Available online: August 31, 2018

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 1, Pages 633–642 (2019)
DOI: 10.31801/cfsuasmas.456433
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

A NEW REPRESENTATION OF CONSTANT ANGLE SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$ WITH SPLIT QUATERNIONS

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Abstract. In this paper we study surfaces in $\mathbb{H}^2 \times \mathbb{R}$ for which the unit normal makes a constant angle with the $\mathbb{R}$-direction. The main idea is to show that constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$ can be obtained by split quaternion product and the matrix representations. Also we give some related examples with their projections of figures.

1. Introduction

An interesting problem of differential geometry of submanifolds, intensively studied in last years, consists in classification and characterization of surfaces whose unit normal vector field forms a constant angle with a fixed field of directions of the ambient space. These surfaces are called helix surfaces or constant angle surfaces and they have been studied in all the 3-dimensional geometries. This kind of surfaces are strictly related to describe some phenomena in physics of interfaces in liquids crystals and of layered fluids [7]. The early results were obtained by studying surfaces isometrically immersed in product spaces of type $\mathbb{M}^2 \times \mathbb{R}$, namely taken $\mathbb{M}^2$ to be the unit 2-sphere $\mathbb{S}^2$, the hyperbolic plane $\mathbb{H}^2$ in [8, 9]. The angle was considered between the unit normal of the surface M and the tangent direction to $\mathbb{R}$. An interesting classification of surfaces in the 3-dimensional Heisenberg group making a constant angle with the fibers of the Hopf- fibration was obtained in [25]. Moreover, Munteanu and Nistor obtained a classification of all surfaces in Euclidean 3-space for which the unit normal makes a constant angle with a fixed vector direction being the tangent direction to $\mathbb{R}$ [18]. In [19], it is also classified certain special ruled surfaces in $\mathbb{R}^3$ under the general theorem of characterization of constant angle surfaces. A classification is given of special developable surfaces and some conical surfaces from the point of view the constant angle property in [22]. Also some characterization are given for a curve lying on a surface for which

Received by the editors: February 09, 2018; Accepted: March 08, 2018.
2010 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.
Key words and phrases. Constant angle surfaces, product manifold, split quaternions, Minkowski space.

the unit normal makes a constant angle with a fixed direction [22]. These curves are called isophote curves in literature.

On the other hand, several authors have studied constant angle surfaces in Minkowski 3-space. Lopez and Munteanu investigated spacelike surfaces with the constant timelike direction [17] Atalay et al. by choosing the constant spacelike direction, they obtained different parametrization for the spacelike surfaces [1]. Also, the classifications are given for the timelike surfaces whose the normal makes a constant angle with a constant direction by Guler et al in [13]. In another recent paper [26] it is defined constant angle spacelike and timelike surfaces in the three-dimensional Heisenberg group and equipped with a 1-parameter family of Lorentzian metrics.

A rotation in $\mathbb{R}^3$ about an axis through the origin can be represented by a 3×3 orthogonal matrix with determinant 1. However, the matrix representation seems redundant because only four of its nine elements are independent. Also the geometric interpretation of such a matrix is not clear until we carry out several steps of calculation to extract the rotation axis and angle. Furthermore, to compose two rotations, we need to compute the product of the two corresponding matrices, which requires twenty-seven multiplications and eighteen additions.

Quaternions are very efficient for analyzing situations where rotations in $\mathbb{R}^3$ are involved. A quaternion is a 4-tuple, which is a more concise representation than a rotation matrix. Its geometric meaning is also more obvious as the rotation axis and angle can be trivially recovered. The quaternion algebra to be introduced will also allow us to easily compose rotations. This is because quaternion composition takes merely sixteen multiplications and twelve additions [27]. So quaternionic approach is a very important method for obtaining surfaces. For example in recent years several authors used this method for obtaining canal surfaces and constant slope surfaces [11, 15, 2, 3, 4, 5, 6].

