CONSTRUCTIVE MATHEMATICAL ANALYSIS
8 (2025), No. SI, pp. 57-67
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

On the second boundary value problem for the system of thermoelasticity with microtemperatures

VITA LEONESSA* AND ANGELICA MALASPINA

ABSTRACT. In this work, we investigate the second BVP (boundary value problem) associated with the linear equilibrium theory of thermoelasticity with microtemperatures. We obtain a solution of the second BVP in terms of a double-layer thermoelastic potential, unlike the results reported in the literature, where a solution is represented by a single-layer thermoelastic potential.

Keywords: Equilibrium theory, thermoelasticity with microtemperatures, boundary value problems, potential theory, integral representations, boundary integral equations.

2020 Mathematics Subject Classification: 74F05, 45F15, 35J57, 31B10.

1. INTRODUCTION

The theory of thermoelasticity with microtemperatures is a generalized extension of classical thermoelasticity, developed to more accurately describe the thermo-mechanical behaviour of materials with internal microstructures, such as composites, porous materials, and nanostructured metals. Unlike classical theory, where temperature is considered as a uniform macroscopic scalar quantity, microtemperature models introduce new internal thermal variables to represent heat transport phenomena and deformations at the microscopic scale.

In [20], Iesan and Quintanilla introduced a three-dimensional model of thermoelasticity, incorporating microtemperatures as additional internal variables. In their formulation, each material point is characterized not only by a macroscopic temperature field, but also by a vector of microthermal variables, which provide a more refined description of local heat transfer at the microscopic scale. In [20], the authors also derive existence and uniqueness theorems, as well as results on the asymptotic behavior of solutions. Further developments of this model can be found in [18, 19]. See also [11, 21] and the references therein.

Several works have contributed to the mathematical formulation and development of the theory due to Iesan and Quintanilla. In particular, Svanadze [28] derived fundamental solutions for the equations of the equilibrium and steady oscillations related to this theory. Green's formulae within the linear equilibrium theory of the considered thermoelasticity framework, as well as uniqueness theorems for both internal and external basic BVPs, were established in [27]. Also in [27], existence theorems for the internal and external BVPs were proved using the potential method in combination with the theory of singular integral equations. See [26, 29] for the basic BVPs of steady vibrations studied using the potential method. As far as

Received: 09.09.2025; Accepted: 05.12.2025; Published Online: 16.12.2025
*Corresponding author: Vita Leonessa; vita.leonessa@unibas.it
DOI: 10.33205/cma.1800066

the two-dimensional case is concerned, see [2], where specific features of planar problems were analyzed.

The theory of thermoelasticity with microtemperatures has been extended in various ways. For example, a linear theory for microstretch elastic materials with microtemperatures, focusing on bodies without microrotational effects, has been presented in [17], establishing uniqueness and existence theorems for the dynamic behavior of anisotropic materials. Liverani and Quintanilla [24] studied thermoelastic models with microtemperatures including fading-memory, proving well-posedness and exponential energy decay. Other recent contributions include nonlinear models [1] and double and triple-porosity frameworks [16, 22, 30].

In [23], we proposed a boundary integral formulation for the Dirichlet problem in the linear equilibrium theory of thermoelasticity with microtemperatures, based on the single-layer potential ansatz. The approach used stems from the integral method first introduced by Cialdea [3], who proposed a solution of the Dirichlet problem for Laplace's equation, in a bounded connected domain, represented by means of a single-layer potential, instead of the classical double-layer one. A similar approach consists in looking for the solution of the Neumann problem by means of a double-layer potential (see, e.g. [6]). The method relies on the theory of reducible operators and the theory of differential forms, and it has been extended in several contexts (see, e.g. [8, 7, 10, 9, 4, 5]).

The aim of the present work is to employ the double-layer ansatz to represent a solution of the second BVP in the linear equilibrium theory of thermoelasticity with microtemperatures, where the thermal and mechanical fluxes are prescribed on the boundary. Such a BVP arises naturally in applications involving external stress, insulation, or prescribed heat fluxes, and poses distinctive analytical challenges due to the structure of the PDE system and the role of the stress operator in the boundary formulation.

