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POLYANALYTIC EQUATIONS IN DOMAINS FORMED BY THE INTERSECTION OF THREE CIRCLES

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Abstract. In this paper, we study the Schwarz boundary value problem for inhomogeneous polyanalytic equations in a triangular domain in the complex plane defined by intersections of three circles. We introduce a family of higher-order Pompeiu-type integral operators that provide explicit solutions to the problem.

1. Introduction and Preliminaries

Boundary value problems for generalized analytic and polyanalytic equations have been extensively investigated in various domains due to their importance in both complex analysis and applied mathematics. The classical foundations were established by Vekua [1] and Begehr [2], who developed the theory of generalized analytic functions and associated integral operator methods. Operator-theoretic approaches were further advanced by Begehr and Hile [3], as well as Kim and Kang [4]. In recent years, boundary value problems of Schwarz- and Dirichlet-type in triangular domains have been studied in [5], while related problems for polyanalytic and bipolyanalytic equations in half-plane and disc-type domains were analyzed in [6], [7], [8], [9], and [10]. Moreover, Robin-type and Neumann problems for higher-order equations in ring domains were considered in [11], [12] and [13].

The present study is motivated by the growing interest in understanding higher-order complex structures in non-circular geometries, such as triangles, which naturally arise in conformal mapping theory, potential theory, and models of physical processes involving piecewise-homogeneous media. The main theorem presented here extends the previously known results by providing explicit integral representations for the Schwarz problem in the triangular domain T, together with uniqueness and solvability conditions that may serve as a basis for further analytical and numerical investigations in complex boundary problems.

Motivated by these developments, we now introduce the triangular domain T in the complex plane, which serves as the setting for our main results.

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Let T be a triangular domain in the complex plane $\mathbb{C}$, see [5], defined by

$$T = \left\{ z \in \mathbb{C} : |z - \epsilon_k| < \sqrt{2}, \quad k = 1, 2, 3 \right\},$$

where

$$\epsilon_1 = 1, \quad \epsilon_2 = -1, \quad \epsilon_3 = -i\sqrt{3}.$$

The sets

$$C_1 = \left\{ z \in \mathbb{C} : |z - 1| = \sqrt{2} \right\}, \quad C_2 = \left\{ z \in \mathbb{C} : |z + 1| = \sqrt{2} \right\}, \quad C_3 = \left\{ z \in \mathbb{C} : |z + i\sqrt{3}| = \sqrt{2} \right\}$$

represent three circles in the complex plane centered at ϵ_1, ϵ_2, and ϵ_3, each having radius $\sqrt{2}$. The triangular domain T is the intersection of the interiors of these three circles.

For a point $z \in T$, define

$$z_2 = -\frac{1}{z}, \quad z_3 = \frac{-(1 + i\sqrt{3})z + 1 - i\sqrt{3}}{(1 - i\sqrt{3})z + 1 + i\sqrt{3}}, \quad z_4 = \frac{(i\sqrt{3} - 1)z - 1 - i\sqrt{3}}{(-1 + i\sqrt{3})z + 1 - i\sqrt{3}},$$

$$z_5 = \frac{\overline{z} + 1}{\overline{z} - 1}, \quad z_6 = \frac{-\overline{z} + 1}{\overline{z} + 1}, \quad z_7 = \frac{-i\sqrt{3}\,\overline{z} - 1}{\overline{z} - i\sqrt{3}}, \quad z_8 = \frac{\overline{z} - i\sqrt{3}}{i\sqrt{3}\,\overline{z} + 1}.$$

Then

$$z_2, z_3, z_4, z_5, z_6, z_7, z_8 \notin T.$$

The boundary of T, denoted by ∂T, consists of three circular arcs. The vertices of T are given by

$$a = -\tfrac{1}{2}(\sqrt{3} - 1)(1 + i), \quad b = \tfrac{1}{2}(\sqrt{3} - 1)(1 - i), \quad c = -i,$$

which correspond to the pairwise intersections of C_1, C_2, and C_3.

We now formulate the Schwarz boundary value problem for the inhomogeneous polyanalytic equation in T. The problem involves finding a function w satisfying a given differential equation in T and boundary conditions on ∂T.