A similar relation to the relationship between quaternions and rotations in the Euclidean space exists between split quaternions and rotations in the Minkowski 3-space. Split quaternions are identified with the semi-Euclidean space $\mathbb{E}_2^4$. Kula and Yaylı showed that algebra of split quaternions has a scalar product that allows us to identify it with semi-Euclidean space $\mathbb{E}_2^4$. They also showed that two unit split quaternions q and p determine a rotation in [16]. Ozdemir and Ergin examined properties of spatial rotations in the Minkowski 3-space via unit time-like quaternions. Moreover, they represented Lorentz rotation matrix using a timelike quaternion in [21].

The main idea in this paper is to show that constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$ can be obtained by split quaternion product and the matrix representations with similar methods of the paper [5]. Finally, some examples of these surfaces are given with their projections of figures by using the Mapple Programme. But, before this, we remind some concepts of split quaternions and the Lorentzian space.

2. Preliminary

In this section we introduce the notion of constant angle surfaces in $\widetilde{M} = \mathbb{H}^2 \times \mathbb{R}$ and give some first characterizations. Let $\mathbb{H}^2 \times \mathbb{R}$ be the Riemannian product of $\left(\mathbb{H}^2(-1), g_H\right)$ and $\mathbb{R}$ with the standard Euclidean metric, where $\mathbb{H}^2(-1)$ denotes the hyperbolic plane of constant curvature -1. Denoted by $\widetilde{g} = g_H + dt^2$ the product metric and $\widetilde{\nabla}$ the Levi-Civita connection of $\widetilde{g}$. Denote by t the global coordinate on $\mathbb{R}$ and $\partial_t = \frac{\partial}{\partial_t}$ is the unit vector field in the tangent bundle $T\left(\mathbb{H}^2 \times \mathbb{R}\right)$ that is tangent to the $\mathbb{R}-$direction.

Now consider a surface M in $\widetilde{M} = \mathbb{H}^2 \times \mathbb{R}$. Let us denote with η a unit normal to M. Then we can decompose ∂_t as

$$\partial_t = T + \cos\theta\eta$$

where T is the projection of ∂_t on the tangent space of M and θ is the angle function defined by

$$\cos\theta = \widetilde{g}\left(\partial_t, \eta\right).$$

By a constant angle surface M in $\mathbb{H}^2 \times \mathbb{R}$, we mean a surface for which the angle function θ is constant on M. There are two trivial cases, $\theta = 0$ and $\theta = \frac{\pi}{2}$. The condition $\theta = 0$ means that ∂_t is always normal, so M is an open part of $\mathbb{H}^2 \times \{t_0\}$, $t_0 \in \mathbb{R}$. In the second case ∂_t is always tangent. This corresponds to the Riemannian product of a curve in $\mathbb{H}^2$and $\mathbb{R}$.

There are many models for the hyperbolic plane (e.g. the Klein model, the Poincare disk, the upper half plane H^+, the Minkowski model $\mathcal{H}$), cf.[23]. The study of the constant angle surfaces was done by the authors in [10] by using the upper half plane model of the hyperbolic plane. In the following we will deal with the Minkowski model or the hyperboloid model for $\mathbb{H}^2$.

We denote by $\mathbb{R}^3_1$ the Minkowski 3-space with coordinates x, y and z, endowed with the Lorentzian metric tensor

$$< ., . >_L = -dx^2 + dy^2 + dz^2.$$

In Minkowski 3-space, the vectors are characterized by Lorentzian inner product. For a vector $v = (v_1, v_2, v_3) \in \mathbb{R}^3_1$, the vector v is said to be a spacelike if $< v, v >_L > 0$ or $v = 0$, timelike if $< v, v >_L < 0$, lightlike (or null) if $< v, v >_L = 0$. Furthermore, curves are classified depending on their tangent vectors. A curve is called spacelike, timelike or lightlike (or, null) if the tangent vector of the curve is always spacelike, timelike or lightlike, respectively.

