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Deforming the Nephroid into the Cayley Sextic and Beyond

Francisco Javier González Vieli

(Communicated by Donghe Pei)

ABSTRACT

In $\mathbb{R}^2$, let Γ be a fixed circle with centre C and radius r, and ℓ a straight line at distance d of C. We study the curve which is the envelope of the circles whose centre lies on Γ and which are tangent to ℓ. When $d = 0$ this curve is a nephroid, when $d = 3r/2$ it is a Cayley sextic.

Keywords: *Envelope, circles, tangent line, nephroid, Cayley sextic.*
AMS Subject Classification (2020): *Primary: 53A04 ; Secondary: 58K05.*

1. Introduction.

The nephroid is an epicycloid with two cusps; it is the trajectory of a point on a circle of radius ρ wich rolls around a fixed circle of radius 2ρ [3, p.170], [6, p.64]. It can also be defined as the envelope of straight lines through two points which run with constant speed on a fixed circle, the speed of one point being three times that of the other point [1], [3, p.166]. In [2, p.115], Bruce and Giblin lamented that envelopes had quietly dropped from texts on differential geometry because they tend to have singular points. This is changing: see [8] or [9], which studies four fundamental problems about envelopes of circles. Now, the nephroid can also be defined as an envelope of circles: it is the envelope of the circles whose centre lies on a fixed circle and wich are tangent to a diameter of the fixed circle [6, p.63].

Here we study the envelope of the circles whose centre lies on a fixed circle Γ and which are tangent to a straight line ℓ at distance d of the center of Γ. In section 2 we find the parametrization of this curve. In section 3 we study the number of its selfintersections. In section 4 we are able to calculate its length, and we calculate the area its surrounds in section 5. Finally, in section 7 we study the position of its possible singular points, after having considered in section 6 an alternative definition of the envelope.

2. Envelope and curve.

For simplicity we may suppose that Γ is the circle with centre $(0, 0)$ and radius 1, and ℓ the straight line with equation $y = -d$ where $d \geq 0$. The circle γ with centre $(\cos t, \sin t)$ on Γ which is tangent to ℓ has the equation

$$(x - \cos t)^2 + (y - \sin t)^2 = (\sin t + d)^2.$$

Hence the set of these circles is the zero set of the function

$$\begin{aligned} F_d(t, x, y) &= (x - \cos t)^2 + (y - \sin t)^2 - (\sin t + d)^2 \\ &= x^2 - 2x\cos t + y^2 - 2y\sin t + \cos^2 t - 2d\sin t - d^2. \end{aligned}$$

It follows that

$$\frac{\partial F_d}{\partial t}(t, x, y) = 2x\sin t - 2y\cos t - 2\cos t\sin t - 2d\cos t.$$

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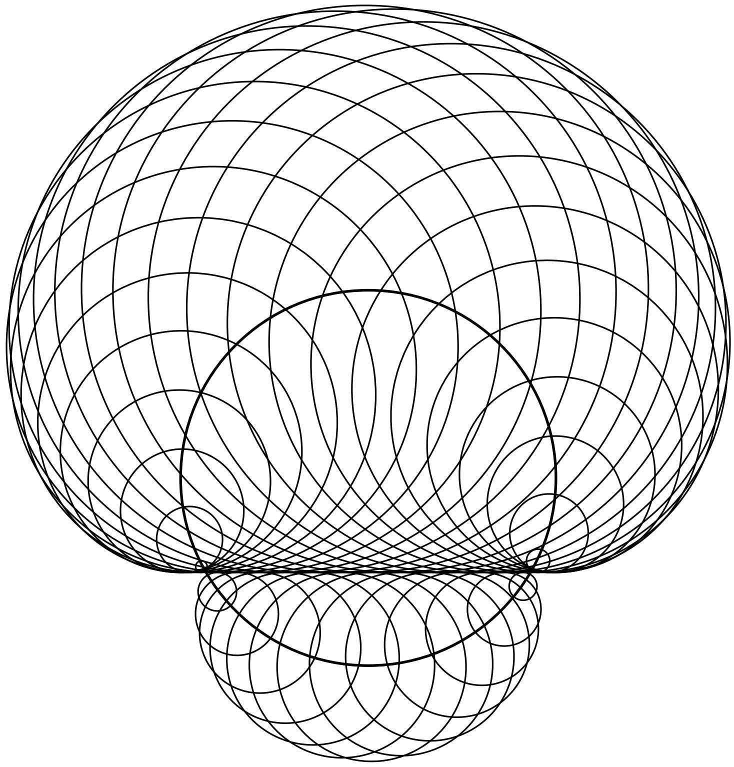

Figure 1. The case $d = 0.5$.