Theorem 1.1 ([14]). *The Schwarz boundary value problem for the inhomogeneous polyanalytic equation in the triangular domain T,*

$$\partial_{\bar{z}}^n w(z) = f(z) \quad \text{in } T, \qquad \operatorname{Re} \partial_{\bar{z}}^s w(z) = \vartheta_s \quad \text{on } \partial T,$$

together with the additional conditions

$$\sum_{j=1}^{3} \frac{2}{\pi i} \int_{\partial T \cap C_j} \operatorname{Im}\left(\partial_{\bar{z}}^s w(\varsigma)\right) \frac{d\varsigma}{\varsigma - \epsilon_j} = c_s, \quad s = 0, \dots, n-1,$$

is uniquely solvable. The solution is given by

$$w(z) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \sum_{j=1}^{3} \frac{1}{\pi i} \int_{\partial T \cap C_j} (\varsigma - z + \overline{\varsigma - z})^k \vartheta_k(\varsigma) \left(K_1(z, \varsigma) - \frac{2}{\varsigma - \epsilon_j} \right) d\varsigma$$

$$+ \frac{(-1)^n}{(n-1)!\pi} \iint_T (\varsigma - z + \overline{\varsigma - z})^{n-1} \left[f(\varsigma) K_1(z, \varsigma) - \overline{f(\varsigma) K_2(z, \varsigma)} \right] d\kappa d\delta + \sum_{s=0}^{n-1} \frac{(z + \bar{z})^s}{s!} i c_s \tag{1.1}$$

where

$$K_1(z, \varsigma) = \sum_{k=1}^{4} \frac{1}{\varsigma - z_k}, \quad K_2(z, \varsigma) = \sum_{k=5}^{8} \frac{1}{\varsigma - z_k}, \quad \varsigma = \kappa + i\delta. \tag{1.2}$$

2. A Class of Integral Operators Related To The Schwarz Problem

To introduce a class of integral operators associated with the Schwarz problem, we define

$$\tilde{T}_{1,T} f(z) := \frac{-1}{\pi} \iint_T \left[f(\varsigma) K_1(z,\varsigma) - \overline{f(\varsigma) K_2(z,\varsigma)} \right] d\kappa d\delta \tag{2.1}$$

for $f \in L_{p,2}(T),\ p > 2$. This operator provides a particular solution to the Schwarz problem for the inhomogeneous Cauchy–Riemann equation in the special case $n = 1$ of Theorem 1.1, under homogeneous boundary conditions. By utilizing the differentiability properties of the well-known Pompeiu operator T introduced by Vekua [1], we establish the following properties for $\tilde{T}_{1,T}$:

(i) The operator $\tilde{T}_{1,T} f$ admits a generalized first-order derivative given by

$$\frac{\partial}{\partial \bar{z}} \tilde{T}_{1,T} f = f, \tag{2.2}$$

for all $f \in L_{p,2}(T),\ p > 2$.

(ii) The generalized derivative of $\tilde{T}_{1,T}$ with respect to z is defined by

$$\frac{\partial}{\partial z} \tilde{T}_{1,T} f := \tilde{\Pi}_{1,T} f = \frac{-1}{\pi} \iint_T \left[f(\varsigma) \frac{\partial}{\partial z} K_1(z,\varsigma) - \overline{f(\varsigma)} \frac{\partial}{\partial z} \overline{K_2}(z,\varsigma) \right] d\kappa d\delta, \quad z \in T, \tag{2.3}$$

where $\tilde{\Pi}_{1,T}$ is a strongly singular integral operator.

The explicit solution (1.1) to the above Schwarz problem with homogeneous boundary conditions motivates the introduction of the following higher-order Pompeiu-type operators.

Definition 2.1. *Let $n \in \mathbb{N}$ and $f \in L_{p,2}(T),\ p > 2$. For $z \in T$, define*

$$\tilde{T}_{n,T} f(z) := \frac{(-1)^n}{(n-1)!\,\pi} \iint_T (\varsigma - z + \overline{\varsigma - z})^{n-1} \left[f(\varsigma) K_1(z,\varsigma) - \overline{f(\varsigma) K_2(z,\varsigma)} \right] d\kappa d\delta, \tag{2.4}$$

with the convention

$$\tilde{T}_{0,T} f := f.$$

We will show inductively that the operator $\tilde{T}_{n,T}$ can be expressed as an n-fold composition of $\tilde{T}_{1,T}$.

Theorem 2.1. *For $n \in \mathbb{N}$, $f \in L_{p,2}(T),\ p > 2$, and $z \in T$, it holds that*

$$\tilde{T}_{n,T} f(z) = \left(\tilde{T}_{1,T} \right)^n f(z), \tag{2.5}$$

where $\left(\tilde{T}_{1,T} \right)^n$ denotes the n-fold composition of $\tilde{T}_{1,T}$.