The paper is organized as follows. Section 2 reviews some key notation, introduces the thermoelasticity system with microtemperatures, the single- and double-layer thermoelastic potentials, and summarizes the main result obtained in [23] for the Dirichlet problem. Section 3 is devoted to some boundary integral operators and their kernels. Finally, in Section 4 we address the second BVP, represent the solution in terms of double-layer thermoelastic potentials, and prove solvability via a suitable Fredholm system.

2. Preliminaries

Throughout the article, Ω denotes a bounded domain (open connected set) of $\mathbb{R}^3$, with complement $\mathbb{R}^3 \setminus \overline{\Omega}$ connected, and such that its boundary Σ is a Lyapunov surface, i.e. $\Sigma \in C^{1,\alpha}$, $\alpha \in]0,1]$. The vector $n(x)$ stands for the outward unit normal vector at the point $x \in \Sigma$, and $D = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$.

For $h \in \mathbb{N}$, let $C^h(\Omega)$ denote the space of all real-valued continuous functions defined on Ω whose partial derivatives exist and are continuous up to order h. Moreover, $C^{h,\beta}(\Omega)$ $(0 < \beta \leq 1)$ denotes the space of all functions defined on Ω possessing continuous derivatives up to order h, with all derivatives of order h being Hölder continuous with exponent β.

The subspace of $C^h(\Omega)$ of functions whose partial derivatives can be continuously extended to $\overline{\Omega}$ is denoted by $C^h(\overline{\Omega})$.

Likewise, $C^{h,\beta}(\overline{\Omega})$ stands for the subspace of $C^h(\overline{\Omega})$ whose derivatives of order h are Hölder continuous with exponent β, for $0 < \beta \leq 1$.

From now on, let p be a real number such that $1 < p < +\infty$. As usual, the symbol $L^p(\Sigma)$ stands for the space of all real-valued measurable functions u for which $|u|^p$ is integrable over Σ. We denote by $L^p_1(\Sigma)$ the space of differential forms of degree 1 (in short, 1-forms) whose components belong to $L^p(\Sigma)$. The Sobolev space $W^{1,p}(\Sigma)$ can be viewed as the space of all functions in $L^p(\Sigma)$ whose weak differential belongs to $L^p_1(\Sigma)$. If $u \in [W^{1,p}(\Sigma)]^7$, then du denotes the vector $(du_1, \dots, du_7)$. For details about differential forms, we refer the reader to [14, 15].

In order to distinguish the limit obtained by approaching the boundary Σ from Ω and $\mathbb{R}^3 \setminus \overline{\Omega}$, respectively, we will use the following notations:

$$u^+(z) = \lim_{\substack{x \to z \\ x \in \Omega}} u(x) \quad \text{and} \quad u^-(z) = \lim_{\substack{x \to z \\ x \in \mathbb{R}^3 \setminus \overline{\Omega}}} u(x), \quad z \in \Sigma.$$

Let us now introduce some additional concepts and preliminary results that will be useful later. For reader's convenience, we proceed to split up the section in several subsections.

2.1. **The system of the linear equilibrium theory of thermoelasticity with microtemperatures.** Suppose that the bounded domain Ω is occupied by an isotropic elastic material with microstructure. Following [26, 27], the system of homogeneous equations of the linear equilibrium theory of thermoelasticity for such materials is

$$\begin{cases} \mu \Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u - \beta \operatorname{grad} \theta = 0, \\ k_6 \Delta w + (k_4 + k_5) \operatorname{grad} \operatorname{div} w - k_3 \operatorname{grad} \theta - k_2 w = 0, \\ k \Delta \theta + k_1 \operatorname{div} w = 0, \end{cases} \tag{2.1}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector, $w = (w_1, w_2, w_3)$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 ($T_0 > 0$), and $\lambda, \mu, \beta, k, k_1, \dots, k_6$ are constitutive coefficients.

Let us assume that the following conditions are satisfied:

$$\begin{aligned} &\mu > 0,\ 3\lambda + 2\mu > 0, \\ &3k_4 + k_5 + k_6 > 0,\ k_6 + k_5 > 0,\ k_6 - k_5 > 0, \\ &k > 0,\ (k_1 + T_0 k_3)^2 < 4 T_0 k k_2. \end{aligned} \tag{2.2}$$

Observe that conditions (2.2) are involved in the uniqueness theorems for the basic BVPs (see [27]). Moreover, (2.2) assure the ellipticity of system (2.1).