The envelope of the circles γ is the set of points (x, y) such that there exists t with $F_d(t, x, y) = 0$ and $(\partial F_d/\partial t)(t, x, y) = 0$ [2, Definition 5.3 p.102]. So we must solve the system

$$\begin{cases} x^2 - 2x\cos t + y^2 - 2y\sin t + \cos^2 t - 2d\sin t - d^2 = 0 \\ 2x\sin t - 2y\cos t - 2\cos t\sin t - 2d\cos t = 0. \end{cases} \tag{2.1}$$

Assuming $\cos t \neq 0$, we deduce from the second equation of (2.1) that

$$y = \frac{x\sin t}{\cos t} - \sin t - d$$

and substitute in the first equation of (2.1):

$$x^2 - 2x\cos t + \left(\frac{x\sin t}{\cos t} - \sin t - d\right)^2 - 2\left(\frac{x\sin t}{\cos t} - \sin t - d\right)\sin t + \cos^2 t - 2d\sin t - d^2 = 0$$

which we simplify to

$$x^2 + \frac{x^2\sin^2 t}{\cos^2 t} - 4\,\frac{x\sin^2 t}{\cos t} - 2d\,\frac{x\sin t}{\cos t} - 2x\cos t + 2d\sin t + 3\sin^2 t + \cos^2 t = 0. \tag{2.2}$$

This is a quadratic equation of the form $ax^2 + bx + c = 0$ with

$$a = 1 + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

$$b = -4\,\frac{\sin^2 t}{\cos t} - 2d\,\frac{\sin t}{\cos t} - 2\cos t = \frac{-2\sin^2 t - 2d\sin t - 2}{\cos t}$$

$$c = 2d\sin t + 3\sin^2 t + \cos^2 t = 2d\sin t + 2\sin^2 t + 1.$$

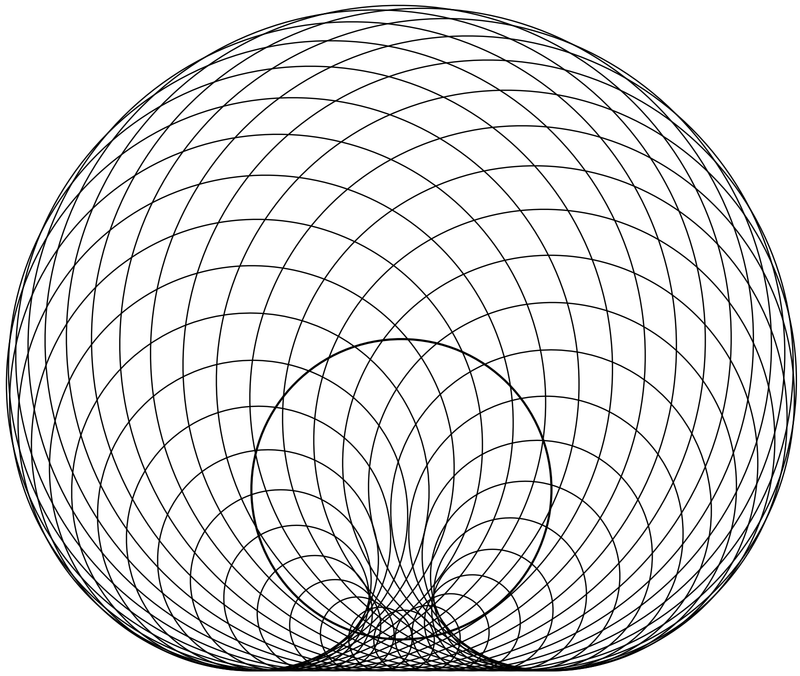

Figure 2. The case $d = 1.2$.

The discriminant of (2.2) is

$$\frac{4\sin^4 t + 4d^2\sin^2 t + 4 + 8d\sin^3 t + 8\sin^2 t + 8d\sin t}{\cos^2 t} - 4\,\frac{2d\sin t + 2\sin^2 t + 1}{\cos^2 t}$$
$$= \frac{4\sin^4 t + 8d\sin^3 t + 4d^2\sin^2 t}{\cos^2 t} = \left(\frac{2\sin^2 t + 2d\sin t}{\cos t}\right)^2.$$

The solutions of (2.2) are

$$x_1 = \cos t \quad \text{and} \quad x_2 = 2\sin^2 t\cos t + 2d\sin t\cos t + \cos t.$$

The corresponding y are

$$y_1 = -d \quad \text{and} \quad y_2 = 2\sin^3 t + 2d\sin^2 t - d.$$

In case $\cos t = 0$ (and $\sin t = \pm 1$), we obtain solutions coherent with these.