Proof. We proceed by induction on n.

Base case: For $n = 1$, the statement is trivial since $\tilde{T}_{1,T} = \tilde{T}_{1,T}$.

Inductive step: Assume the statement holds for some $n - 1 \geq 1$, i.e., $\tilde{T}_{n-1,T} = \left(\tilde{T}_{1,T} \right)^{n-1}$.

Using the definition (2.4) and the property of generalized derivatives, we compute the $\bar{z}$-derivative of $\tilde{T}_{n,T}f$:

$$\frac{\partial}{\partial \bar{z}}\tilde{T}_{n,T}f(z) = \frac{(-1)^n}{(n-1)!\pi}\iint\limits_T \frac{\partial}{\partial \bar{z}}\left[(\varsigma - z + \overline{\varsigma - z})^{n-1}\right]\left[f(\varsigma)K_1(z,\varsigma) - \overline{f(\varsigma)K_2(z,\varsigma)}\right] d\kappa d\delta. \tag{2.6}$$

Noting that

$$\frac{\partial}{\partial \bar{z}}(\varsigma - z + \overline{\varsigma - z})^{n-1} = -(n-1)(\varsigma - z + \overline{\varsigma - z})^{n-2}, \tag{2.7}$$

we get

$$\frac{\partial}{\partial \bar{z}}\tilde{T}_{n,T}f(z) = \frac{(-1)^n}{(n-1)!\pi}\cdot\big(-(n-1)\big)\iint\limits_T (\varsigma - z + \overline{\varsigma - z})^{n-2}\left[f(\varsigma)K_1(z,\varsigma) - \overline{f(\varsigma)K_2(z,\varsigma)}\right] d\kappa d\delta. \tag{2.8}$$

Simplifying we obtain,

$$\frac{\partial}{\partial \bar{z}}\tilde{T}_{n,T}f(z) = \frac{(-1)^{n-1}}{(n-2)!\pi}\iint\limits_T (\varsigma - z + \overline{\varsigma - z})^{n-2}\left[f(\varsigma)K_1(z,\varsigma) - \overline{f(\varsigma)K_2(z,\varsigma)}\right] d\kappa d\delta = \tilde{T}_{n-1,T}f(z). \tag{2.9}$$

This shows that $\tilde{T}_{n,T}f$ is a right inverse (in the sense of generalized derivatives) of $\partial_{\bar{z}}$ applied n times.

On the boundary ∂T, since $\varsigma \in \partial T$ implies that

$$\varsigma - z + \overline{\varsigma - z} = 2\,\mathrm{Re}(\varsigma - z), \tag{2.10}$$

we have

$$\tilde{T}_{n,T}f(z) = \frac{(-1)^n}{(n-1)!\pi}\iint\limits_T \big(2\,\mathrm{Re}(\varsigma - z)\big)^{n-1}\left[f(\varsigma)K_1(z,\varsigma) - \overline{f(\varsigma)K_2(z,\varsigma)}\right] d\kappa d\delta. \tag{2.11}$$

Since the expression inside the integral is purely imaginary on the boundary (due to the kernels K_1, K_2 and conjugation), it follows that

$$\mathrm{Re}\,\tilde{T}_{n,T}f(z) = 0, \quad z \in \partial T. \tag{2.12}$$

Also, by the properties of the kernels and the integrals over the boundary arcs,

$$\sum_{j=1}^{3}\frac{2}{\pi i}\int_{\partial T \cap C_j} \mathrm{Im}\,\tilde{T}_{n,T}f(\varsigma)\frac{d\varsigma}{\varsigma - \epsilon_j} = 0. \tag{2.13}$$

Applying Theorem 1.1, we get the integral representation

$$\tilde{T}_{n,T}f(z) = \frac{-1}{\pi}\iint\limits_T \left[\tilde{T}_{n-1,T}f(\varsigma)K_1(z,\varsigma) - \overline{\tilde{T}_{n-1,T}f(\varsigma)}\overline{K_2(z,\varsigma)}\right] d\kappa d\delta, \tag{2.14}$$

which can be rewritten as

$$\tilde{T}_{n,T}f(z) = \tilde{T}_{1,T}\left(\tilde{T}_{n-1,T}f\right)(z). \tag{2.15}$$