Then $\mathbb{H}^2$ can be considered as the upper sheet $(x > 0)$ of the hyperboloid

$$\left\{(x, y, z) \in \mathbb{R}^3_1 : -x^2 + y^2 + z^2 = -1\right\}.$$

The external unit normal to $\mathcal{H}$ in a point $p \in \mathcal{H} \subset \mathbb{R}^3_1$ is $N = p$ and we have $< N, N >_L = -1$.

We recall the notion of the Lorentzian cross-product (see e.g.[23]): For any $a=(a_1,a_2,a_3)$, $b=(b_1,b_2,b_3)\in\mathbb{R}^3_1$, Lorentzian cross product is defined by

$$a\wedge_L b=(a_3b_2-a_2b_3,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1)\,.$$

As analogue to the vector cross product in the Euclidean space, it has similar algebraic and geometric properties:

(i) $a\wedge_L b$ is perpendicular to a and b, i.e. $<a\wedge_L b,a>_L=<a\wedge_L b,b>_L=0$;

(ii) $a\wedge_L b=-b\wedge_L a$;

(iii) $<a\wedge_L b,a\wedge_L b>_L=-<a,a>_L<b,b>_L+<a,b>^2_L$, for all $a,b\in\mathbb{R}^3_1$.

Let M be a 2-dimensional surface in $\mathcal{H}\times\mathbb{R}\subset\mathbb{R}^3_1\times\mathbb{R}$. On the ambient space we consider the product metric: $g_0=-dx^2+dy^2+dz^2+dt^2$.

The characterization of constant angle surface in $\mathbb{H}^2\times\mathbb{R}$ was given in [9]. The main result is the following:

Theorem 1. *A surface M in $\mathcal{H}\times\mathbb{R}$ is a constant angle surface if and only if the position vector F is up to isometries of $\mathcal{H}\times\mathbb{R}$, locally given by*

$$F:M\longrightarrow\mathcal{H}\times\mathbb{R}:(u,v)\to F(u,v),$$

where

$$F(u,v)=(\cosh\xi(u)f(v)+\sinh\xi(u)f(v)\wedge_L f'(v),u\sin\theta)\,,$$

f is a unit speed curve on $\mathcal{H}$, $\xi(u)=u\cos\theta$ and θ is the constant angle [9].

Let us recall some details about the main theorem. Since $f(v)$ lies on the hyperboloid it follows that $f'(v)$ is spacelike. But the curve f has unit speed so, $\langle f'(v),f'(v)\rangle_L=1$. In each point of the curve f one has an orthonormal basis, namely $\{f(v),f'(v),f(v)\wedge_L f'(v)\}$. Taking into account that $\langle f'(v),f''(v)\rangle_L=0$ for all v, one can express $f''(v)$ as linear combination of $f(v)$ and $f(v)\wedge_L f'(v)$.

Now let us give some basic concepts about the split quaternions. The split quaternions or coquaternions are basic elements of four dimensional associative algebra defined by James Cockle. Like the real quaternions defined by Hamilton in 1843, the split quaternions have some different properties (eg. nilpotent elements, nontrivial idempotents, nonzero zero divisors) from real quaternions. Whereas 3-dimensional Euclidean spatial rotations can be expressed by real quaternions, Minkowski spatial rotations can be stated by split quaternions. The set Q' of all split quaternions is given by

$$Q'=\left\{q=d+ai+bj+ck:a,b,c,d\in\mathbb{R},i^2=-1,j^2=k^2=ijk=1\right\}.$$

A split quaternion can be written as $q=d+ai+bj+ck$ or $q=S_q+V_q$, where $S_q=d\in\mathbb{R}$ is the scalar component of q and $V_q=ai+bj+ck$ is the vectorial component of q. We also write following four-tuple notation to represent a split quaternion:

$$\begin{aligned} q &= (a,b,c,d)\,, \\ q &= (w,d)\,, \end{aligned}$$

where $S_q = d \in \mathbb{R}$, $V_q = w \in \mathbb{R}^3_1$. If $S_q = 0$, the split quaternion is called pure split quaternion. Addition of two split quaternions, multiplication of a split quaternion with a scalar $\lambda \in \mathbb{R}$ and conjugate of a split quaternion, $\overline{q}$, can be given in Q' as follows:

$$\begin{aligned} q + p &= (S_q + S_p) + (V_q + V_p), \\ \lambda q &= \lambda S_q + \lambda V_q, \\ \overline{q} &= S_q - V_q. \end{aligned}$$

By using dot and cross-product in the Minkowski 3-space we can give the quaternion product of two split quaternions p and q as:

$$q \times_L p = S_q S_p + \langle V_q, V_p \rangle_L + S_q V_p + S_p V_q + V_q \wedge_L V_p. \tag{2.1}$$

Moreover the following relation between a split quaternion and its conjugate is satisfied, $I_q = q \times_L \overline{q} = \overline{q} \times_L q = d^2 + a^2 - b^2 - c^2$. We say that a split quaternion q is a spacelike, timelike or lightlike quaternion, if $I_q < 0$, $I_q > 0$ or $I_q = 0$ respectively. The norm of a split quaternion $q \in Q'$ is given by $N_q = \sqrt{|I_q|}$. If $I_q = \pm 1$, then the split quaternion q is said to be a unit split quaternion. A non-lightlike split quaternion $q \neq 0$ has an inverse quaternion denoted by q^{-1} according to split quaternion product which satisfies the following property $q \times_L \frac{\overline{q}}{I_q} = \frac{\overline{q}}{I_q} \times_L q = 1$, which gives us that the inverse of q can be given as

$$q^{-1} = \frac{\overline{q}}{I_q},$$

[21]. The vector part of any spacelike quaternions is spacelike but the vector part of any timelike quaternion can be spacelike or timelike. Polar forms of the split quaternions are given by following classification:

(i) Let q be a unit spacelike split quaternion. Then it can be expressed in the form $q = \sinh\theta + v\cosh\theta$, where v is a unit spacelike vector in $\mathbb{R}^3_1$.

(ii) Let q be a unit timelike split quaternion with the spacelike vector part. Then it can be expressed in the form $q = \cosh\theta + v\sinh\theta$, where v is a unit spacelike vector in $\mathbb{R}^3_1$.

(iii) Let q be a unit timelike split quaternion with the timelike vector part. Then, it can be expressed as $q = \cos\theta + v\sin\theta$, where v is a unit timelike vector in $\mathbb{R}^3_1$ [16, 21, 28].

Unit timelike quaternions are used to perform rotations in the Minkowski 3-space. Let $\Phi : \mathbb{R}^3_1 \to \mathbb{R}^3_1$ be a linear mapping and $\Phi = q \times_L v \times_L q^{-1}$, where q is a unit timelike quaternion and v is a pure split quaternion (that is, a vector in $\mathbb{R}^3_1$). So, for every timelike unit quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$, we can give matrix representation M_q of Φ as follows:

$$M_q = \begin{bmatrix} a_0^2 + a_1^2 + a_2^2 + a_3^2 & 2a_0a_3 - 2a_1a_2 & -2a_0a_2 - 2a_1a_3 \\ 2a_0a_3 + 2a_1a_2 & a_0^2 - a_1^2 - a_2^2 + a_3^2 & -2a_0a_1 - 2a_2a_3 \\ -2a_0a_2 + 2a_1a_3 & 2a_0a_1 - 2a_2a_3 & a_0^2 - a_1^2 + a_2^2 - a_3^2 \end{bmatrix}.$$

It can be seen that all rows of this matrix are orthogonal in the Lorentzian mean. Therefore the unit timelike quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$ is equivalent to 3×3 orthogonal rotational matrix M_q. The matrix represents a rotation in Minkowski 3-space under the condition that $\det M_q = 1$. This is possible with unit timelike quaternions. Also causal character of vector part of the unit timelike quaternion q is important. If the vector part of q is timelike or spacelike then the rotation angle is spherical or hyperbolic, respectively [21].

