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A Quantum Space and Some Associated Quantum Groups

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Abstract: In the present paper, we first introduce a quantum n-space on which the algebra of coordinates is η-commutative. Further, it is shown that there are some σ-twisted derivations acting on this algebra, and the algebra of such derivations is a quantum group. Morever, we show that a bicovariant differential calculus on this space can be constructed by using σ-twisted derivations. Finally, the quantum Lie algebra is obtained by using this bicovariant differential calculus.

Bir Kuantum Uzay ve Bazı İlişkili Kuantum Gruplar

Anahtar Kelimeler
Kuantum uzaylar,
Kuantum gruplar,
Türev cebirleri

Özet: Bu makalede önce koordinatlar cebiri η-değişmeli olan bir kuantum uzay tanımlandı. Ayrıca bu cebire etki eden σ-bükümlü türevlerin varlığı gösterildi ve bu türevlerin oluşturduğu cebirin değişmeli ve eşdeğişmeli olmayan bir Hopf cebiri olduğu ispat edildi. Üstelik, söz konusu kuantum uzay üzerinde bir bikovaryant diferansiyel hesabın σ-bükümlü türevlerin yardımı ile elde edilebileceği gösterildi. En son kuantum Lie cebiri bu diferansiyel hesap yardımı ile oluşturuldu.

1. Introduction

It is well known from [1] that the notion of quantum group is used to specify deformed versions of some classical Lie groups in the classical differential geometry. In addition, as the most concrete examples of noncommutative spaces, quantum spaces(groups) are also regarded by many as a pattern in generalizing quantum deformed physics(for example, see [1–4]). In this context many efforts were displayed to introduce differential calculus on quantum spaces(groups) (see [5–14]). Differential calculus for quantum groups can be realized by using the setting of Hopf algebra [11], and this differential calculus is extended to the graded differential Hopf algebra [2].

In this study we first introduce a quantum n-space on which the corresponding coordinate algebra is an η-commutative algebra. Morever it is shown that this η-commutative algebra admits some σ-twisted derivations and the algebra of these derivations is a non-cocommutative Hopf algebra, namely, a quantum group. Further it is seen that the η-commutative algebra has a proper cocommutative Hopf algebra structure making possible a bicovariant differential calculus obtained by means of σ-twisted derivations on the quantum n-space. Finally, based on this bicovariant differential calculus, the right invariant Maurer-Cartan forms and the corresponding vector fields are given, and it is seen that the algebra of these vector fields has a non-cocommutative Hopf algebra structure.

2. Preliminaries

Throughout the paper, the complex numbers $\mathbb{C}$ will be always ground field for all our objects. A triple $(\mathscr{A}, m, \mathrm{u})$ stands for an associative algebra with multiplicative identity $1_{\mathscr{A}}$ where $\mathscr{A}$ is a linear space, $m : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ is the multiplication mapping on $\mathscr{A}$ and $\mathrm{u} : \mathbb{C} \to \mathscr{A}$ is the unit mapping defined as $\mathrm{u}(k) = k.1_{\mathscr{A}}$ such that these algebraic mappings hold the axioms of algebra. A coassociative *coalgebra* $(\mathscr{A}, \Delta, \varepsilon)$ is defined by inverting all arrows in the diagrams of data and axioms for $(\mathscr{A}, m, \mathrm{u})$. This is equivalent to saying that we have a coproduct $\Delta : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ and a counit $\varepsilon : \mathscr{A} \to \mathbb{C}$ satisfying two axioms

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \tag{1}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta. \tag{2}$$

where id is the identity mapping.
A *bialgebra* $\mathscr{A}$ is a coalgebra such that Δ and ε are both algebraic homomorphisms such that $\Delta(1_{\mathscr{A}}) = 1_{\mathscr{A}} \otimes 1_{\mathscr{A}}$ and $\varepsilon(1_{\mathscr{A}}) = 1_K$. Finally, a *Hopf algebra* is a bialgebra $\mathscr{A}$ endowed with an algebra antihomomorphism called "antipode mapping" $S : \mathscr{A} \to \mathscr{A}$ enjoying

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \mathrm{u} \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta, \tag{3}$$

Let $\mathscr{A}$ be an algebra. A $\mathbb{C}$-vector space Ω is called a right $\mathscr{A}$-module if there exists a linear mapping $\varphi_R : \Omega \otimes \mathscr{A} \to \Omega$ with $\varphi_R \circ (\varphi_R \otimes \mathrm{id}) = \varphi_R \circ (\mathrm{id} \otimes m)$ and $\varphi_R \circ (\mathrm{id} \otimes \mathrm{u}) = \mathrm{id}$. Similarly, the vector space Ω is called a left

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$\mathscr{A}$-module if there is a linear mapping $\varphi_L : \mathscr{A} \otimes \Omega \to \Omega$ satisfying the conditions $\varphi_L \circ (\mathrm{id} \otimes \varphi_L) = \varphi_L \circ (m \otimes \mathrm{id})$ and $\varphi_L \circ (\mathrm{u} \otimes \mathrm{id}) = \mathrm{id}$. If the Ω is both a right $\mathscr{A}$-module and a left $\mathscr{A}$-module such that the relevant actions φ_R and φ_L commute, that is, $\varphi_L \circ (\mathrm{id} \otimes \varphi_R) = \varphi_R \circ (\varphi_L \otimes \mathrm{id})$, then we call Ω an $\mathscr{A}$-bimodule. Let $\mathscr{A}$ be a Hopf algebra. A *right comodule* over $\mathscr{A}$ is a $\mathbb{C}$-vector space Ω equipped with a linear function $\Delta_R : \Omega \longrightarrow \Omega \otimes \mathscr{A}$ satisfying

$$\begin{aligned} (\mathrm{id} \otimes \Delta) \circ \Delta_R &= (\Delta_R \otimes \mathrm{id}) \circ \Delta_R \\ \mathrm{id} &= m \circ (\mathrm{id} \otimes \varepsilon) \circ \Delta_R. \end{aligned} \tag{4}$$