By the induction hypothesis, this equals

$$\tilde{T}_{1,T}\left(\tilde{T}_{1,T}^{n-1}f\right)(z) = \tilde{T}_{1,T}^{n}f(z). \tag{2.16}$$

This completes the induction and the proof. $\square$

Remark. *By the Leibniz rule, for $0 \le l \le n-1$, the following formula holds:*

$$\partial_z^l \tilde{T}_{n,T} f(z) = \frac{1}{\pi} \iint\limits_T \sum_{j=0}^{l} \binom{l}{j} \frac{(-1)^{n-j}}{(n-1-j)!} (\varsigma - z + \overline{\varsigma - z})^{n-1-j} \left[f(\varsigma) \partial_z^{l-j} K_1(z,\varsigma) - \overline{f(\varsigma)} \partial_z^{l-j} \overline{K_2(z,\varsigma)} \right] d\kappa d\delta. \tag{2.17}$$

Alternatively, it can be written as

$$\partial_z^l \tilde{T}_{n,T} f(z) = \frac{1}{\pi} \iint\limits_T \sum_{j=0}^{l} \binom{l}{j} \frac{(-1)^{n-j}}{(n-1-j)!} (\varsigma - z + \overline{\varsigma - z})^{n-1-j} \times$$
$$\left[f(\varsigma) \sum_{k=1}^{4} \frac{A_k^j(\varsigma, z_k)}{(\varsigma - z_k)^{l-j+1}} - \overline{f(\varsigma)} \sum_{k=5}^{8} \frac{A_k^j(\bar\varsigma, \bar z_k)}{(\bar\varsigma - \bar z_k)^{l-j+1}} \right] d\kappa d\delta, \tag{2.18}$$

where the coefficients $A_k^j(\varsigma, z_k)$ (or $A_k^j(\bar\varsigma, \bar z_k)$) are defined as follows: after computing the derivatives $\partial_z^{l-j} K_1(z,\varsigma)$ (or $\partial_z^{l-j}\overline{K_2(z,\varsigma)}$), rewrite them as sums of terms whose denominators are powers of $(\varsigma - z_k)$ (or $(\bar\varsigma - \bar z_k)$), then the numerators of these terms give $A_k^j(\varsigma, z_k)$ (or $A_k^j(\bar\varsigma, \bar z_k)$).

For the case $l = 0$, we set

$$A_k^0(\varsigma, z_k) = K_1(z,\varsigma), \quad \textit{and} \quad A_k^0(\bar\varsigma, \bar z_k) = \overline{K_2(z,\varsigma)}.$$

Taking $l = n-1$ and differentiating with respect to z, we obtain

$$\tilde{\Pi}_{n,T} f(z) := \partial_z^n \tilde{T}_{n,T} f(z) = \frac{(-1)^n}{\pi (n-1)!} \iint\limits_T (\varsigma - z + \overline{\varsigma - z})^{n-1} \left[f(\varsigma)\, \partial_z^n K_1(z,\varsigma) - \overline{f(\varsigma)}\, \partial_z^n \overline{K_2(z,\varsigma)} \right] d\kappa d\delta$$
$$+ \frac{1}{\pi} \iint\limits_T \sum_{m=0}^{n-2} \left\{ \binom{n-1}{m} + \binom{n-1}{m+1} \right\} \frac{(-1)^{n-(m+1)}}{(n-(m+2))!} (\varsigma - z + \overline{\varsigma - z})^{n-(m+2)} \times$$
$$\left[f(\varsigma)\, \partial_z^{n-(m+1)} K_1(z,\varsigma) - \overline{f(\varsigma)}\, \partial_z^{n-(m+1)} \overline{K_2(z,\varsigma)} \right] d\kappa d\delta \tag{2.19}$$

Hence, as a direct consequence of Theorem 2.1, we can formulate the following result, which summarizes the main analytical properties of the operators $\tilde{T}_{n,T}$.

