\ \sigma & \frac{-\nu}{x} \end{bmatrix} \begin{bmatrix} J_{\nu-1}(\sigma x) \\ J_\nu(\sigma x) \end{bmatrix}. \tag{29}$$

Consequently the vector $u(x) = \left[J_{\nu-1}(\sigma x), J_\nu(\sigma x)\right]$ satisfy the differential condition (4) with corresponding matrix

$$A = \begin{bmatrix} \frac{\nu-1}{x} & -\sigma \\ \sigma & \frac{-\nu}{x} \end{bmatrix}. \tag{30}$$

So that kernel based method is applied to approximate the integrals of the form

$$\int_a^b \left(f_1(x) J_{\nu-1}(\sigma x) + f_2(x) J_\nu(\sigma x) \right) dx. \tag{31}$$

The local kernel based approximate scheme of the ODE system (15), corresponding to the integral (25)

$$\begin{bmatrix} p_1' + \left(\frac{\nu-1}{x}\right) p_1 & \sigma p_2 \\ -\sigma p_1 & p_2' - \left(\frac{\nu}{x}\right) p_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{32}$$

For $\nu = 1$, the above system can be given by

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{33}$$

where the block matrices are given by $d_{11} = D_x$, $d_{12} = \sigma A_{rbf}$, $d_{21} = -\sigma A_{rbf}$, $d_{22} = D_x - diag\left(\frac{1}{x}\right) A_{rbf}$.

The $N \times N$ matrices D_x and A_{rbf} when $L \equiv I$ can be obtained from (24). The functions $p_1(x), p_2(x), f_1(x) \equiv f(x)$, and $f_2(x) \equiv 0$ are $N \times 1$ vectors. Hence the approximation to the integral (24) is given by the following numerical scheme

$$I_n = \sum_{i=1}^{2} \left[u_i(b) p_i(b) - u_i(a) p_i(a) \right]. \tag{34}$$

5.1. VOLTERRA INTEGRAL EQUATION

The results of the present method corresponding to the integral equation

$$\int_a^x J_0\left(\sigma(x-t)\right) y(t)\, dt = x \sin x, \tag{35}$$

by the kernel based method using Lemma (5.0.4) the solution is given by

$$y(b) = h'(b) + \sigma^2 \int_a^b F(x) J_0(\sigma x) dx + \sigma \int_a^b G(x) J_1(\sigma x) dx \ , \ b \in [a, T], \tag{36}$$

With $F(x) = h(b-x)$, $G(x) = -h'(b-x)$.
To compute the integral

$$\int_a^b F(x)\, J_0(\sigma x) dx \ , \tag{37}$$

set $f_1 = F$ and $f_2 = 0$ while for computing the integral

$$\int_a^b G(x)\, J_1(\sigma x) dx \ , \tag{38}$$

we used $f_1 = 0$ and $f_2 = G$ in the numerical scheme (33) respectively. The results are shown in Table (1). The results show very fast convergence rate even for small number of collocation points and a very large oscillation parameter. The present numerical scheme is well equipped to approximate the integral equations for large oscillation parameters. We used the compactly supported radial kernels $\phi(r) = (1-\varepsilon r)^6_+ \left[35(\varepsilon r)^2 + 18\varepsilon r + 3\right]$ to approximate the solution with GK- method. In order to get sparse differential matrix with a compact support, we used $\varepsilon = 0.2$. Similarly we used the LK-method and the results are shown in Table (1). We used different number of nodes n in local domain $\Omega_i \subset \Omega$ and N in global domain Ω. The advantage of the local method over the global method is that the resultant differentiation matrix is sparse while that obtained with the global method is dense. The $N \times N$ sparse matrix is assembled by solving small size matrices of order $n \times n$ in each local sub-domain $\Omega_i, i = 1, 2, ..., N$, where $n \leq N$. The LK-method can be used for large number of collocations points, while the GK-method cannot be used for a large number of points due to dense differentiation matrix.
The LK-method $\rightarrow$ GK-method when $n \rightarrow N$.

Approximate value of the integral equation using the global kernel based method (GK-method) and local kernel based method (LK-method) for different values of oscillation parameters σ and various number of collocations nodes N with $a=1, x=2$.

Table 1. *Methods*

(GK-Method)

σ	$N=3$	$N=5$	$N=9$
5	1.6000e-003	2.9824e-005	5.1503e-006
10	2.7828e-005	1.5892e-005	1.2086e-006
50	2.7471e-006	9.7379e-007	2.4281e-007
100	2.8806e-008	5.9128e-009	5.2042e-009

(LK-Method)

σ	$(n,N)=(3,3)$	$(n,N)=(3,5)$	$(n,N)=(3,9)$
5	9.6794e-004	3.1499e-004	8.3331e-004
10	3.5693e-004	3.1307e-005	9.0688e-005
50	7.1936e-007	2.5468e-007	2.6829e-007
100	3.1063e-007	9.5544e-009	1.8529e-008

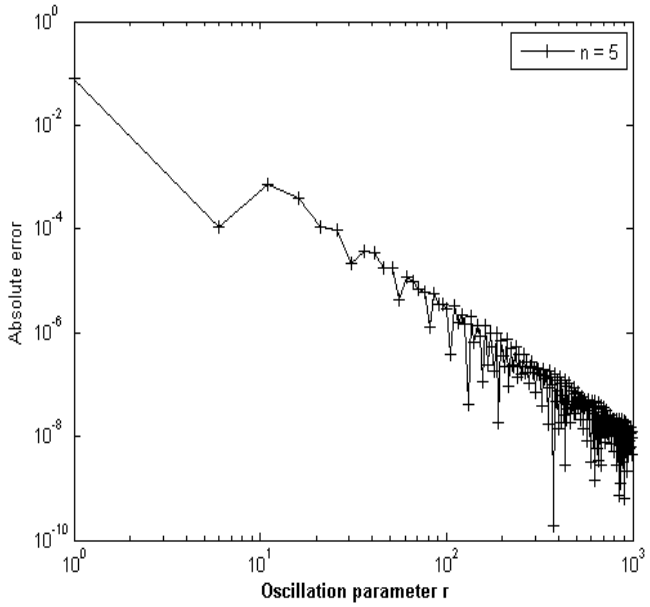

Figure 1. *Error versus oscillation parameter σ, corresponding to integral equation (1)*

6. CONCLUSION

In the present work a kernel based numerical method which is proposed is the extension of [7] in the context of radial kernel functions. The use of radial kernels is very much suitable for computing these types of integral equation. Because in the present collocation method the differentiation matrix to be solved with great accuracy. The real benefit of the kernel based method is that we can extend it to compute such types integral equations of highly oscillatory kernels for large oscillation parameters.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors**.**

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