In conclusion, the envelope of the circles γ consists of two curves. The first curve has the parametrization $(\cos t, -d)$; its image is the horizontal segment with extremities $(1, -d)$ and $(-1, -d)$. The second curve, on which we shall concentrate our investigations, has the parametrization

$$\alpha_d(t) = \begin{pmatrix} (2\sin^2 t + 2d\sin t + 1)\cos t \\ 2\sin^3 t + 2d\sin^2 t - d \end{pmatrix},$$

and we shall call it, for convenience, an envelope of the circles γ. It is periodic with period 2π and goes through the points $(1, -d)$, $(0, d+2)$, $(-1, -d)$ and $(0, d-2)$ for $t = 0$, $\pi/2$, π and $3\pi/2$ respectively. Since the figure formed by the circle Γ and the straight line ℓ is symmetric with respect to the y-axis, so is the curve α_d. This can also be seen in the parametrization: $x(\pi - t) = -x(t)$ and $y(\pi - t) = y(t)$, where $(x(t), y(t)) = \alpha_d(t)$.

Figures 1, 2 and 3 illustrate the cases $d = 0.5$, $d = 1.2$ and $d = 1.6$ respectively — only the unit circle Γ and some of the circles γ are drawn.

As already mentioned, α_0 is a nephroid; and we shall now see that the curve $\alpha_{3/2}$ is a Cayley sextic [5, p.178]. For that we translate $\alpha_{3/2}$ by 1/2 vertically, to get the parametrization

$$\gamma(t) = \begin{pmatrix} (2\sin^2 t + 3\sin t + 1)\cos t \\ 2\sin^3 t + 3\sin^2 t - 1 \end{pmatrix},$$

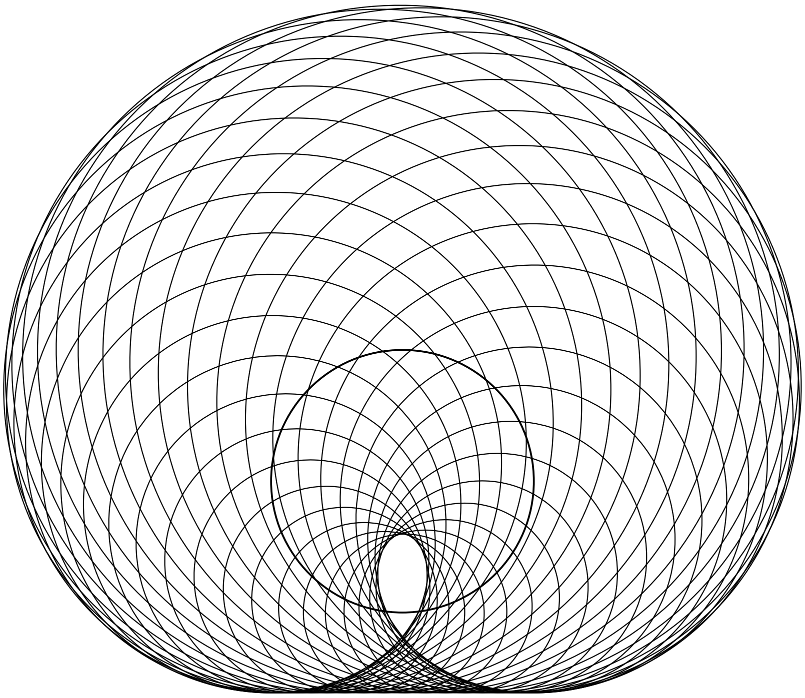

Figure 3. The case $d = 1.6$.

and we shall prove that this curve γ satisfies the Cartesian equation of the Cayley sextic:

$$4(x^2 + y^2 - y)^3 = 27(x^2 + y^2)^2. \tag{2.3}$$

We factorize both coordinates of $\gamma(t)$:

$$x(t) = (2\sin t + 1)(\sin t + 1)\cos t \quad \text{and} \quad y(t) = (2\sin t - 1)(\sin t + 1)^2.$$

Hence

$$\begin{aligned} x^2(t) + y^2(t) &= (2\sin t + 1)^2(\sin t + 1)^2(1 - \sin^2 t) + (2\sin t - 1)^2(\sin t + 1)^4 \\ &= (\sin t + 1)^3\Big((2\sin t + 1)^2(1 - \sin t) + (2\sin t - 1)^2(\sin t + 1)\Big). \end{aligned}$$

But $(2z + 1)^2(1 - z) + (2z - 1)^2(1 + z) = 2$ for any z, so we get

$$x^2(t) + y^2(t) = 2(\sin t + 1)^3.$$

Then

$$\begin{aligned} x^2(t) + y^2(t) - y(t) &= 2(\sin t + 1)^3 - (2\sin t - 1)(\sin t + 1)^2 \\ &= (\sin t + 1)^2\Big(2(\sin t + 1) - (2\sin t - 1)\Big) \\ &= (\sin t + 1)^2 \cdot 3 \end{aligned}$$

and equation (2.3) follows.