3. A New Approach On Constant Angle Surface in $\mathbb{H}^2 \times \mathbb{R}$ with Split Quaternions

In this section we consider unit timelike quaternions with the spacelike vector parts

$$Q(u,v) = \cosh \xi(u) - \sinh \xi(u) f'(v)$$

defines a 2-dimensional surface in $\mathcal{H} \times \mathbb{R} \subset \mathbb{R}^3_1 \times \mathbb{R}$, where $f'(v) = (f'_1(v), f'_2(v), f'_3(v))$ and f is a unit speed spacelike curve on $\mathcal{H}$. As we gave earlier, for the unitary quaternion $Q(u,v)$, the matrix representation of the map $\Phi : \mathbb{R}^3_1 \to \mathbb{R}^3_1$ is given by

$$M_Q = \begin{bmatrix} \cosh^2 \xi + \left(f_1'^2 + f_2'^2 + f_3'^2\right) \sinh^2 \xi & -2\left(f'_3 \cosh \xi + f'_1 f'_2 \sinh \xi\right) \sinh \xi & 2\left(f'_2 \cosh \xi - f'_1 f'_3 \sinh \xi\right) \sinh \xi \\ 2\left(-f'_3 \cosh \xi + f'_1 f'_2 \sinh \xi\right) \sinh \xi & \cosh^2 \xi + \left(-f_1'^2 - f_2'^2 + f_3'^2\right) \sinh^2 \xi & 2\left(f'_1 \cosh \xi - f'_2 f'_3 \sinh \xi\right) \sinh \xi \\ 2\left(f'_2 \cosh \xi + f'_1 f'_3 \sinh \xi\right) \sinh \xi & -2\left(f'_1 \cosh \xi + f'_2 f'_3 \sinh \xi\right) \sinh \xi & \cosh^2 \xi + \left(-f_1'^2 + f_2'^2 - f_3'^2\right) \sinh^2 \xi \end{bmatrix}.$$

We are now ready to show main result of this paper:

Theorem 2. *Let $F : M \longrightarrow \mathcal{H} \times \mathbb{R} : (u,v) \to F(u,v)$ be an immersion up to isometries of $\mathcal{H} \times \mathbb{R}$. Then the constant angle surface M can be reparametrized by*

$$F(u,v) = u \sin \theta + Q(u,v) \times_L Q_1(u,v), \tag{3.1}$$

where "$\times_L$" is the split quaternion product, $Q_1(u,v) = f(v)$ is a unit speed spacelike curve on $\mathcal{H}$ and a pure split quaternion and θ is the constant angle.

Proof. Since $Q(u,v) = \cosh \xi(u) - \sinh \xi(u) f'(v)$ and $Q_1(u,v) = f(v)$, we obtain

$$\begin{aligned} Q(u,v) \times_L Q_1(u,v) &= (\cosh \xi(u) - \sinh \xi(u) f'(v)) \times_L f(v) \\ &= \cos \xi(u) f(v) - \sin \xi(u) f'(v) \times_L f(v). \end{aligned} \tag{3.2}$$

By using the Eq.(2.1), we get

$$f'(v) \times_L f(v) = \langle f'(v), f(v) \rangle_L + f'(v) \Lambda_L f(v).$$

We know that $\langle f'(v), f(v) \rangle_L = 0$ since f is a unit speed spacelike curve on $\mathcal{H}$. Thus

$$\begin{aligned} f'(v) \times_L f(v) &= f'(v) \Lambda_L f(v) \\ &= -f(v) \Lambda_L f'(v). \end{aligned} \tag{3.3}$$