It is convenient to rewrite the system (2.1) in a matrix form:

$$A(D)U(x) = 0, \quad x \in \Omega, \tag{2.3}$$

where $U = (u, w, \theta)^T$, superscript T denoting transposition, and $A(D)$ is the 7×7 matrix

$$A(D) = \begin{pmatrix} A^{(1)}(D) & A^{(2)}(D) & A^{(5)}(D) \\ A^{(3)}(D) & A^{(4)}(D) & A^{(6)}(D) \\ A^{(7)}(D) & A^{(8)}(D) & A^{(9)}(D) \end{pmatrix}$$

whose entries are, for $l, j = 1, 2, 3$,

$$A^{(1)}_{lj}(D) = \mu \Delta \delta_{lj} + (\lambda + \mu) \frac{\partial^2}{\partial x_l \partial x_j}, \quad A^{(2)}_{lj}(D) = A^{(3)}_{lj}(D) = 0,$$

$$A^{(4)}_{lj}(D) = (k_6 \Delta - k_2)\delta_{lj} + (k_4 + k_5) \frac{\partial^2}{\partial x_l \partial x_j}, \quad A^{(5)}_{l\,7}(D) = -\beta \frac{\partial}{\partial x_l},$$

$$A^{(6)}_{l\,7} = -k_3 \frac{\partial}{\partial x_l}, \quad A^{(7)}_{7\,j}(D) = 0, \quad A^{(8)}_{7\,j}(D) = k_1 \frac{\partial}{\partial x_j}, \quad A^{(9)}(D) = A_{7\,7} = k\Delta,$$

δ_{lj} being the Kronecker delta.

In what follows, we denote by $\Gamma(x)$ the fundamental solution of the equilibrium system of thermoelasticity with microtemperatures. The explicit construction of $\Gamma(x)$ can be found in [28, formulae (52)-(53)]. Here, we just recall that each column of $\Gamma(x)$ satisfies system (2.3) at every point $x \in \mathbb{R}^3$ except the origin, in which $\Gamma(x)$ is singular. For more details about the behavior of $\Gamma(x)$ in a neighborhood of the origin, see [23, Lemma 3.1].

2.2. **Thermoelastic potentials.** In order to define the thermoelastic potentials and collect some useful properties for them, we introduce the stress operator $P(D,n)$, that is the 7×7 matrix

$$P(D,n) = \begin{pmatrix} P^{(1)}(D,n) & 0 & -\beta n \\ 0 & P^{(2)}(D,n) & 0 \\ 0 & k_1 n & k\frac{\partial}{\partial n} \end{pmatrix} \tag{2.4}$$

of differential operators

$$P^{(1)}_{lj}(D,n) = \mu\delta_{lj}\frac{\partial}{\partial n} + \mu n_j \frac{\partial}{\partial x_l} + \lambda n_l \frac{\partial}{\partial x_j},$$
$$P^{(2)}_{lj}(D,n) = k_6\delta_{lj}\frac{\partial}{\partial n} + k_5 n_j \frac{\partial}{\partial x_l} + k_4 n_l \frac{\partial}{\partial x_j}$$

$(l, j = 1,2,3)$. Moreover, we also consider the matrix $\widetilde{P}(D,n)$, defined as

$$\widetilde{P}(D,n) = \begin{pmatrix} P^{(1)}(D,n) & 0 & 0 \\ 0 & P^{(2)}(D,n) & 0 \\ 0 & k_3 n & k\frac{\partial}{\partial n} \end{pmatrix}. \tag{2.5}$$

Then, we can define the thermoelastic potentials as follows.

The single-layer thermoelastic potential $U[\psi]$, with density ψ, is given by

$$U[\psi](x) = \int_\Sigma \Gamma(x-y)\psi(y)d\sigma_y, \quad x \in \Omega. \tag{2.6}$$

The double-layer thermoelastic potential $W[\phi]$, with density ϕ, is defined as

$$W[\phi](x) = \int_\Sigma [\widetilde{P}(D_y,n)\Gamma^T(x-y)]^T\phi(y)d\sigma_y, \quad x \in \Omega. \tag{2.7}$$

Now, we recall some properties of thermoelastic potentials, that we use in what follows. For a proof of the first result, see [27, Theorem 20].