Theorem 2.2. *Let $f \in L_{p,2}(T)$ with $p > 2$ and $n \in \mathbb{N}$. Then, for each $0 \le l \le n-1$, the function $\tilde{T}_{n,T} f$ satisfies:*

$$\partial_{\bar z}^l \tilde{T}_{n,T} f(z) = \tilde{T}_{n-1,T} f(z), \tag{2.20}$$

$$\operatorname{Re} \partial_{\bar z}^l \tilde{T}_{n,T} f(z) = 0 \quad \textit{on} \quad \partial T, \tag{2.21}$$

$$\sum_{j=1}^{3} \frac{2}{\pi i} \int_{\partial T \cap C_j} \operatorname{Im} \left(\partial_{\bar z}^l \tilde{T}_{n,T} f(\varsigma) \right) \frac{d\varsigma}{\varsigma - \epsilon_j} = 0 \quad \textit{on} \quad \partial T. \tag{2.22}$$

The first property shows that successive $\bar z$-derivatives of $\tilde{T}_{n,T} f$ reproduce lower-order operators in the same family, reflecting an inductive structure. The second property expresses that the real part of these derivatives vanishes on the boundary ∂T, which corresponds to a homogeneous boundary condition in the Schwarz problem. The third property ensures that certain boundary integrals involving the imaginary part of these derivatives vanish, which is linked to the orthogonality conditions in the solution representation.

Moreover, $\partial_z^l \tilde{T}_{n,T} f(z)$ is a *weakly singular* integral for $0 \le l \le n-1$, while for $l = n$ it becomes a *strongly singular* integral.

In the next subsection, we focus on establishing the *boundedness* and *continuity* of the operators $\tilde{T}_{n,T}$ on suitable function spaces, which will be essential for further applications to the Schwarz problem.

Lemma 2.3. *Let $n \in \mathbb{N}$ and $f \in L_{p,2}(T)$ with $p > 2$. Then the following estimates hold:*

(i) *If $0 \le l \le n-1$, we have*

$$\left|\partial_z^l \tilde{T}_{n,T} f(z)\right| \le C(l,p)\, \|f\|_{L_{p,2}(T)}. \tag{2.23}$$

(ii) *For the difference estimates:*

- *If $0 \le l < n-1$,*

$$\left|\partial_z^l \tilde{T}_{n,T} f(z^*) - \partial_z^l \tilde{T}_{n,T} f(z^{**})\right| \le C(l,p)\, \|f\|_{L_{p,2}(T)}\, |z^* - z^{**}|. \tag{2.24}$$

- *If $l = n-1$,*

$$\left|\partial_z^l \tilde{T}_{n,T} f(z^*) - \partial_z^l \tilde{T}_{n,T} f(z^{**})\right| \le C(l,p)\, \|f\|_{L_{p,2}(T)}\, |z^* - z^{**}|^{\frac{p-2}{p}}. \tag{2.25}$$

Here, $z^*, z^{**} \in T$ and $C(l,p)$ denotes a constant depending only on the differentiation order l and the space parameter p.

Proof. **(i)** Using the definition and Leibniz rule, we have

$$\left|\partial_z^l \tilde{T}_{n,T} f(z)\right| = \left|\frac{1}{\pi}\iint\limits_T \sum_{j=0}^{l}\binom{l}{j}\frac{(-1)^{n-j}}{(n-1-j)!}(\varsigma - z + \overline{\varsigma - z})^{n-1-j}\left[f(\zeta)\partial_z^{l-j}K_1(z,\varsigma) - \overline{f(\varsigma)}\partial_z^{l-j}\overline{K_2(z,\varsigma)}\right] d\kappa d\delta\right|.$$

This is bounded by

$$\le \sum_{j=0}^{l}\binom{l}{j}\frac{1}{\pi}\iint\limits_T |\varsigma - z + \overline{\varsigma - z}|^{n-1-j}|f(\varsigma)|\left(\sum_{k=1}^{4}\frac{|A_k^j(\varsigma, z_k)|}{|\varsigma - z_k|^{\,l-j+1}} + \sum_{k=5}^{8}\frac{|A_k^j(\bar{\varsigma}, \bar{z}_k)|}{|\bar{\varsigma} - \bar{z}_k|^{\,l-j+1}}\right) d\kappa d\delta.$$

Since the points z_k lie outside the domain T and $z \in T$, we have

$$|\varsigma - z| < |\varsigma - z_k|, \quad k = 1, \dots, 8,$$

which allows us to estimate

$$\le \sum_{j=0}^{l}\binom{l}{j}\frac{1}{\pi}\iint\limits_T |\varsigma - z + \overline{\varsigma - z}|^{n-1-j}|f(\varsigma)|\frac{\sum_{k=1}^{4}|A_k^j(\varsigma, z_k)| + \sum_{k=5}^{8}|A_k^j(\bar{\varsigma}, \bar{z}_k)|}{|\varsigma - z|^{l-j+1}} d\kappa d\delta.$$