3. Selfintersection.

As Figure 3 shows, the envelope α_d may have points of selfintersection on the y-axis. Therefore we must study the zeros of its first coordinates:

$$(2\sin^2 t + 2d\sin t + 1)\cos t = 0.$$

The equation $\cos t = 0$ gives $t = \pi/2$ or $t = 3\pi/2$ and the points $(0, d+2)$ or $(0, d-2)$: no selfintersection. The equation

$$2\sin^2 t + 2d\sin t + 1 = 0 \tag{3.1}$$

is of degree 2 in $\sin t$ and its discriminant is $4d^2 - 8$. We have then three cases.

When $0 \leq d < \sqrt{2}$, (3.1) has no solution and α_d no selfintersection.

When $d = \sqrt{2}$, (3.1) has one solution: $\sin t = -\sqrt{2}/2$ and $t = 5\pi/4$ or $t = 7\pi/4$. The curve $\alpha_{\sqrt{2}}$ has a double point at $(0, -\sqrt{2}/2)$, but the two arcs are here vertical and so tangent (see (4.1) below for a proof and Figure 4 for an illustration).

When $d > \sqrt{2}$, (3.1) has the solutions

$$\sin t = \frac{-d + \sqrt{d^2 - 2}}{2} \quad \text{and} \quad \sin t = \frac{-d - \sqrt{d^2 - 2}}{2}.$$

We begin with the first possibility. Clearly $\sqrt{d^2 - 2} - d < 0$. Next $(\sqrt{d^2 - 2} - d)/2 \geq -1$ is equivalent to $\sqrt{d^2 - 2} \geq d - 2$. This last is trivial if $\sqrt{2} < d < 2$, and if $d \geq 2$ we can square the inequality to get $d^2 - 2 \geq d^2 - 4d + 4$ or $4d \geq 6$ i.e. $d \geq 3/2$, which shows that in fact $(\sqrt{d^2 - 2} - d)/2 > -1$. In other words, there exist $\pi < t_1 < t_2 < 2\pi$ with $\sin t_j = (\sqrt{d^2 - 2} - d)/2$ for $j = 1, 2$.

For the second possibility, clearly $-d - \sqrt{d^2 - 2} < 0$. Next $(-d - \sqrt{d^2 - 2})/2 \geq -1$ is equivalent to $\sqrt{d^2 - 2} \leq 2 - d$. This last is false if $d > 2$, and if $\sqrt{2} < d \leq 2$ we can square the inequality to get $d^2 - 2 \leq 4 - 4d + d^2$ or $4d \leq 6$ i.e. $d \leq 3/2$. Hence there exist $\pi < t_3 \leq t_4 < 2\pi$ with $\sin t_j = (-d - \sqrt{d^2 - 2})/2$ for $j = 3, 4$ if and only if $\sqrt{2} < d \leq 3/2$; moreover $t_3 = t_4$ only when $d = 3/2$. Note also that, since $(-d - \sqrt{d^2 - 2})/2 < (-d + \sqrt{d^2 - 2})/2$, we have $t_1 < t_3 \leq t_4 < t_2$.

In short, when $\sqrt{2} < d < 3/2$ the curve α_d has two selfintersections, and when $d \geq 3/2$ only one.

4. Arclength.

We calculate $\|\dot{\alpha}_d(t)\|^2$. First

$$\begin{aligned}
\dot{x}(t) &= (4\sin t\cos t + 2d\cos t)\cos t - (2\sin^2 t + 2d\sin t + 1)\sin t \\
&= 4\sin t\cos^2 t + 2d\cos^2 t - 2\sin^3 t - 2d\sin^2 t - \sin t \\
&= 4\sin t - 4\sin^3 t + 2d - 2d\sin^2 t - 2\sin^3 t - 2d\sin^2 t - \sin t \\
&= -6\sin^3 t - 4d\sin^2 t + 3\sin t + 2d
\end{aligned}$$

and

$$\begin{aligned}
(\dot{x}(t))^2 &= 36\sin^6 t + 16d^2\sin^4 t + 9\sin^2 t + 4d^2 + 48d\sin^5 t - 36\sin^4 t - 24d\sin^3 t \\
&\quad - 24d\sin^3 t - 16d^2\sin^2 t + 12d\sin t \\
&= 36\sin^6 t + 48d\sin^5 t + (16d^2 - 36)\sin^4 t - 48d\sin^3 t + (9 - 16d^2)\sin^2 t + 12d\sin t + 4d^2.
\end{aligned}$$