A *left comodule* over $\mathscr{A}$ is a $\mathbb{C}$-vector space Ω equipped with a linear function $\Delta_L : \Omega \longrightarrow \mathscr{A} \otimes \Omega$ such that

$$\begin{aligned} (\Delta \otimes \mathrm{id}) \circ \Delta_L &= (\mathrm{id} \otimes \Delta_L) \circ \Delta_L \\ \mathrm{id} &= m \circ (\varepsilon \otimes \mathrm{id}) \circ \Delta_L. \end{aligned} \tag{5}$$

Let Ω be left and right comodule over $\mathscr{A}$ with the relevant linear mappings Δ_L and Δ_R. If Δ_L and Δ_R commute, then one says that Ω is a *bicomodule* over $\mathscr{A}$. A *bicovariant bimodule* Ω is a bicomodule Ω where Δ_L and Δ_R hold the compatibility condition:

$$\begin{aligned} \Delta_L(apb) &= \Delta(a)\Delta_L(p)\Delta(b), \\ \Delta_R(apb) &= \Delta(a)\Delta_R(p)\Delta(b) \end{aligned}$$

for all $a, b \in \mathscr{A}$ and $p \in \Omega$.

Let Ω^1 be a bimodule over an algebra $\mathscr{A}$ and d be a linear mapping from $\mathscr{A}$ to Ω^1. The pair (Ω^1, d) is called a *first order differential calculus* over $\mathscr{A}$ if the Leibniz rule $\mathsf{d}(ab) = \mathsf{d}(a)b + a\mathsf{d}(b)$ holds for all $a, b \in \mathscr{A}$. Note that Ω^1 is the linear span of elements $a\mathsf{d}(b)$ or $\mathsf{d}(a')b', a, a', b, b' \in \mathscr{A}$. Morever, it is said that a first order differential calculus (Ω^1, d) is bicovariant if Ω^1 is bicovariant together with the coactions $\Delta_L(\mathsf{d}(a)) = (\mathrm{id} \otimes \mathsf{d})(\Delta(a))$ and $\Delta_R(\mathsf{d}(a)) = (\mathsf{d} \otimes \mathrm{id})(\Delta(a)), a \in \mathscr{A}$, where $\Delta_L|_{\mathscr{A}} = \Delta$ and $\Delta_R|_{\mathscr{A}} = \Delta$(see [11]). Let Ω^n be the space of differential n-forms. Let any element of Ω^n be denoted by $a\mathsf{d}(a_1) \wedge \mathsf{d}(a_2) \wedge \ldots \wedge \mathsf{d}(a_n)$ or $\mathsf{d}(a_1) \wedge \mathsf{d}(a_2) \wedge \ldots \wedge \mathsf{d}(a_n)a$ for a, a_i's$\in \mathscr{A}$ where the multiplication $\wedge$ is defined as $u \wedge v \in \Omega^{m+n}$ for $u \in \Omega^m$ and $v \in \Omega^n$. Thus, the exterior algebra of all higher order differential forms (or differential graded algebra) is a $\mathbb{N}_0$-graded algebra $\Omega^\wedge = \bigoplus_{n \geq 0} \Omega^n$, $\Omega^0 = \mathscr{A}$, with the exterior mapping $\mathsf{d} : \Omega^\wedge \to \Omega^\wedge$ of degree 1 satisfying $\mathsf{d}^2 = 0$ and the graded Leibniz rule $\mathsf{d}(w_1 \wedge w_2) = \mathsf{d}(w_1) \wedge w_2 + (-1)^n w_1 \wedge \mathsf{d}(w_2)$, $w_1 \in \Omega^n$, $w_2 \in \Omega^\wedge$. Thus we extend the bicovariant first order differential calculus to the differential graded algebra $\Omega^\wedge$ in a similar way. Morever, $\Omega^\wedge$ can be endowed with a Hopf algebra structure derived from the coproduct $\hat{\Delta} = \Delta_L + \Delta_R$ (for more details see [2, 15]).

Let σ be an automorphism of an algebra $\mathscr{A}$. For $\mathscr{A}$, a σ-derivation is a linear mapping $\partial : \mathscr{A} \to \mathscr{A}$ with $\partial(ab) = \partial(a)b + \sigma(a)\partial(b)$ for all $a, b \in \mathscr{A}$.

Let us recall that a quantum n-space [8, 9] is an associative $\mathbb{C}$-algebra generated by $x_1, x_2, \ldots, x_n$ satisfying the following commutation relations:

$$x_i x_j = q_{ij} x_j x_i, \; i, j = 1, 2, \ldots, n, \tag{6}$$

where q_{ij}'s are nonzero complex numbers with $q_{ij}^{-1} = q_{ji}$.