Theorem 2.1. *Let $U[\psi]$ be a single-layer thermoelastic potential with density $\psi \in [C^{0,\alpha'}(\Sigma)]^7$, $0 < \alpha' < \alpha \le 1$. Then, the following properties hold:*

(i) *$A(D)U[\psi] = 0$ in Ω;*
(ii) *$U[\psi] \in [C^{1,\alpha'}(\overline{\Omega})]^7 \cap [C^\infty(\Omega)]^7$;*
(iii) *the integral*
$$P(D,n)U[\psi](z) = \int_\Sigma [P(D_z,n)\Gamma(z-y)]\psi(y)d\sigma_y$$
is singular for $z \in \Sigma$;
(iv) *$\{P(D,n)U[\psi]\}^\pm(z) = \mp\frac{1}{2}\psi(z) + P(D,n)U[\psi](z)$, $z \in \Sigma$.*

Next theorem was proved in [27, Theorem 21].

Theorem 2.2. *Let $W[\phi]$ be a double-layer thermoelastic potential with density $\phi \in [C^{0,\alpha'}(\Sigma)]^7$, $0 < \alpha' < \alpha \le 1$. Then,*

(i) *$A(D)W[\phi] = 0$ in Ω;*
(ii) *$W[\phi] \in [C^{0,\alpha'}(\overline{\Omega})]^7 \cap [C^\infty(\Omega)]^7$;*

(iii) *the integral*

$$W[\phi](z) = \int_\Sigma [\widetilde{P}(D_y, n)\Gamma^T(z-y)]^T \phi(y) d\sigma_y$$

is singular for $z \in \Sigma$*;*

(iv) $\{W[\phi]\}^{\pm}(z) = \pm\frac{1}{2}\phi(z) + W[\phi](z)$, $z \in \Sigma$*;*

(v) $\{P(D,n)W[\phi]\}^{+}(z) = \{P(D,n)W[\phi]\}^{-}(z)$, $z \in \Sigma$.

2.3. **On the Dirichlet problem.** Here, we summarize the main results contained in [23], where the Dirichlet problem for the thermoelastic system with microtemperatures (2.1) has been considered.

First of all, we define the class $\mathcal{S}^p$ of all functions which can be represented by a single-layer thermoelastic potential (2.6) with density in $[L^p(\Sigma)]^7$. In this class, we are interested to solve the Dirichlet problem

$$\begin{cases} U \in \mathcal{S}^p, \\ A(D)U = 0 & \text{in } \Omega, \\ U^+ = f & \text{on } \Sigma, \end{cases} \tag{2.8}$$

where $f \in [W^{1,p}(\Sigma)]^7$.

Following the approach in [3], we apply the exterior differential d to both sides of the integral system of equations of the first kind obtained imposing the boundary condition, and we achieve the singular integral system

$$S\phi(z) = \int_\Sigma d_z[\Gamma(z-y)]\phi(y) d\sigma_y = df(z), \qquad z \in \Sigma, \tag{2.9}$$

where the unknown is the vector $(\phi_1, \ldots, \phi_7)$ of scalar functions and the datum is the vector $(df_1, \ldots, df_7)$ of 1-forms.

A crucial step is to prove that the operator $S : [L^p(\Sigma)]^7 \longrightarrow [L^p_1(\Sigma)]^7$ can be reduced on the left, that is that there exists a continuous linear operator $S' : [L^p_1(\Sigma)]^7 \to [L^p(\Sigma)]^7$ such that $S'S = I + T$, where I stands for the identity operator on $[L^p(\Sigma)]^7$ and $T : [L^p(\Sigma)]^7 \to [L^p(\Sigma)]^7$ is a compact operator. The construction of S' is made in [23, Proposition 4.4].

From one of the main properties of reducible operators, it follows that for system (2.9), briefly $S\phi = df$, the Fredholm alternative theorem holds (see, e.g. [13, 25]). Then, $S\phi = df$ is always solvable since df satisfies the corresponding compatibility conditions, which are $\langle \gamma_i, df_i \rangle = 0$ for every $\gamma \in [L^q_1(\Sigma)]^7$, $p + q = pq$, such that each component γ_i of γ is a weakly closed 1-form. This leads to the main result [23, Theorem 5.6], that we mention below for the reader convenience.