Next $\dot{y}(t) = 6\sin^2 t\cos t + 4d\sin t\cos t$ and

$$\begin{aligned}
(\dot{y}(t))^2 &= 36\sin^4 t\cos^2 t + 48d\sin^3 t\cos^2 t + 16d^2\sin^2 t\cos^2 t \\
&= 36\sin^4 t - 36\sin^6 t + 48d\sin^3 t - 48d\sin^5 t + 16d^2\sin^2 t - 16d^2\sin^4 t.
\end{aligned}$$

We get

$$\|\dot{\alpha}_d(t)\|^2 = (\dot{x}(t))^2 + (\dot{y}(t))^2 = 9\sin^2 t + 12d\sin t + 4d^2 = (3\sin t + 2d)^2.$$

When $d > 3/2$, $3\sin t + 2d > 0$ for all t: the envelope α_d is regular and its length is equal to

$$\int_0^{2\pi} \|\dot{\alpha}_d(t)\|\, dt = \int_0^{2\pi} (3\sin t + 2d)\, dt = \Big[-3\cos t + 2dt\Big]_0^{2\pi} = -3 + 4\pi d + 3 = 4\pi d.$$

When $d = 3/2$, $3\sin t + 2d \geq 0$ for all t, with equality if and only if $\sin t = -1$, so for $t = 3\pi/2$. Hence $\alpha_{3/2}$ has one singular point and its length can be calculated as above.

When $0 \le d < 3/2$, $3\sin t + 2d = 0$ for example at

$$s_1 = \arcsin(-2d/3), \quad s_2 = \pi - s_1, \quad s_3 = s_1 + 2\pi,$$

where $-\pi/2 < s_1 \le 0$. We have $3\sin t + 2d \ge 0$ when $s_1 \le t \le s_2$ and $3\sin t + 2d \le 0$ when $s_2 \le t \le s_3$. Therefore the length of α_d is

$$\begin{aligned}\int_0^{2\pi} \|\dot\alpha_d(t)\|\,dt &= \int_{s_1}^{s_3} \|\dot\alpha_d(t)\|\,dt\\ &= \int_{s_1}^{s_2} (3\sin t + 2d)\,dt + \int_{s_2}^{s_3} -(3\sin t + 2d)\,dt\\ &= -3\cos s_2 + 2ds_2 + 3\cos s_1 - 2ds_1 + 3\cos s_3 - 2ds_3 - 3\cos s_2 + 2ds_2\\ &= 3\cos s_1 + 2d(\pi - s_1) + 3\cos s_1 - 2ds_1 + 3\cos s_1 - 2d(s_1 + 2\pi) + 3\cos s_1 + 2d(\pi - s_1)\\ &= 12\cos s_1 - 8ds_1.\end{aligned}$$

Since $-\pi/2 < s_1 \le 0$, $\cos s_1 > 0$ and so $\cos s_1 = \sqrt{1-(-2d/3)^2} = \sqrt{9-4d^2}/3$. Finally, when $0 \le d < 3/2$ the length of α_d is equal to

$$4\sqrt{9-4d^2} + 8d\arcsin(2d/3).$$

Letting d tend to $3/2$, we get $4\cdot 0 + 8\cdot(3/2)\cdot(\pi/2) = 6\pi$, which coincides with the length found before. In the case $d = 0$, α_0 is a nephroid symmetric with respect to the x-axis; its length is then equal to $2\int_0^{\pi} \|\dot\alpha_0(t)\|\,dt = 2\int_0^{\pi} 3\sin t\,dt = 12$, which coincides with $4\sqrt{9-0} + 8\cdot 0$.

In summary, as a function of d the length L of α_d is given by

$$L(d) = \begin{cases} 4\sqrt{9-4d^2} + 8d\arcsin(2d/3) & \text{if } 0 \le d < 3/2\\ 4\pi d & \text{if } d \ge 3/2.\end{cases}$$

We have seen that L is continuous on $[0, +\infty)$. In fact it is C^1 on $[0, +\infty)$. Indeed, on $[0, 3/2)$, $L'(d) = 8\arcsin(2d/3)$ and $\lim_{d\to 3/2-} L'(d) = 8\cdot(\pi/2) = 4\pi$.