3. A Quantum n-Space

In this section we concern with a quantum n-space on which the algebra of noncommuting coordinates is η-commutative. The construction of such a quantum n-space can be achieved by using some means such as bicharacter and 2-cocycle on additive group $\mathbb{Z}^n$ as used in [5]. However, our quantum n-space differs from that of [5] in two aspects. First, we make a bicharacter which yields a commutation relation that is not only based on the powers of generators but also their indices, and second, we take a grouplike generator into account. Let $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$ be any two integer n-tuples, and consider a mapping $* : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ defined through

$$\alpha * \beta = \sum_{j=1}^{n-1} \sum_{i>j} (j-i) \alpha_i \beta_j. \tag{7}$$

One can easily show that the mapping $*$ holds the following distributive laws:

$$\begin{aligned} (\alpha + \beta) * \gamma &= \alpha * \gamma + \beta * \gamma, \\ \alpha * (\beta + \gamma) &= \alpha * \beta + \alpha * \gamma. \end{aligned}$$

The mapping $*$ also satisfies

$$\varepsilon_i * \beta = \sum_{s<i} (s-i) \beta_s \; (1 \leq i \leq n),$$

$$\beta * \varepsilon_i = \sum_{s>i} (i-s) \beta_s \; (1 \leq i \leq n),$$

$$(\varepsilon_i - \varepsilon_{i+1}) * \beta = \sum_{s=1}^{i} \beta_s \; (1 \leq i < n),$$

$$\beta * (\varepsilon_i - \varepsilon_{i+1}) = - \sum_{s=i+1}^{n} \beta_s \; (1 \leq i < n),$$

where $\varepsilon_i = (\delta_{1i}, \ldots, \delta_{ni})$ $(1 \leq i \leq n)$ is a basis of $\mathbb{Z}^n$ as a $\mathbb{Z}$-module. Let q be a nonzero complex constant. Now we define a mapping $\eta : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{C}^*$ as follows

$$\eta(\alpha, \beta) = q^{\alpha * \beta - \beta * \alpha}. \tag{8}$$

In particular, we can verify that $\eta(\varepsilon_i, \varepsilon_j) = q^{j-i}$. In fact, since the following properties hold

$$\eta(\alpha + \beta, \gamma) = \eta(\alpha, \gamma)\eta(\beta, \gamma) \tag{9}$$

$$\eta(\alpha, \beta + \gamma) = \eta(\alpha, \beta)\eta(\alpha, \gamma) \tag{10}$$

$$\eta(\alpha, 0) = 1 = \eta(0, \alpha) \tag{11}$$

$$\eta(\alpha, \beta)\eta(\beta, \alpha) = 1 = \eta(\alpha, \alpha), \tag{12}$$

the mapping η is a bicharacter of the additive group $\mathbb{Z}^n$. We are now in a position to give our quantum n-space. This quantum n-space is generated by $x_1, \ldots, x_n$ having commutation relations which we write as

$$x_i x_j = \eta(\varepsilon_i, \varepsilon_j) x_j x_i, \; 1 \leq i, j \leq n. \tag{13}$$

This space or, rather, the polynomial function ring is formally defined by the following ring

$$\mathscr{A}_q(n) = \mathbb{C}\langle x_1, \ldots, x_n \rangle / \mathscr{I}, \tag{14}$$

where $\mathbb{C}\langle x_1,...,x_n\rangle$ means an associative $\mathbb{C}$-algebra freely generated by $x_1,...,x_n$ and $\mathscr{I}$ is an ideal generated by the relations (13). This is a deformation of the usual commutative space corresponding to the case $q=1$. Let $\mathbb{Z}_+^n$ show the set of nonnegative-integer n-tuples and $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ any nonzero monomial in $\mathscr{A}_q(n)$ where $\alpha=(\alpha_1,...,\alpha_n)\in\mathbb{Z}_+^n$. So we assume that the quantum n-space has a PBW basis whose elements are of the ordered form $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. That is, the vector space $\mathscr{A}_q(n)$ can be given as a direct sum of the vector subspaces consisting of the above ordered monomials of homogeneity degree m. From now on, our all linear mappings will be defined by taking into account the above ordering. Morever, we easily derive the commutation relation between two monomials x^α, x^β with the help of η as follows:

$$x^\alpha x^\beta = \eta(\alpha,\beta)x^\beta x^\alpha.$$

We note at this point that since η holds the conditions $(9-12)$, η-commutativity is well defined, and η is a 2-cocycle on the additive group $\mathbb{Z}^n$, meaning that it satisfies

$$\eta(\alpha,\beta)\eta(\alpha+\beta,\gamma)=\eta(\beta,\gamma)\eta(\alpha,\beta+\gamma), \qquad (15)$$

which ensures the compatibility of η-commutativity with associativity rule. So the above construction shows that the quantum n-space $\mathscr{A}_q(n)$ is an associative η-commutative algebra if we set that $x^0=1$ and $x^\alpha=0$ for $\alpha\notin\mathbb{Z}_+^n$. For more details on more general algebras such as $\Gamma-$graded $\eta-$commutative algebras, see the series of papers [3, 16].