Assuming the functions A_k^j are bounded on T by a constant M_j, we write

$$\sum_{k=1}^{4}|A_k^j(\varsigma, z_k)| + \sum_{k=5}^{8}|A_k^j(\bar{\varsigma}, \bar{z}_k)| \le M_j.$$

Also, note that

$$|\varsigma - z + \overline{\varsigma - z}| = 2|\operatorname{Re}(\varsigma - z)| \le 2|\varsigma - z|.$$

Therefore,

$$\left|\partial_z^l \tilde{T}_{n,T} f(z)\right| \le C\sum_{j=0}^{l}\iint_T \frac{|\varsigma - z|^{n-1-j}}{|\varsigma - z|^{l-j+1}}|f(\varsigma)| d\kappa d\delta = C\sum_{j=0}^{l}\iint_T |\varsigma - z|^{n-l-2}|f(\varsigma)| d\kappa d\delta,$$

where C is a positive constant depending on n, l, and the bounds M_j.

Finally, applying Hölder's inequality with exponents $p > 2$ and its conjugate q such that $1/p + 1/q = 1$, we get

$$\left|\partial_z^l \tilde{T}_{n,T} f(z)\right| \leq C \left(\iint_T |\varsigma - z|^{q(n-l-2)} d\kappa d\delta \right)^{1/q} \|f\|_{L_{p,2}(T)}.$$

The integral on the right-hand side is finite for appropriate choices of q because the singularity is integrable on the triangular domain T.

Thus,

$$\left|\partial_z^l \tilde{T}_{n,T} f(z)\right| \leq C(l,p)\|f\|_{L_{p,2}(T)},$$

which completes the proof.

(ii) Consider the difference

$$\left|\partial_z^l \tilde{T}_{n,T} f(z^*) - \partial_z^l \tilde{T}_{n,T} f(z^{**})\right|.$$

Using the integral representation and the notation introduced in part (i), we have

$$\left|\partial_z^l \tilde{T}_{n,T} f(z^*) - \partial_z^l \tilde{T}_{n,T} f(z^{**})\right| \leq M^*(k,j) \iint_T \left| |\varsigma - z^*|^{n-l-2} - |\varsigma - z^{**}|^{n-l-2} \right| |f(\varsigma)|\, d\kappa d\delta. \tag{2.26}$$

Now, by applying Lemma 4.5 from [3] (specifically inequality (4.14)), which gives a Hölder-type control for differences of singular kernels, we obtain

$$||\varsigma - z^*|^\alpha - |\varsigma - z^{**}|^\alpha| \leq C(\alpha)|z^* - z^{**}|^\beta$$

for some $\beta > 0$ depending on α and the integrability exponent p.

In our case, $\alpha = n - l - 2$, which is negative since $n > l$, so the kernel is singular but integrable.

Thus,

$$\left|\partial_z^l \tilde{T}_{n,T} f(z^*) - \partial_z^l \tilde{T}_{n,T} f(z^{**})\right| \leq M^*(k,j) C(l,p) |z^* - z^{**}|^{\frac{p-2}{p}} \|f\|_{L_{p,2}(T)}, \tag{2.27}$$

where $\frac{p-2}{p} \in (0,1)$ due to $p > 2$.

Hence, the operator $\partial_z^l \tilde{T}_{n,T}$ is Hölder continuous with exponent $\frac{p-2}{p}$ for $l = n-1$.

For the case $l < n-1$, the kernel singularity is milder, and the Hölder exponent improves to 1, giving Lipschitz continuity:

$$\left|\partial_z^l \tilde{T}_{n,T} f(z^*) - \partial_z^l \tilde{T}_{n,T} f(z^{**})\right| \leq C(l,p)|z^* - z^{**}|\|f\|_{L_{p,2}(T)}.$$

This completes the proof of the difference estimates. □

We now state the boundedness property of the operator $\tilde{\Pi}_{n,T}$ in the following lemma.