Theorem 2.3. *The Dirichlet problem* (2.8) *admits a unique solution* $U \in \mathcal{S}^p$*. In particular, the density* ϕ *of* U *can be written as* $\phi = \phi_0 + \psi_0$*, where* ψ_0 *is the density of a single-layer thermoelastic potential which is constant on* Σ *and* ϕ_0 *solves the singular integral system*

$$\int_\Sigma d_x\left[\Gamma_{ij}(y-x)\right]\phi_{0j}(y) d\sigma_y = df_i(x), \quad i = 1, \ldots, 7, \quad a.e.\ x \in \Sigma.$$

Note that every constant function belongs to $\mathcal{S}^p$ (see [23, Lemma 5.5]). As quoted in the introduction, this formulation is different from that one obtained in [27], where a double-layer thermoelastic approach has been used, and shows the effectiveness of the single-layer method also in the context of thermoelasticity with microtemperatures.

3. Boundary integral operators

Let $K : [L^p(\Sigma)]^7 \to [L^p(\Sigma)]^7$ and $K^* : [L^q(\Sigma)]^7 \to [L^q(\Sigma)]^7$ $(p+q=pq)$ be the boundary integral operators defined by

$$K\phi(z) = \int_\Sigma [\widetilde{P}(D_y, n)\Gamma^T(z-y)]^T\phi(y)d\sigma_y, \quad z \in \Sigma \tag{3.10}$$

and

$$K^*\psi(z) = \int_\Sigma [P(D_z, n)\Gamma(z-y)]\psi(y)d\sigma_y, \quad z \in \Sigma, \tag{3.11}$$

where $P(D,n)$ and $\widetilde{P}(D,n)$ are the boundary differential operators defined by (2.4) and (2.5), respectively.

The operators K and K^* are adjoint with respect to the duality

$$< \psi, K\phi >=< K^*\psi, \phi >,$$

where $< f, g >$ is the bilinear form

$$\int_\Sigma fg d\sigma = \int_\Sigma \sum_{j=1}^{7} f_j g_j d\sigma.$$

Moreover, the operators K and K^* are singular, as stated in Theorems 2.1 and 2.2. Further, K and K^* are Fredholm operators and hence the index is equal to zero (see [27, Theorem 24]). For the definition of Fredholm operator see, e.g. [12, Chapter 5].

In what follows, we are interested in the kernels of the operators $\pm\frac{1}{2}I + K$ and $\pm\frac{1}{2}I + K^*$. It is known that (see [27, p. 746])

$$\ker\Big(\tfrac{1}{2}I + K\Big) = \{0\} \quad \text{and} \quad \ker\Big(\tfrac{1}{2}I + K^*\Big) = \{0\}. \tag{3.12}$$

On the contrary, the kernels of the operators $-\frac{1}{2}I + K$ and $-\frac{1}{2}I + K^*$ are not trivial. To describe them, let us denote by $\mathcal{H}$ the space of solutions of the homogeneous second problem

$$\begin{cases} h \in [C^1(\overline{\Omega})]^7 \cap [C^2(\Omega)]^7 & \\ A(D)h = 0 & \text{in } \Omega, \\ \{P(D,n)h\}^+ = 0 & \text{on } \Sigma. \end{cases} \tag{3.13}$$

As shown in [23, Lemma 5.2], the space $\mathcal{H}$ is not trivial and contains vector functions $h = (u, w, \theta)^T$ of the form

$$u(x) = a + b \wedge x + c_0 c_1 x, \qquad w(x) = 0, \qquad \theta(x) = c_1, \qquad x \in \Omega, \tag{3.14}$$

with $a, b \in \mathbb{R}^3$ arbitrary constant vectors, $c_1 \in \mathbb{R}$ an arbitrary constant, and $c_0 = \beta/(3\lambda + 2\mu)$.

Then, we have the following result.

Proposition 3.1. *Let K and K^* be as in (3.10) and (3.11), respectively. The homogeneous boundary integral systems*

$$\Big(-\tfrac{1}{2}I + K\Big)\phi = 0 \quad \textit{on } \Sigma, \tag{3.15}$$
$$\Big(-\tfrac{1}{2}I + K^*\Big)\psi = 0 \quad \textit{on } \Sigma$$

admit seven linearly independent solutions, forming a complete system.