4. Derivation Algebra on $\mathscr{A}_q(n)$

In this section, we will show that, in presence of automorphisms, some derivation operators on $\mathscr{A}_q$ with deformed Leibniz rules given by the automorphism are related with a bicovariant differential calculus on $\mathscr{A}_q$. Let us define a linear mapping $\partial_q/\partial x_i$ of $\mathscr{A}_q$, defined through

$$\frac{\partial_q}{\partial x_i}(x^\alpha)=\eta(\overleftarrow{\alpha_i},\varepsilon_i)\alpha_i x^{\alpha-\varepsilon_i}\ (i=1,...,n), \qquad (16)$$

where $\overleftarrow{\alpha_i}=(\alpha_1,..\alpha_{i-1},0,...,0)$ with $\overleftarrow{\alpha_1}=0$. Note that $\frac{\partial_q}{\partial x_i}(1)=0$, and even if we extend $\mathscr{A}_q$ by x_1^{-1}, the the algebra is again η-commutative. This says that the rule (16) can be also defined for x_1^z where z is a negative integer, for example, $\frac{\partial_q}{\partial x_1}(x_1^z)=zx_1^{z-1}$ as the usual one. For simplicity of notation, we let ∂_i denote briefly $\partial_q/\partial x_i$. From the definition (16) one has the following commutation rule between two mappings ∂_i,∂_j with respect to their composition as follows:

$$\partial_i\partial_j=\eta(\varepsilon_i,\varepsilon_j)\partial_j\partial_i. \qquad (17)$$

We also note that the mapping ∂_i reduces to the usual partial derivative operator with respect to x_i when $q\to1$. It is intriguing at this point to ask whether the mappings $\partial_i, i=1,...,n$ are deformed derivations acting on $\mathscr{A}_q(n)$. To answer this, we investigate Leibniz rule for the each mapping. Really, each ∂_i has the following deformed Leibniz rule when it acts on $x^\alpha g$, where g is any element of $\mathscr{A}_q$:

$$\partial_i(x^\alpha g)=\partial_i(x^\alpha)g+\sigma_i(x^\alpha)\partial_i(g), \qquad (18)$$

where for any $\beta\in\mathbb{Z}^n$, the mapping $\sigma_\beta:\mathscr{A}_q(n)\to\mathscr{A}_q(n)$ is an algebra automorphism acting on x^α

$$\sigma_\beta(x^\alpha)=\eta(\alpha,\beta)x^\alpha \ \text{ for all } \ x^\alpha\in\mathscr{A}_q(n), \qquad (19)$$

and σ_i denotes $\sigma(\varepsilon_i), i=1,...,n$. Note that this rule can be also defined for negative powers of x_1.
From the properties of η, it is obviously seen that for all $\alpha,\beta\in\mathbb{Z}^n$, the relation between automorphisms $\sigma_\alpha,\sigma_\beta$ appears as commutative one

$$\sigma_\alpha\sigma_\beta=\sigma_{\alpha+\beta}=\sigma_\beta\sigma_\alpha \qquad (20)$$

$$\sigma(0)=1,\ \ \sigma_\alpha^{-1}=\sigma_{-\alpha}, \qquad (21)$$

and the relation of an automorphism $\sigma(\alpha)$ with the mapping $\partial_i, i=1,...,n$ is of the form

$$\sigma_\alpha\partial_i=\eta(\alpha,\varepsilon_i)\partial_i\sigma_\alpha \qquad (22)$$

Considering the above constructions, we give an algebra $\mathscr{D}_q(2n)$ freely generated by $\partial_1,....,\partial_n,\sigma_1,...,\sigma_n$ enjoying the relations (17), (20) and (22). Here we note that $\mathscr{D}_q(2n)$ is a deformation of the algebra $\mathscr{D}(n)$ generated by the usual partial derivations $\partial_1,...,\partial_n$ because the mappings σ_i reduce to the idendity mapping when $q\to1$. It is well known that $\mathscr{D}(n)$ has a Hopf algebra structure via the coproduct Δ, counit ε and antipode S acting on the generators as $\Delta(\partial_i)=\partial_i\otimes\mathrm{id}+\mathrm{id}\otimes\partial_i$, $\varepsilon(\partial_i)=0, S(\partial_i)=-\partial_i$. Based on the above investigations, one can investigate whether the deformed derivation algebra $\mathscr{D}_q(2n)$ has a Hopf algebra structure as a deformation of the Hopf algebra structure for $\mathscr{D}(n)$. In fact, the answer is hidden in the structure of automorphisms and Leibniz rule of the derivations. That is, using the definitions $m(\Delta(\partial_a)(f\otimes g)):=\partial_a(fg)$ and $m(\Delta(\sigma_a)(f\otimes g)):=\sigma_a(fg)=\sigma_a(f)\sigma_a(g)$ together with (18) implies that the coproduct Δ acts on the generators as $\Delta(\sigma_i)=\sigma_i\otimes\sigma_i$ and $\Delta(\partial_i)=\partial_i\otimes id+\sigma_i\otimes\partial_i$. Morever we observe that $\mathscr{D}_q(2n)$ is nothing else than $\mathscr{U}_q$ mentioned in Subsection 3.3 of [5]. Therefore, $\mathscr{D}_q(2n)$ can be equipped with a Hopf algebra structure by the following comappings:

$$\begin{aligned}\Delta(\sigma_i)&=\sigma_i\otimes\sigma_i\\ \Delta(\partial_i)&=\partial_i\otimes\mathrm{id}+\sigma_i\otimes\partial_i,\end{aligned} \qquad (23)$$

$$\varepsilon(\sigma_i)=1,\ \varepsilon(\partial_i)=0, \qquad (24)$$

$$S(\sigma_i)=\sigma_i^{-1},\ S(\partial_i)=-\sigma_i^{-1}\partial_i, \qquad (25)$$

where $i=1,...,n$ and the multiplication in $\mathscr{D}_q(2n)\otimes\mathscr{D}_q(2n)$ is the usual one, that is, $(\alpha\otimes\gamma)(\beta\otimes\theta)=\alpha\beta\otimes\gamma\theta$.