We note that for $d = \sqrt{2}$ and $\sin t = -\sqrt{2}/2$,

$$\dot x(t) = -6\left(\frac{-\sqrt{2}}{2}\right)^3 - 4\sqrt{2}\left(\frac{-\sqrt{2}}{2}\right)^2 + 3\left(\frac{-\sqrt{2}}{2}\right) + 2\sqrt{2} = 0 \tag{4.1}$$

as asserted above.

5. Area.

We calculate the area of the region in the plane surrounded by the envelope α_d. For this, we use Leibniz sector formula [4, Satz 4 p.247]:

$$F(\alpha_d) = \frac{1}{2}\int_0^{2\pi} \big(x(t)\dot y(t) - y(t)\dot x(t)\big)\,dt.$$

First,

$$\begin{aligned}x(t)\dot y(t) &= [(2\sin^2 t + 2d\sin t + 1)\cos t]\cdot[6\sin^2 t\cos t + 4d\sin t\cos t]\\ &= 12\sin^4 t\cos^2 t + 12d\sin^3 t\cos^2 t + 6\sin^2 t\cos^2 t + 8d\sin^3 t\cos^2 t + 8d^2\sin^2 t\cos^2 t + 4d\sin t\cos^2 t\\ &= 12\sin^4 t - 12\sin^6 t + 20d\sin^3 t - 20d\sin^5 t + (6+8d^2)\sin^2 t - (6+8d^2)\sin^4 t + 4d\sin t - 4d\sin^3 t\\ &= -12\sin^6 t - 20d\sin^5 t + (6-8d^2)\sin^4 t + 16d\sin^3 t + (6+8d^2)\sin^2 t + 4d\sin t.\end{aligned}$$

Next

$$\begin{aligned}y(t)\dot x(t) &= [2\sin^3 t + 2d\sin^2 t - d]\cdot[-6\sin^3 t - 4d\sin^2 t + 3\sin t + 2d]\\ &= -12\sin^6 t - 8d\sin^5 t + 6\sin^4 t + 4d\sin^3 t - 12d\sin^5 t - 8d^2\sin^4 t + 6d\sin^3 t + 4d^2\sin^2 t\\ &\quad + 6d\sin^3 t + 4d^2\sin^2 t - 3d\sin t - 2d^2\\ &= -12\sin^6 t - 20d\sin^5 t + (6-8d^2)\sin^4 t + 16d\sin^3 t + 8d^2\sin^2 t - 3d\sin t - 2d^2.\end{aligned}$$

We get

$$x(t)\dot{y}(t) - y(t)\dot{x}(t) = 6\sin^2 t + 7d\sin t + 2d^2.$$

So

$$\begin{aligned}F(\alpha_d) &= \frac{1}{2}\int_0^{2\pi} (6\sin^2 t + 7d\sin t + 2d^2)\, dt\\ &= \frac{1}{2}\Big[3(t - \sin t\cos t) - 7d\cos t + 2d^2 t\Big]_0^{2\pi}\\ &= \frac{1}{2}\big(6\pi - 7d + 4\pi d^2 + 7d\big)\\ &= 3\pi + 2\pi d^2.\end{aligned}$$

Note that in the presence of an inner loop, the area inside this inner loop is counted twice by Leibniz sector formula.

For example, when $d \geq 3/2$ the area inside the inner loop is

$$A_d = \frac{1}{2}\int_{\pi-a}^{a+2\pi} (6\sin^2 t + 7d\sin t + 2d^2)\, dt$$

where $a = \arcsin\big((\sqrt{d^2-2} - d)/2\big)$, by the findings in section 3. Then

$$\begin{aligned}A_d = \frac{1}{2}\Big(&3(a+2\pi) - 3\sin(a+2\pi)\cos(a+2\pi) - 7d\cos(a+2\pi) + 2d^2(a+2\pi)\\ &-3(\pi-a) + 3\sin(\pi-a)\cos(\pi-a) + 7d\cos(\pi-a) - 2d^2(\pi-a)\Big)\\ = \frac{1}{2}\Big(&3a + 6\pi - 3\sin a\cos a - 7d\cos a + 2d^2 a + 4d^2\pi - 3\pi + 3a - 3\sin a\cos a - 7d\cos a - 2d^2\pi + 2d^2 a\Big)\\ = 3a + &\frac{3\pi}{2} - 3\sin a\cos a - 7d\cos a + 2d^2 a + d^2\pi.\end{aligned}$$