In particular, the family $\{\phi^{(1)}, \ldots, \phi^{(7)}\}$ *defined on* Σ *by*

$$
\begin{array}{ll}
\phi^{(1)}(z) = (1,0,0,0,0,0,0), & \phi^{(2)}(z) = (0,1,0,0,0,0,0),\\
\phi^{(3)}(z) = (0,0,1,0,0,0,0), & \phi^{(4)}(z) = (0,-z_3,z_2,0,0,0,0),\\
\phi^{(5)}(z) = (z_3,0,-z_1,0,0,0,0), & \phi^{(6)}(z) = (-z_2,z_1,0,0,0,0,0),\\
\phi^{(7)}(z) = (c_0 z_1, c_0 z_2, c_0 z_3, 0,0,0,1), &
\end{array}
\tag{3.16}
$$

with $c_0 = \beta/(3\lambda+2\mu)$*, is a basis for* $\ker(-\frac{1}{2}I + K)$*.*

Proof. As quoted before, vector functions of type (3.14) are solution of the problem (3.13). From this, it follows that the functions on Ω

$$
\begin{array}{ll}
\phi^{(1)}(x) = (1,0,0,0,0,0,0), & \phi^{(2)}(x) = (0,1,0,0,0,0,0),\\
\phi^{(3)}(x) = (0,0,1,0,0,0,0), & \phi^{(4)}(x) = (0,-x_3,x_2,0,0,0,0),\\
\phi^{(5)}(x) = (x_3,0,-x_1,0,0,0,0), & \phi^{(6)}(x) = (-x_2,x_1,0,0,0,0,0),\\
\phi^{(7)}(x) = (c_0 x_1, c_0 x_2, c_0 x_3, 0,0,0,1), &
\end{array}
$$

with $c_0 = \beta/(3\lambda+2\mu)$, are linearly independent solutions of (3.13). In particular, we have that $\{P(D,n)\phi^j\}^+(z) = 0$ for $z \in \Sigma$. Thanks to the Somigliana formula in the equilibrium theory of thermoelasticity with microtemperatures (see [27, Theorem 17]), we get $\phi^j(x) = W[\phi^j](x)$ for $x \in \Omega$. Then, for $x \to z \in \Sigma$, using the jump relations for the double-layer thermoelastic potential (see Theorem 2.2), we get $\{\phi^j\}^+(z) = \frac{1}{2}\phi^j(z) + W[\phi^j](z)$, and hence the functions (3.16) form a set of linearly independent solutions of the boundary integral equation (3.15). The rest of the proof proceeds as in [23, Lemma 5.2]. □

We conclude this section defining the following space

$$H = \{h_{|\Sigma} : h \in \mathcal{H}\} \tag{3.17}$$

that we use in the next result.

Proposition 3.2. *We have that*

$$\ker(-\tfrac{1}{2}I + K) = H.$$

Proof. Let $\phi \in \ker(-\frac{1}{2}I + K)$ and consider the double-layer thermoelastic potential $W[\phi]$ defined by (2.7). Then $\{W[\phi]\}^- = 0$ on Σ and, by virtue of the uniqueness of the exterior first problem (see [27, Theorem 8]), we have that $W[\phi] = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. On the other hand $\{P(D,n)W[\phi]\}^-(z) = 0$, and hence $\{P(D,n)W[\phi]\}^+(z) = 0$ on Σ too, because of Theorem 2.2, (v). In other words, $W[\phi]$ is a solution of the problem (3.13). Finally, taking Theorem 2.2, (iv), into account, $\phi = \{W[\phi]\}^+(z) - \{W[\phi]\}^-(z) = \{W[\phi]\}^+(z)$, and hence $\phi \in H$.