4.1. A bicovariant differential calculus over $\mathscr{A}_q(n)$

This subsection is devoted to the construction of a bicovariant differential calculus of $\mathscr{A}_q(n)$ by using σ_i-derivations $\partial_i, i=1,...,n$. For this, we first need to show that $\mathscr{A}_q(n)$ has a proper Hopf algebra structure making possible the existence of bicovariant differential calculus obtained by means of σ_i-derivations ∂_i.

Lemma 4.1. *Let us extend $\mathscr{A}_q(n)$ by the inverse x_1^{-1}. Given two algebra homomorphisms $\Delta : \mathscr{A}_q(n) \to \mathscr{A}_q(n) \otimes \mathscr{A}_q(n)$, $\varepsilon : \mathscr{A}_q(n) \to \mathbb{C}$ and an algebra antihomomorphism $S : \mathscr{A}_q(n) \to \mathscr{A}_q(n)$, acting on the generators $x_1, x_2, ..., x_n$ as follows:*

$$\begin{aligned} &\Delta(x_1^{\pm 1}) = x_1^{\pm 1} \otimes x_1^{\pm 1}, \\ &\Delta(x_i) = x_i \otimes x_1 + x_1 \otimes x_i, \ 1 < i \leq n, \\ &\varepsilon(x_1) = 1, \ \varepsilon(x_i) = 0, \ 1 < i \leq n, \\ &S(x_1) = x_1^{-1}, \ S(x_i) = -x_1^{-1} x_i x_1^{-1}, \ 1 < i \leq n, \end{aligned} \tag{26}$$

where the multiplication in $\mathscr{A}_q(n) \otimes \mathscr{A}_q(n)$ is given by $(a \otimes b)(c \otimes d) = ac \otimes bd$. Then the extended quantum space $\mathscr{A}_q(n)$ is a cocommutative Hopf algebra together with the mappings Δ, counit ε and antipode S.

Proof. For any basis element x^α of the extended quantum space $\mathscr{A}_q(n)$, let $\alpha \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$. Thus, it is clear that the extended $\mathscr{A}_q(n)$ is again η-commutative algebra in the assumption that $x^0 = 1$ and $x^\alpha = 0$ if the last (n-1)-tuple of α $(\alpha_{n-1}, ..., \alpha_n) \notin \mathbb{Z}_+^n$. So, it must be first checked whether the mappings Δ, ε and S leave invariant the commutation relation (13). For ε, it is clear. For Δ, we shall check whether $\Delta(x_i x_j - \eta(\varepsilon_i, \varepsilon_j) x_j x_i) = 0$ holds for all $i, j = 1, 2, ..., n$. It is seen in the following:

$$\begin{aligned} \Delta(x_i x_j) &= (x_i \otimes x_1 + x_1 \otimes x_i)(x_j \otimes x_1 + x_1 \otimes x_j) \\ &= x_i x_j \otimes x_1^2 + x_i x_1 \otimes x_1 x_j + x_1 x_j \otimes x_i x_1 \\ &\quad + x_1^2 \otimes x_i x_j \\ &= \eta(\varepsilon_i, \varepsilon_j)(x_j x_i \otimes x_1^2 + x_1 x_i \otimes x_j x_1) \\ &+ \eta(\varepsilon_i, \varepsilon_j)(x_j x_1 \otimes x_1 x_i + x_1^2 \otimes x_j x_i) \\ &= \eta(\varepsilon_i, \varepsilon_j)\Delta(x_j)\Delta(x_i) \end{aligned} \tag{27}$$

for $1 < i, j \leq n$ with $i \neq j$ and

$$\begin{aligned} \Delta(x_1 x_i) &= (x_1 \otimes x_1)(x_i \otimes x_1 + x_1 \otimes x_i) \\ &= x_1 x_i \otimes x_1^2 + x_1^2 \otimes x_1 x_i \\ &= \eta(\varepsilon_1, \varepsilon_i)(x_i x_1 \otimes x_1^2 + x_1^2 \otimes x_i x_1) \\ &= \eta(\varepsilon_1, \varepsilon_i)\Delta(x_i)\Delta(x_1) \end{aligned}$$

and

$$\begin{aligned} \Delta(x_1^{-1} x_i) &= (x_1^{-1} \otimes x_1^{-1})(x_i \otimes x_1 + x_1 \otimes x_i) \\ &= x_1^{-1} x_i \otimes 1 + 1 \otimes x_1^{-1} x_i \\ &= \eta(\varepsilon_i, \varepsilon_1)(x_i x_1^{-1} \otimes 1 + 1 \otimes x_i x_1^{-1}) \\ &= \eta(\varepsilon_i, \varepsilon_1)\Delta(x_i)\Delta(x_1^{-1}) \end{aligned}$$