If we substitute Eq.(3.3) into Eq.(3.2), we get

$$Q(u,v) \times_L Q_1(u,v) = \cosh \xi(u) f(v) + \sinh \xi(u) f(v) \Lambda_L f'(v).$$

Then the immersion $F : M \to \mathcal{H} \times \mathbb{R}$ is given by

$$\begin{aligned} F(u,v) &= u\sin\theta + \cosh(u\cos\theta)\, f(v) + \sinh(u\cos\theta)\, f(v)\Lambda_L f'(v), \\ &= \left(\cosh(u\cos\theta)\, f(v) + \sinh(u\cos\theta)\, f(v)\Lambda_L f'(v), u\sin\theta\right), \end{aligned}$$

as we wished to prove.

Remark 1. *Theorem 1 says that a unit speed curve $f(v)$ in $\mathcal{H}$, is rotated by $Q(u,v)$ through the hyperbolic angle $\xi(u)$ about the spacelike axis $S_p\{f'(v)\}$.*

□

As a consequence of this theorem, we get the following corollary.

Corollary 1. *Let M_Q be the matrix representation of the map $\Phi : \mathbb{R}^3_1 \to \mathbb{R}^3_1$ for the unit timelike quaternions with the spacelike vector parts $Q(u,v)$. Then, for the pure quaternion $Q_1(u,v)$, the constant angle surface in $\mathcal{H} \times \mathbb{R}$ can be written as*

$$F(u,v) = u\sin\theta + M_Q Q_1(u,v).$$

Remark also that the two trivial cases are included in the parametrization (3.1).
(i) If $\theta = 0$, then $\xi(u) = u$, $Q(u,v) = \cosh u - \sinh u f'(v)$, (3.1) becomes

$$F(u,v) = Q(u,v) \times_L Q_1(u,v)$$

which gives us $\mathbb{H}^2 \times \{0\}$.
(ii) If $\theta = \frac{\pi}{2}$, then $\xi(u) = 0$, $Q(u,v) = 1$, (3.1) becomes.

$$F(u,v) = u + Q_1(u,v)$$

This clearly gives the Riemannian product of a curve in $\mathbb{H}^2$ and $\mathbb{R}$.

Example 1. *Let us consider unit speed spacelike curve in $\mathbb{H}^2$ defined by $f(v) = (\cosh v, 0, \sinh v)$ and taking $\theta = 0$. Then the constant angle surface M can be parametrized by*

$$\begin{aligned} F(u,v) &= Q(u,v) \times_L Q_1(u,v) \\ &= \cosh u\,(\cosh v, 0, \sinh v) + \sinh u\,(0, -1, 0) \\ &= \left(\cosh u \cosh v, -\sinh u, \cosh u \sinh v, 0\right), \end{aligned}$$

(see Figure 1).

Example 2. *Let $f(v) = \left(\cosh v, \frac{\sqrt{3}}{2}\sinh v, \frac{1}{2}\sinh v\right)$ is a unit speed spacelike curve in $\mathbb{H}^2$ and $\theta = \frac{\pi}{2}$. Then the constant angle surface M can be parametrized by*

$$\begin{aligned} F(u,v) &= u + Q_1(u,v) \\ F(u,v) &= \left(\cosh v, \frac{\sqrt{3}}{2}\sinh v, \frac{1}{2}\sinh v, u\right), \end{aligned}$$

(see Figure 2).