Lemma 2.4. *Let $n \in \mathbb{N}$ and $f \in L_{p,2}(T)$ with $p > 2$. Then the operator $\tilde{\Pi}_{n,T}$ is bounded on $L_p(T)$, and there exists a constant $C_p > 0$ such that*

$$\|\tilde{\Pi}_{n,T} f\|_{L_p(T)} \leq C_p \|f\|_{L_p(T)}. \tag{2.28}$$

Proof.

$$\tilde{\Pi}_{1,T} := \frac{-1}{\pi} \iint\limits_T \left[f(\varsigma)\, \partial_z K_1(z,\varsigma) - \overline{f(d\kappa d\delta)}\, \partial_z \overline{K_2(z,\varsigma)} \right] d\kappa d\delta \tag{2.29}$$

$$= \frac{-1}{\pi}\iint_T \left[f(\varsigma)\sum_{k=1}^{4}\frac{z_k'}{(\varsigma - z_k)^2} - \overline{f(\varsigma)}\sum_{k=5}^{8}\frac{\bar{z_k}'}{(\bar{\varsigma}-\bar{z_k})^2}\right] d\kappa d\delta \tag{2.30}$$

This can be considered as the summation of the strongly singular integral operators

$$\Pi f(z) = \frac{-1}{\pi}\iint_T \frac{f(\varsigma)}{(\varsigma - z_k)^2}\, d\kappa d\delta, \quad k = 1, 2, 3, 4, \tag{2.31}$$

and

$$\Pi f(z) = \frac{-1}{\pi}\iint_T \frac{f(\varsigma)}{(\bar{\varsigma} - \bar{z_k})^2}\, d\kappa d\delta, \quad k = 5, 6, 7, 8. \tag{2.32}$$

(see [1], [4]). These operators are known to be L_p-bounded. Similar arguments to those in the proof of (i) in the previous theorem apply.

$$\left|\tilde{\Pi}_{n,T} f(z)\right| = \left| \frac{(-1)^n}{\pi(n-1)!}\iint_T (\varsigma - z + \overline{\varsigma - z})^{n-1}\left[f(\varsigma)\,\partial_z^n K_1(z,\varsigma) - \overline{f(\varsigma)}\,\partial_z^n \overline{K_2(z,\varsigma)}\right] d\kappa d\delta \right|$$

$$+ \left| \frac{1}{\pi}\iint_T \sum_{m=0}^{n-2}\left\{\binom{n-1}{m} + \binom{n-1}{m+1}\right\}\frac{(-1)^{n-(m+1)}}{(n-(m+2))!}(\varsigma - z + \overline{\varsigma - z})^{n-(m+2)} \right| \times$$

$$\left| f(\varsigma)\,\partial_z^{n-(m+1)} K_1(z,\varsigma) - \overline{f(\varsigma)}\,\partial_z^{n-(m+1)}\overline{K_2(z,\varsigma)}\right| d\kappa d\delta \tag{2.33}$$

$$\leq \frac{1}{\pi}\iint_T |\varsigma - z + \overline{\varsigma - z}|^{n-1}\,|f(\varsigma)| \left(\sum_{k=1}^{4}\frac{|A_k^j(\varsigma, z_k)|}{|\zeta - z_k|^{n+1}} + \sum_{k=5}^{8}\frac{|A_k^j(\bar{\varsigma}, \bar{z}_k)|}{|\bar{\varsigma} - \bar{z}_k|^{n+1}}\right) d\kappa d\delta$$

$$+ \sum_{m=0}^{n-2}\left\{\binom{n-1}{m} + \binom{n-1}{m+1}\right\}\iint_T |\varsigma - z + \overline{\varsigma - z}|^{n-(m+2)}\,|f(\varsigma)|\left(\sum_{k=1}^{4}\frac{|A_k^j(\varsigma, z_k)|}{|\varsigma - z_k|^{n-m}} + \sum_{k=5}^{8}\frac{|A_k^j(\bar{\varsigma}, \bar{z}_k)|}{|\bar{\varsigma} - \bar{z}_k|^{n-m}}\right) d\kappa d\delta \tag{2.34}$$

$$\leq M_1(k,j)\iint_T \frac{|f(\varsigma)|}{|\varsigma - z|^2}\, d\kappa d\delta + M_2(k,j)\iint_T \frac{|f(\varsigma)|}{|\varsigma - z|^2}\, d\kappa d\delta \tag{2.35}$$

These are well-known Π-operators and are L_p-bounded; hence, we obtain the desired result. □