for $1 < i \leq n$. From the definition of Δ, it is readily apparent that Δ holds the rule of cocommutativity $\Delta = \tau \circ \Delta$, where τ is the twisting mapping defined by $\tau(a \otimes b) = b \otimes a$. From the actions of Δ, ε, S on the generators, it is also clear that the mappings Δ, ε, S fulfill the Hopf algebra axioms (1-3). Thus we can note that S is an anti-homomorphism at the level of coalgebra, meaning that

$$\tau \circ (S \otimes S) \circ \Delta = \Delta \circ S. \tag{28}$$

Indeed, from (26),

$$\begin{aligned} &(\tau \circ (S \otimes S) \circ \Delta)(x_i) \\ &= (\tau \circ (S \otimes S))(x_i \otimes x_1 + x_1 \otimes x_i), \\ &= \tau\left(-x_1^{-1} x_i x_1^{-1} \otimes x_1^{-1} - x_1^{-1} \otimes x_1^{-1} x_i x_1^{-1}\right) \\ &= -x_1^{-1} \otimes x_1^{-1} x_i x_1^{-1} - x_1^{-1} x_i x_1^{-1} \otimes x_1^{-1} \end{aligned} \tag{29}$$

Also

$$\begin{aligned} &(\Delta \circ S)(x_i) \\ &= \Delta\left(-x_1^{-1} x_i x_1^{-1}\right) \\ &= -\left(x_1^{-1} \otimes x_1^{-1}\right)(x_i \otimes x_1 + x_1 \otimes x_i)\left(x_1^{-1} \otimes x_1^{-1}\right) \\ &= -x_1^{-1} \otimes x_1^{-1} x_i x_1^{-1} - x_1^{-1} x_i x_1^{-1} \otimes x_1^{-1}. \end{aligned} \tag{30}$$

We see from (29) and (30) that the equality (28) exists for the generator x_i, $1 < i \leq n$. It is also shown in a similar manner that $(\tau \circ (S \otimes S) \circ \Delta)(x_1) = (\Delta \circ S)(x_1)$. Finally we observe equalities $\varepsilon \circ S = \varepsilon$ and $S^2 = \mathrm{id}$. □

Theorem 4.2. *Let Ω^1 be an $\mathscr{A}_q(n)$-bimodule with basis elements $\mathsf{d}x_1, ..., \mathsf{d}x_n$ whose relations with the generators $x_1, ..., x_n$ are of the form*

$$x_i \mathsf{d}x_j = \eta(\varepsilon_i, \varepsilon_j)\mathsf{d}x_i x_j, \ i, j = 1, ..., n. \tag{31}$$

If a mapping $\mathsf{d} : \mathscr{A}_q(n) \to \Omega^1$ is defined by

$$\mathsf{d}(f) = \mathsf{d}x_1 \partial_1(f) + \cdots + \mathsf{d}x_n \partial_n(f) = \tag{32}$$

then the pair (Ω^1, d) is a first order differential calculus on $\mathscr{A}_q(n)$.

Proof. By $\mathsf{d} : \mathscr{A}_q(n) \to \Omega^1$, it is clear that $\mathsf{d}(x_i)$ =$\mathsf{d}x_i$. Thus, to show that the pair (Ω^1, d) is a first order differential calculus, it is sufficient to prove that $\mathsf{d}(fg) = \mathsf{d}(f)g + f\mathsf{d}(g)$, where $f = x^\alpha$ and g is any element of $\mathscr{A}_q(n)$. For this goal we first need the following commutation relation obtained by (31):

$$\mathsf{d}x_i \sigma_i(f) = f\mathsf{d}x_i, \tag{33}$$

By substituting fg into (32) and using (16), we have the following

$$\begin{aligned} \mathsf{d}(fg) &= \mathsf{d}x_1(\partial_1(f)g + \sigma_1(f)\partial_1(g)) \\ &\quad + \cdots + \mathsf{d}x_n(\partial_n(f)g + \sigma_n(f)\partial_n(g)). \end{aligned} \tag{34}$$

Making use (33) in (34), we have

$$\begin{aligned} \mathsf{d}(fg) &= (\mathsf{d}x_1\partial_1 + \cdots + \mathsf{d}x_n\partial_n)(f)g \\ &\quad + f(\mathsf{d}x_1\partial_1 + \cdots + \mathsf{d}x_n\partial_n)(g), \end{aligned} \tag{35}$$

which turns out that $\mathsf{d}(fg) = \mathsf{d}(f)g + f\mathsf{d}(g)$. Thus the pair (Ω^1, d) is a first order differential calculus on $\mathscr{A}_q(n)$. Morever, since $\mathscr{A}_q(n)$ is a Hopf algebra, using Woronowicz's approach, we can define the right covariant bimodule structure by a mapping $\Delta_R : \Omega^1 \to \Omega^1 \otimes \mathscr{A}_q(n)$, defined as $\Delta_R = (\mathsf{d} \otimes \mathrm{id}) \circ \Delta$, and the left covariant bimodule structure by a mapping $\Delta_L : \Omega^1 \to \mathscr{A}_q(n) \otimes \Omega^1$, given by $\Delta_L = (\mathrm{id} \otimes \mathsf{d}) \circ \Delta$. Note that Δ_R and Δ_L act on $\mathscr{A}_q(n)$ as the coproduct given in (26). Finally, it is remain to show that the mappings Δ_L and Δ_R preserve the commutation relations (31). For example, in the case $1 < i \leq n, 1 \leq j \leq n$, for Δ_R, we can see this from the following straightforward calculation:

$$\begin{aligned}&\Delta_R(x_i \mathrm{d}x_j)\\&=(x_1\otimes x_i+x_i\otimes x_1)(\mathrm{d}x_1\otimes x_j+\mathrm{d}x_j\otimes x_1)\\&=x_1\mathrm{d}x_1\otimes x_ix_j+x_1\mathrm{d}x_j\otimes x_ix_1+x_i\mathrm{d}x_1\otimes x_1x_j\\&\quad+x_i\mathrm{d}x_j\otimes x_1x_1\\&=\eta(\varepsilon_i,\varepsilon_j)((\mathrm{d}x_1\otimes x_j+\mathrm{d}x_j\otimes x_1)(x_1\otimes x_i+x_i\otimes x_1)\\&=\eta(\varepsilon_i,\varepsilon_j)\Delta_R(\mathrm{d}x_j)\Delta_R(x_i).\end{aligned}$$

In the other cases, it can be similarly shown for both Δ_R and Δ_L. Thus the differential calculus (Ω^1,d) is a bicovariant one on $\mathscr{A}_q(n)$. □

Note that the action of d on a negative integer power of x_1 is computed by using $\mathrm{d}(x_1^{-1}) := -\mathrm{d}(x_1)x_1^{-2}$. Based on the above differential calculus (Ω^1,d), one can extend d to the exterior operator as follows

$$\mathscr{A}\cong\Omega_0\xrightarrow{\mathrm{d}}\Omega_1\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}\Omega_n\xrightarrow{\mathrm{d}}\Omega_{n+1}\xrightarrow{\mathrm{d}}\cdots \tag{36}$$

$$\mathrm{d}\circ\mathrm{d}:=\mathrm{d}^2=0 \tag{37}$$

$$\mathrm{d}(u\wedge v)=(\mathrm{d}u)\wedge v+(-1)^k u\wedge(\mathrm{d}v), \tag{38}$$

where $u\in\Omega_k$ and Ω_k is the space of differential k-forms. Thus, taking into account (31) with (37) and(38), we get the relation of differentials $\mathrm{d}x_i$ and $\mathrm{d}x_j, i,j=1,\ldots,n$ as follows:

$$\mathrm{d}x_i\wedge\mathrm{d}x_j=(\delta_{ij}-\eta(\varepsilon_i,\varepsilon_j))\mathrm{d}x_j\wedge\mathrm{d}x_i. \tag{39}$$

At this position we note that the above relation is consistent with the nilpotency rule (37) when we take into account $\mathrm{d}:=(\mathrm{d}x_1\partial_1+\cdots+\mathrm{d}x_n\partial_n)$.

As the final part of this section we can obtain the relation of mapping ∂_i and generator x_j by using the Leibniz property of d and the relations in (31)

$$\partial_i x_j=\delta_{ij}+\eta(\varepsilon_j,\varepsilon_i)x_j\partial_i, 1\le i,j\le n, \tag{40}$$

which complete the scheme of Weyl algebra corresponding to $\mathscr{A}_q(n)$ together with the relation (17).

5. Space of Maurer-Cartan 1-Forms on $\mathscr{A}_q(n)$

In the framework of the Hopf algebra $\mathscr{A}_q(n)$, the right-invariant Maurer-Cartan form for any $f\in\mathscr{A}_q(n)$ is defined by through the formula [11]

$$w_f:=m((\mathrm{d}\otimes S)\Delta(f)) \tag{41}$$

where m stands for the multiplication. Thus, for the non-commuting coordinates of $\mathscr{A}_q(n)$,

$$\begin{aligned}\omega_{x_1}&=m((\mathrm{d}\otimes S)\Delta(x_1))\\&=m((\mathrm{d}\otimes S)(x_1\otimes x_1))\\&=m(\mathrm{d}(x_1)\otimes S(x_1))=\mathrm{d}x_1x_1^{-1},\\ \omega_{x_i}&=m((\mathrm{d}\otimes S)\Delta(x_i))\\&=m((\mathrm{d}\otimes S)(x_1\otimes x_i+x_i\otimes x_1))\\&=\mathrm{d}x_i\,x_1^{-1}-\mathrm{d}x_1\,x_1^{-1}x_ix_1^{-1},\end{aligned}$$

where $1<i\le n$. Let $\omega_i=\omega_{x_i}$. Then (41) implies that for any $f\in\mathscr{A}_q(n)$, ω_f could be written as a linear combination of all ω_i's of the form $\omega_f=f_1\omega_1+\cdots+f_n\omega_n$ where $f_i\in\mathscr{A}_q(n), i=1,2,3$. Now, using (13), (31) and (39), we obtain some relations, which will be used in the following sections, such as commutation relations of ω_i's with the generators x_i's as

$$\begin{aligned}x_i\omega_1&=\omega_{x_1}x_i,\ 1\le i\le n\\ x_i\omega_j&=q^{j-1}\omega_jx_i,\ 1\le i\le n,\ 1<j\le n,\end{aligned} \tag{42}$$

and one between any ω_i and ω_j as follows:

$$\omega_i\wedge\omega_j=-(1-\delta_{ij})\omega_j\wedge\omega_i \tag{43}$$

6. Vector Fields

In this section we will obtain the quantum Lie algebra of vector fields, denoted by $\mathscr{T}$, corresponding to the right-invariant Maurer-Cartan forms obtained in the Section 4. To express the generators of $\mathscr{T}$ by the derivation mappings ∂_i we first obtain the Maurer-Cartan forms as follows

$$\mathrm{d}x_1=\omega_1x_1,\ \mathrm{d}x_i=\omega_1x_i+\omega_ix_1,\ 1<i\le n \tag{44}$$