Example 3. *Let $Q_1(u,v) = f(v) = (\cosh v, \sinh v, 0)$ is a unit speed spacelike curve in $\mathbb{H}^2$, $\xi(u) = u\cos\theta$ and $Q(u,v) = \cos\xi(u) - \sin\xi(u)(\sinh v, \cosh v, 0)$. Then the constant angle surface M can be parametrized by*

$$\begin{aligned} F(u,v) &= u\sin\theta + Q(u,v) \times_L Q_1(u,v) \\ &= u\sin\theta + \cosh\xi(u) f(v) + \sin\xi(u) f(v) \times_L f'(v) \\ &= u\sin\theta + \cosh\xi(u)(\cosh v, \sinh v, 0) + \sinh\xi(u)(0,0,1) \\ &= u\sin\theta + (\cosh\xi(u)\cosh v, \cosh\xi(u)\sinh v, \sinh\xi(u)). \end{aligned}$$

Up to parametrization we get

$$F(u,v) = (\cosh u \cosh v, \cosh u \sinh v, \sinh u, u\tan\theta),$$

(see Figure 3).

4. Visualization

The $3D$-surfaces geometric modeling are very important in the surface modeling systems. We give the visualization of the surfaces with the parametrization

$$F(u,v) = (x(u,v), y(u,v), z(u,v), w(u,v))$$

in $\mathbb{R}^4$ by use of Maple Programme. We plot the graph of the surface with plotting command

$$plot3d\,(x, y, z + w)$$

We construct the geometric model of the constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$ defined in Example 1 (see, Figure 1), Example 2 (see, Figure 2) and Example 3 (see, Figure 3). obtained for $u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $v \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

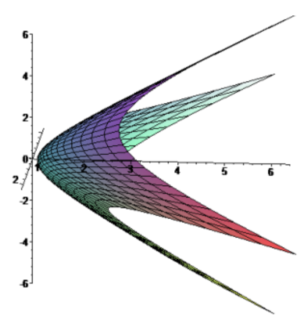

Figure 1. The projections of constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$ obtained for $u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $v \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

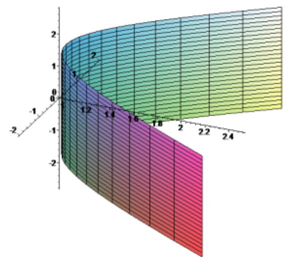

FIGURE 2. The projections of constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$ obtained for $u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $v \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

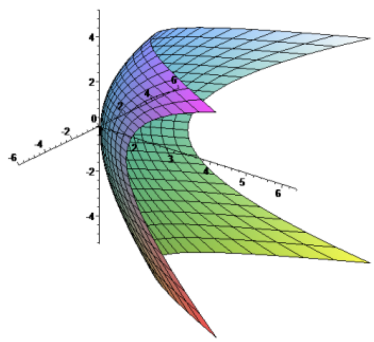

FIGURE 3. The projections of constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$ obtained for $\theta = \frac{\pi}{3}, u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], v \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

References

[1] Atalay, G.Ş., Güler, F. and Kasap, E., Spacelike Constant Angle Surfaces in Minkowski 3-space, *J. Math. Comput. Sci.* 2 (2012), No. 3, 451-461.

[2] Aslan, S. and Yaylı, Y., Canal surfaces with quaternions. *Adv. Appl. Clifford Algebras* (2015).

[3] Aslan, S. and Yaylı, Y., Split quaternions and canal surfaces in Minkowski 3-space. *Int. J. Geom.*5(2), (2016), 51–61.

[4] Babaarslan, M. and Yaylı, Y., A new approach to constant slope surfaces with quaternion. *ISRN Geom.* 2012, Article ID 126358, p. 8 (2012). doi:10.5402/ 2012/126358

[5] Babaarslan, M. and Yaylı, Y., Split quaternions and spacelike constant slope surfaces in Minkowski 3-space. *Int. J. Geom.* Vol. 2, No.1 (2013) 23–33.

[6] Babaarslan, M. and Yaylı, Y., Split quaternions and timelike constant slope surfaces in Minkowski 3-space, arXiv:1206.3935 [math-ph]

[7] Cermelli, P. and Di Scala A., J., Constant angle surfaces in liquid crystals. *Phylos Magazine.* 87, (2007) 1871-1888.