3. Schwarz Problem for a Linear Equation

We consider the following linear elliptic complex partial differential equation in the domain T:

$$\partial_{\bar{z}}^n w + \sum_{j=1}^{n}\left(q_{1j}(z)\partial_z^j\partial_{\bar{z}}^{n-j} w + q_{2j}(z)\partial_z^{n-j}\partial_{\bar{z}}^j\overline{w}\right) + \sum_{l=0}^{n-1}\sum_{m=0}^{l}\left(a_{ml}(z)\partial_{\bar{z}}^{l-m}\partial_z^m w + b_{ml}(z)\partial_z^{l-m}\partial_{\bar{z}}^m\overline{w}\right) = f(z), \tag{3.1}$$

where the coefficients $a_{ml}, b_{ml}, f \in L_{p,2}(T)$, and q_{1j}, q_{2j} are measurable bounded functions satisfying the smallness condition

$$\sum_{j=1}^{n}|q_{1j}(z)| + |q_{2j}(z)| < q_0 < 1.$$

We formulate the following boundary value problem:

Schwarz Problem. Find a function $w \in W^{n,p}(T)$ that satisfies equation (3.1) and the boundary conditions

$$\operatorname{Re} \partial_{\bar z}^{s} w(z) = 0 \quad \text{on } \partial T, \quad \text{and} \quad \sum_{j=1}^{3} \frac{2}{\pi i} \int_{\partial T \cap C_j} \operatorname{Im}\left(\partial_{\bar z}^{s} w(\varsigma)\right) \frac{d\varsigma}{\varsigma - \epsilon_j} = 0, \quad s = 0, \dots, n-1, \tag{3.2}$$

where C_j and ϵ_j denote appropriate contours and points related to the domain boundary.

The following theorem establishes an equivalent integral equation formulation of the problem:

Theorem 3.1. *The Schwarz problem defined by equations* (3.1) *and* (3.2) *is equivalent to the singular integral equation*

$$(I + \hat{\Pi} + \hat{K})g = f, \tag{3.3}$$

where the solution w is represented as

$$w = \tilde{T}_{n,T} g,$$

and the operators $\hat{\Pi}$ and $\hat{K}$ are defined by

$$\hat{\Pi} g = \sum_{j=1}^{n} \left(q_{1j}(z) \tilde{\Pi}_{j,T} g + q_{2j}(z) \overline{\tilde{\Pi}_{j,T} g} \right), \tag{3.4}$$

$$\hat{K} g = \sum_{l=0}^{n-1} \sum_{m=0}^{l} \left(a_{ml}(z) \frac{\partial^m \tilde{T}_{n-l+m,T} g}{\partial z^m} + b_{ml}(z) \frac{\partial^m \overline{\tilde{T}_{n-l+m,T} g}}{\partial \bar{z}^m} \right). \tag{3.5}$$

Next, we provide conditions under which the singular integral equation (3.3) is of Fredholm type and the Fredholm alternative applies.

Theorem 3.2. *Suppose the following smallness condition holds:*

$$q_0 \max_{1 \le j \le n} \|\tilde{\Pi}_{j,T}\|_{L_p(T)} < 1. \tag{3.6}$$

Then, the Schwarz problem (3.1) *with boundary conditions* (3.2) *admits a solution of the form $w = \tilde{T}_{n,T} g$, where $g \in L_{p,2}(T), p > 2$ solves the singular integral equation* (3.3)*.*

Proof. By Lemmas 2.3 and 2.4 and the Arzelà-Ascoli theorem, the operator $\hat{K}$ is compact on $L_p(T)$. Additionally, $\hat{\Pi}$ is a bounded linear operator on $L_p(T)$. The smallness condition (3.6) guarantees the invertibility of the operator $I + \hat{\Pi}$.

Since $\hat{K}$ is compact, the operator $I + \hat{\Pi} + \hat{K}$ is a Fredholm operator with index zero. Therefore, the Fredholm alternative applies to equation (3.3), ensuring the existence of a solution g. Consequently, $w = \tilde{T}_{n,T} g$ provides a solution to the original Schwarz problem. □

4. Conclusion

In this paper, we established explicit integral representations for the Schwarz boundary value problem associated with inhomogeneous polyanalytic equations in a circular triangle domain. The obtained formulas extend classical approaches by guaranteeing both uniqueness and solvability under Schwarz boundary conditions, thereby enriching the general theory of boundary value problems in complex domains.

Beyond their theoretical contribution, these results provide analytical tools that may be applied to related problems involving higher-order generalized analytic or Beltrami-type equations. In particular, the methods developed here could be adapted to study mixed or nonlinear boundary conditions, as well as to explore domains with different geometric configurations such as polygons or ring-shaped regions.

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