Now we write d in terms of the Maurer-Cartan forms and the generators of $\mathscr{T}$:

$$\mathrm{d}=\sum_{i=1}^n\omega_iT_i \tag{45}$$

where T_i's are generators of $\mathscr{T}$. By inserting (44) to the expression

$$\mathrm{d}=\sum_{i=1}^n\mathrm{d}x_i\partial_i$$

we obtain the generators as the following vector fields:

$$\begin{aligned}T_1&\equiv\sum_{i=1}^n x_i\partial_i\\ T_i&\equiv x_1\partial_i,\ 1<i\le n\end{aligned} \tag{46}$$

Together with the relations (13), (17) and (40), (46) implies that commutation relation between two generators T_i and T_j is of the form

$$T_iT_j-T_jT_i=0,\ 1\le i,j\le n \tag{47}$$

The commutation relation (47) must be consistent with monomials of the algebra $\mathscr{A}$. For this, we get the following relations of the generators of $\mathscr{T}$ with the coordinates x_i's by using the relations in (13) and (40):

$$\begin{aligned}&T_1\,x_j=x_j+x_j\,T_1,\ 1\le j\le n\\&T_i\,x_j=\delta_{ij}x_1+q^{i-1}\,x_j\,T_i,\ 1<i\le n,\ 1\le j\le n\end{aligned}$$

Lemma 6.1. *Let f be any monomial of the form x^α in $\mathscr{A}_q(n)$. For any $g\in\mathscr{A}_q(n)$, the vector fields T_i's have the following q-deformed Leibniz rules when they act on fg :*

$$T_i(fg)=T_i(f)\ g+q^{\lambda_i}fT_i(g),\ \lambda_i=(i-1)\sum_{k=1}^n\alpha_k \tag{48}$$

Proof. The relations in (42) result in the following relation between the form ω_i and the monomial f

$$f\omega_i = q^{\lambda_i}\omega_i f. \tag{49}$$

From the Leibniz rule and the exterior differential operator d given by (45), we have

$$\left(\sum_{i=1}^{n}\omega_i T_i\right)(fg) = \left(\sum_{i=1}^{n}\omega_i T_i\right)(f)g + f\left(\sum_{i=1}^{n}\omega_i T_i\right)(g) \tag{50}$$

Inserting (49) to (50) and collecting according to ω_i, we obtain

$$\left(\sum_{i=1}^{n}\omega_i T_i\right)(fg) = \sum_{i=1}^{n}\omega_i\left(T_i(f)g + q^{\lambda_i} f T_i(g)\right) \tag{51}$$

which results in (48). □

Theorem 6.2. *We have the following q-deformed coproduct for the vector fields T_i, which is consistent with the q-deformed Leibniz rule (48):*

$$\Delta(T_i) = T_i \otimes \mathbf{1} + q^{(i-1)T_1}\otimes T_i \tag{52}$$

Proof. If we consider tensor product of the form

$$(X\otimes Y)(f\otimes g) = X(f)\otimes Y(g),\ f,g\in\mathscr{A},\ X,Y\in\mathscr{T} \tag{53}$$

and $m(\Delta(X)(f\otimes g)) := X(fg)$, then we have, from the q-deformed Leibniz rule (48), the q-deformed coproduct (53). Notice that by the action (16), the vector field T_1 acts on the monomial $f = x^\alpha$ as follows

$$T_1(f) = \left(\sum_{i=1}^{n} x_i\partial_{x_i}\right)(f) = \left(\sum_{i=1}^{n}\alpha_i\right).f$$

Since the action rule (16) holds also for a negative power of x_1, the action of T_i can be extended to a negative power of x_1.

Finally, in order to introduce Hopf algebra for the universal enveloping algebra $\mathscr{U}(\mathscr{T})$, we obtain the counit and antipode corresponding to the coproduct given in (52) as

$$\begin{aligned}\varepsilon(T_i) &= 0,\ 1\le i\le n\\ S(T_i) &= -q^{-(i-1)T_1}T_i,\ 1\le i\le n\end{aligned} \tag{54}$$

Notice that the Hopf algebra derived by (52) is q-deformed version of the usual Hopf algebra with the primitive coproduct $\Delta(T_i) = T_i\otimes\mathbf{1} + \mathbf{1}\otimes T_i$ obtained in the classical case $q = 1$. □

7. Conclusion

In this study we first introduce a quantum n-space whose coordinates yield an η-commutative polynomial algebra. Further it is shown that this η-commutative algebra admits some σ-twisted derivations and the algebra of these derivations is a non-cocommutative Hopf algebra, namely, a quantum group. Morever, we introduce a proper cocommutative Hopf algebra structure on η-commutative algebra such that a bicovariant differential calculus on the quantum n-space can be obtained via σ-twisted derivations. Finally, using this bicovariant differential calculus, the right invariant Maurer-Cartan forms and the corresponding vector fields are given, and it is seen that the algebra of these vector fields has a non-cocommutative Hopf algebra structure.

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