Since $-\pi/2 \leq a < 0$, $\cos a > 0$. But

$$\sin^2 a = \left(\frac{\sqrt{d^2-2} - d}{2}\right)^2 = \frac{d^2 - 2 - 2d\sqrt{d^2-2} + d^2}{4} = \frac{d^2 - 1 - d\sqrt{d^2-2}}{2}.$$

So

$$\cos a = \sqrt{1 - \sin^2 a} = \sqrt{\frac{3 - d^2 + d\sqrt{d^2-2}}{2}}.$$

Therefore

$$A_d = \frac{3\pi}{2} + \pi d^2 + (3 + 2d^2)\arcsin\left(\frac{\sqrt{d^2-2} - d}{2}\right) - 3\,\frac{\sqrt{d^2-2}}{2}\sqrt{\frac{3 - d^2 + d\sqrt{d^2-2}}{2}} - \frac{11d}{2}\sqrt{\frac{3 - d^2 + d\sqrt{d^2-2}}{2}}.$$

In the particular case $d = 3/2$, we have $\sqrt{d^2-2} = 1/2$, $(\sqrt{d^2-2} - d)/2 = -1/2$, $a = \arcsin(-1/2) = -\pi/6$ and

$$A_{3/2} = \frac{3\pi}{2} + \pi\,\frac{9}{4} + \left(3 + 2\,\frac{9}{4}\right)\frac{-\pi}{6} - 3\,\frac{1/2}{2}\,\frac{\sqrt{3}}{2} - \frac{11}{2}\,\frac{3}{2}\,\frac{\sqrt{3}}{2} = \frac{5\pi}{2} - \frac{9\sqrt{3}}{2} \cong 0.059752999914.$$

This is sequence A118309 in [7].

6. An other envelope

When Lockwood defines the nephroid as an envelope of circles [6, p.63], he does not consider the diameter of Γ as part of it. In fact, his definition of envelope seems to be the E_3 definition of envelope, in the terminology of Bruce and Giblin [2, p.110]: it is the boundary ∂R_d of the region R_d in the plane filled by the circles γ:

$$R_d = \{(x, y) \in \mathbb{R}^2 : \exists t \in \mathbb{R}, F_d(x, y, t) = 0\}$$

(Indeed, in Figure 1 we perceive $\partial R_{0.5}$ more than the curve $\alpha_{0.5}$ with its singular points.)

In the case $0 \leq d < 1$, ∂R_d is made by the concatenation of

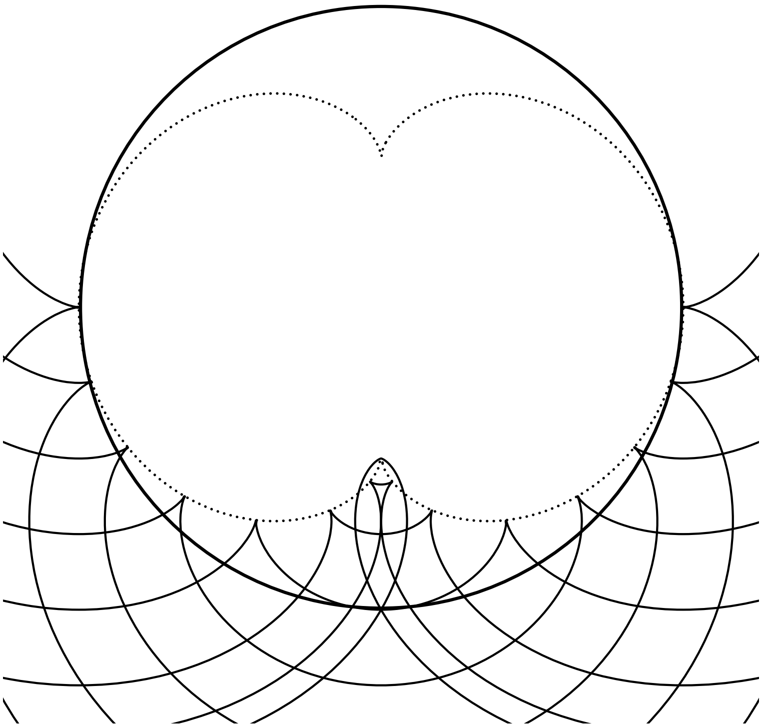

Figure 4. Detail of some of the curves α_d.