[8] Dillen, F., Fastenakels, J., Van der Veken J. and Vrancken, L., Constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$. *Monaths. Math.* 152, (2007) 89–96.

[9] Dillen, F. and Munteanu, M.,I., Constant Angle Surfaces in $\mathbb{H}^2 \times \mathbb{R}$. *Bull. Braz.Math. Soc.* 40, (2009) 85–97.

[10] Dillen, F. and Munteanu, M.I., Surfaces in $H^+ \times \mathbb{R}$, *Pure and Applied Differential Geometry,* PADGE 2007, Berichte aus der Mathematik (Shaker Verlag) Eds. Franki Dillen, Ignace Van de Woestyne, (2007) 185-193.

[11] Gök, İ., Quaternionic approach of canal surfaces constructed by some new ideas, *Adv. Appl. Clifford Algebras*, 27 (2017) 1175–1190.

[12] Gray, A., Abbena, E. and Salamon, S., Modern Geometry of Curves and Surfaces with Mathematica. 3rd Edition. Studies in Advanced Mathematics. 47, 2006.

[13] Güler, F., Şaffak, G. and Kasap, E., Timelike Constant Angle Surfaces in Minkowski Space $\mathbb{R}^3_1$, Int. J. Contemp. Math. Sciences, Vol. 6, no. 44, (2011), 2189–2200.

[14] Hacısalihoğlu, H.H., Hareket Geometrisi ve Kuaterniyonlar Teorisi. Faculty of Sciences and Arts, University of Gazi Press 1983.

[15] Kocakusaklı, E., Tuncer, O.O, Gok, İ. and Yaylı Y., A New Representation of Canal Surfaces with Split Quaternions in Minkowski 3-Space *Adv. Appl. Clifford Algebras*, 27 (2017) 1387–1409.

[16] Kula, L. and Yayli, Y., Split quaternions and rotations in semi Euclidean space $\mathbb{E}^4_2$. *J. Korean Math. Soc.* 44(6), (2007) 1313–1327.

[17] Lopez, R., and Munteanu, M.I., Constant Angle Surfaces in Minkowski Space, *Bulletin of the Belgian Mathematical Society*, Volume18, Number 2 (2011), 271-286.

[18] Munteanu, M.I. and Nistor, A. I., A New Approach on Constant Angle Surfaces in $\mathbb{E}^3$. *Turk. J. Math.* Vol. 33. (2009) 169-178.

[19] Nistor, A., Certain Constant Angle Surfaces Constructed on Curves, I*nt. Electron. J. Geom.,* Vol. 4 No.1(2011), 79–87.

[20] O'Neill, B., Semi-Riemannian Geometry. Academic Press. New York, 1983.

[21] Özdemir, M. and Ergin, A.A., Rotations with unit timelike quaternions in Minkowski 3-space. *J. Geom. Phys.* 56(2), (2006) 322–326.

[22] Özkaldı, S. and Yaylı, Y., Constant Angle Surfaces and Curves in $\mathbb{E}^3$, *Int. Electron. J. Geom.,* Vol. 4, No.1 (2011), 70–78.

[23] Ratcliffe, J. G., Foundatitons of Hyperbolic Manifolds, Springer-Verlag, New York, 1994.

[24] Struik, D.J., Lectures on Classical Differential Geometry. New York. Dover, 1988.

[25] Fastenakels, J., Munteanu M. I. and Van der Veken, J., Constant angle surfaces in the Heisenberg group, *Acta Math. Sinica - Engl. Series,* 27, 4 (2011), 747–756.

[26] Onnis I. and Piu P., Constant angle surfaces in the Lorentzian Heisenberg group, *Arch. Math.* 109 (2017), 575–589.

[27] Jia, Yan-Bin, Quaternion and Rotation, Com S 477/577 Notes Sep 5, 2017.

[28] Ward, J.P., Quaternions and Cayley Numbers. Kluwer Academic Publishers, Boston 1997.

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