Conversely, let $\phi \in H$, that is $\phi = h_{|\Sigma}$ for some $h \in \mathcal{H}$. Then, by applying [27, Theorem 16], we have that

$$\int_\Sigma [\widetilde{P}(D_y,n)\Gamma^T(x-y)]^T \phi(y) d\sigma_y = 0, \quad x \in \mathbb{R}^3 \setminus \overline{\Omega}.$$

Letting now $x \to z \in \Sigma$, we get $-\frac{1}{2}\phi(z) + K(\phi)(z) = 0$ for $z \in \Sigma$, that is $\phi \in \ker(-\frac{1}{2}I + K)$. □

4. Second problem

Let us define the class $\mathcal{D}^p$ of all functions which can be expressed by means of a double-layer thermoelastic potentials (2.7) with density in $[W^{1,p}(\Sigma)]^7$, and consider the second problem in this class, i.e.

$$
\begin{cases}
W \in \mathcal{D}^p, & \\
A(D)W = 0 & \text{in } \Omega,\\
\{P(D,n)W\}^+ = g & \text{on } \Sigma,
\end{cases}
\tag{4.18}
$$

where $P(D,n)$ is the stress operator defined by (2.4). The datum g is assumed to be in $[L^p(\Sigma)]^7$ and to satisfy the following conditions

$$\int_\Sigma g h d\sigma = 0, \qquad \text{for every } h \in H, \tag{4.19}$$

where H is given by (3.17). Note that, because of Proposition 3.2, conditions (4.19) can be rewritten as

$$\int_\Sigma g \phi^{(j)} d\sigma = 0, \qquad \text{for every } j = 1, \dots, 7,$$

where $\phi^{(j)}$ are given by (3.16). First, we prove the following lemma.

Lemma 4.1. *Consider $U \in \mathcal{S}^p$ with density ψ and $W \in \mathcal{D}^p$ with density U, i.e.*

$$W[U[\psi]](x) = \int_\Sigma [\widetilde{P}(D_y,n)\Gamma^T(x-y)]^T U[\psi](y) d\sigma_y, \quad x \in \Omega. \tag{4.20}$$

Then, we have

$$\{P(D,n)W[U[\psi]]\}^+ = -\frac{1}{4}\psi + K^{*2}\psi \qquad \text{a.e. on } \Sigma, \tag{4.21}$$

where K^* is the boundary integral operator defined by (3.11).

Proof. We begin by noting that

$$W[U[\psi]](x) = U[\psi](x) + U[P(D,n)U[\psi]](x), \qquad x \in \Omega. \tag{4.22}$$

Indeed, let ψ_n be a sequence of polynomials such that $\psi_n \to \psi$ in $[L^p(\Sigma)]^7$ and consider

$$U[\psi_n](x) = \int_\Sigma \Gamma(x-y)\psi_n(y) d\sigma_y \qquad x \in \Omega.$$

Thanks to the Somigliana formula in the equilibrium theory of thermoelasticity with microtemperatures (see [27, Theorem 17]), for $x \in \Omega$, we have

$$U[\psi_n](x) = \int_\Sigma [\widetilde{P}(D_y,n)\Gamma^T(x-y)]^T U[\psi_n](y) d\sigma_y - \int_\Sigma \Gamma(x-y) P(D_y,n) U[\psi_n](y) d\sigma_y.$$

Therefore, letting $n \to +\infty$, we gain formula (4.22).

On the other hand (see Theorem 2.1),

$$\{P(D,n)U[\psi]\}^+(z) = -\frac{1}{2}\psi(z) + P(D,n)U[\psi](z) \qquad \text{a.e. } z \in \Sigma. \tag{4.23}$$

Then, on account of (4.22) and (4.23), for a.e. $z \in \Sigma$ we obtain

$$\begin{aligned}
&\{P(D,n)W[U[\psi]]\}^{+}(z)\\
=&\{P(D,n)U[\psi]\}^{+}(z) + \{U[P(D,n)U[\psi]]\}^{+}(z)\\
=&\left(-\frac{1}{2}\psi(z) + P(D,n)U[\psi](z)\right) + \left(-\frac{1}{2}\{P(D,n)U[\psi]\}^{+}(z) + P(D,n)U[P(D,n)U[\psi]](z)\right)\\
=&-\frac{1}{2}\psi(z) + P(D,n)U[\psi](z) - \frac{1}{2}\left(-\frac{1}{2}\psi(z) + P(D,n)U[\psi](z)\right)\\
&+P(D_z,n)\int_{\Sigma}\Gamma(z-y)\{P(D_y,n)U[\psi]\}^{+}(y)d\sigma_y\\
=&-\frac{1}{4}\psi(z) + \frac{1}{2}P(D,n)U[\psi](z) + P(D,n)\int_{\Sigma}\Gamma(z-y)\left(-\frac{1}{2}\psi(y) + P(D,n)U[\psi](y)\right)d\sigma_y\\
=&-\frac{1}{4}\psi(z) + \int_{\Sigma}P(D_z,n)\Gamma(z-y)\int_{\Sigma}P(D_y,n)\Gamma(y-x)\psi(x)d\sigma_x d\sigma_y\\
=&-\frac{1}{4}\psi(z) + K^{*2}\psi(z),
\end{aligned}$$