1. the curve $\alpha_d(t)$ with $0 \le t \le \pi$,
2. the horizontal segment $[(-1,-d),(-\sqrt{1-d^2},-d)]$,
3. the curve $\alpha_d(t)$ with $\pi - \arcsin(-d) \le t \le 2\pi + \arcsin(-d)$,
4. the horizontal segment $[(\sqrt{1-d^2},-d),(1,-d)]$.

We find the length of ∂R_d to be

$$8 + 4\sqrt{1-d^2} + 4d \arcsin d.$$

To calculate the area of R_d, we use again Leibniz sector formula, with $\lambda(t) = (t,-d)$ as the parametrization of both horizontal segments, so that $x(t)\dot{y}(t) - y(t)\dot{x}(t) = d$ on these segments. We find the area of R_d to be

$$3\pi + 8d + 2\pi d^2 - 5d\sqrt{1-d^2} - (3+2d^2)\arcsin d.$$

We verify that the length of ∂R_0 is equal to the length of α_0 found in section 4, and the area of R_0 equal to the area surrounded by α_0 found in section 5, as expected.

In the case $1 \le d \le \sqrt{2}$, the set R_d is bounded by the curve $\alpha_d(t)$ with $0 \le t \le \pi$ and the horizontal segment $[(-1,-d),(1,-d)]$. We find the length of ∂R_d to be

$$8 + 2\pi d$$

and the area of R_d to be

$$(3\pi/2) + 8d + \pi d^2.$$

7. Singular points.

In section 4, we have seen that the envelope α_d has singular points if and only if $0 \le d \le 3/2$. If we draw these curves we observe that the singular points — starting from $(1,0)$ and $(-1,0)$ when $d = 0$ until $(0,-1/2)$

when $d = 3/2$ — seem to move along the (inferior double arc of the) nephroid whose parametrization is

$$\beta(\theta) = \begin{pmatrix} \sin^3\theta \\ (\sin^2\theta + \frac{1}{2})\cos\theta \end{pmatrix}$$

and we shall prove that this is the case. Figure 4 shows a detail of the envelopes α_d for $d = 0$, $1/4$, $1/2$, $3/4$, 1, $5/4$, $\sqrt{2}$, $3/2$, the circle Γ and, as a dotted line, the nephroid β. (Geometrically, β is the image of the nephroid α_0 under a rotation of angle $\pi/2$ and a similitude of factor $1/2$.)

We fix $0 \le d \le 3/2$. The singular points of α_d appear when $\|\dot{\alpha}_d(t)\| = 0$, that is $|3\sin t + 2d| = 0$, hence at $s = \arcsin(-2d/3)$ and $\pi - s$. The coordinates of $\alpha_d(s)$ are:

$$\begin{aligned} x(s) &= (2\sin^2 s + 2d\sin s + 1)\cos s \\ &= \left(2\Big(\frac{-2d}{3}\Big)^2 + 2d\Big(\frac{-2d}{3}\Big) + 1\right)\sqrt{1 - \Big(\frac{-2d}{3}\Big)^2} \\ &= \left(\frac{9 - 4d^2}{9}\right)\sqrt{\frac{9 - 4d^2}{9}} \\ &= \left(\frac{\sqrt{9 - 4d^2}}{3}\right)^3 \end{aligned}$$

and

$$y(s) = 2\sin^3 s + 2d\sin^2 s - d = 2\Big(\frac{-2d}{3}\Big)^3 + 2d\Big(\frac{-2d}{3}\Big)^2 - d = \frac{8d^3 - 27d}{27}.$$

If we want $\alpha_d(s)$ to be on β, we must have $\sqrt{9 - 4d^2}/3 = \sin\theta$ for some θ. We choose

$$\theta = \pi - \arcsin\left(\frac{\sqrt{9 - 4d^2}}{3}\right),$$

so that $\pi/2 \le \theta \le \pi$ and $\cos\theta \le 0$. Hence $\cos\theta = -\sqrt{1 - \sin^2\theta} = -\sqrt{1 - (9 - 4d^2)/9} = -2d/3$. The second coordinate of $\beta(\theta)$ is then

$$\Big(\sin^2\theta + \frac{1}{2}\Big)\cos\theta = \left(\frac{9 - 4d^2}{9} + \frac{1}{2}\right)\frac{-2d}{3} = \frac{27 - 8d^2}{18}\cdot\frac{-2d}{3} = \frac{8d^3 - 27d}{27}.$$

This shows that $\alpha_d(s) = \beta(\theta)$: the singular point $\alpha_d(s)$ of the envelope α_d is on the nephroid β. By symmetry, the singular point $\alpha_d(\pi - s)$ of the envelope α_d is also on β.

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