that is the claim. □

Theorem 4.4. *The second BVP (4.18)-(4.19) is always solvable.*

In particular, a solution can be represented by means of a double-layer thermoelastic potential $W[U[\psi]]$ as in (4.20), where $\psi \in [L^p(\Sigma)]^7$ solves the singular integral system

$$-\tfrac{1}{4}\psi + K^{*2}\psi = g, \tag{4.24}$$

K^* being given by (3.11) and $g \in [L^p(\Sigma)]^7$ satisfying conditions (4.19).

Proof. Let $W[\phi]$ be a double-layer thermoelastic potential with density $\phi \in [W^{1,p}(\Sigma)]^7$. Consider the Dirichlet problem (2.8) with the datum ϕ. Thanks to Theorem 2.3, ϕ can be represented as a single-layer thermoelastic potential, which means that there exists $\psi \in [L^p(\Sigma)]^7$, such that $\phi = U[\psi]$. Hence, $W[\phi]$ is equal to the double-layer thermoelastic potential $W[U[\psi]]$ as is (4.20). By imposing the boundary condition to $W[U[\psi]]$, and using formula (4.21), we have that the density ψ must satisfy the singular integral system (4.24). Then, the claim is equivalent to demonstrating that (4.24) is always solvable. In fact, we shall show that the system (4.24) is solvable if, and only if, g satisfies conditions (4.19).

First of all, we observe that the operator on the left-hand side of (4.24) can be factored as follows:

$$-\tfrac{1}{4}I + K^{*2} = \left(-\tfrac{1}{2}I + K^*\right)\left(\tfrac{1}{2}I + K^*\right).$$

Let us assume that g satisfies conditions (4.19). In other words, g is orthogonal to the elements of the space H which is equal to $\ker\left(-\frac{1}{2}I + K\right)$ (see Proposition 3.2). Therefore, the system

$$\left(-\tfrac{1}{2}I + K^*\right)\gamma = g$$

admits a solution, that we refer to us γ_0. Now, consider the system

$$\left(\tfrac{1}{2}I + K^*\right)\psi = \gamma_0.$$

Since $\ker(\frac{1}{2}I + K) = \{0\}$ (see (3.12)), this system is always solvable. Accordingly, we have proven that the system (4.24) is always solvable.

Conversely, let ψ be a solution of (4.24). Hence, ψ solves

$$\left(-\tfrac{1}{2}I + K^*\right)\left(\tfrac{1}{2}I + K^*\right)\psi = g.$$

In particular, g belongs to the range of the operator $-\frac{1}{2}I + K^*$, and hence g is orthogonal to $\ker(-\frac{1}{2}I + K)$. From Proposition 3.2, it follows that g fulfills conditions (4.19). □

Acknowledgement. The authors are members of the "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni" (GNAMPA) of the "Istituto Nazionale di Alta Matematica" (INdAM). The authors acknowledge the support of the Project funded by the European Union - Next Generation EU under the National Recovery and Resilience Plan (NRRP), Mission 4 Component 2 Investment 1.1 - Call for tender PRIN 2022 No. 104 of February, 2 2022 of Italian Ministry of University and Research; Project code: 2022SENJZ3 (subject area: PE - Physical Sciences and Engineering) "Perturbation problems and asymptotics for elliptic differential equations: variational and potential theoretic methods".

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Vita Leonessa
University of Basilicata
Department of Engineering
Via dell'Ateneo Lucano nr. 10, 85100, Potenza, Italy
ORCID: 0000-0001-9547-8397
Email address: vita.leonessa@unibas.it

Angelica Malaspina
University of Basilicata
Department of Basic and Applied Sciences
Via dell'Ateneo Lucano nr. 10, 85100, Potenza, Italy
ORCID: 0000-0002-1971-0235
Email address: angelica.malaspina@unibas.it