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Prime Ideals and Homoderivations on Rings

Fatih Türkmen [1,a], Zeliha Bedir [2,b,*]

[1] Institute of Science, Sivas Cumhuriyet University, Sivas, Türkiye.
[2] Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University, Sivas, Türkiye.
**Corresponding author e-mail address: zelihabedir@cumhuriyet.edu.tr*

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ABSTRACT

In this paper, we aim to establish a new approach that involves characterizing the commutativity of a quotient ring $\mathcal{L}/\mathfrak{P}$ with homoderivations of $\mathcal{L}$ satisfying some algebraic identities involving the prime ideal $\mathfrak{P}$. In addition, some well-known results regarding the commutativity of prime rings have been developed for homoderivations of the rings. Some of the results obtained in this context are as follows: Let $\mathcal{L}$ be a ring, $\mathfrak{P}$ a prime ideal of $\mathcal{L}$ and ξ a nonzero homoderivation of $\mathcal{L}$. If any one of the following holds then $\xi(\mathcal{L}) \subseteq \mathfrak{P}$ or $\mathcal{L}/\mathfrak{P}$ is commutative integral domain: i) $\xi([\mu_1,\mu_2]) \in \mathfrak{P}$, ii) $\xi(\mu_1 o \mu_2) \in \mathfrak{P}$, iii) $\xi([\mu_1,\mu_2]) - [\mu_1,\mu_2] \in \mathfrak{P}$, iv) $\xi(\mu_1 o \mu_2) - \mu_1 o \mu_2 \in \mathfrak{P}$ v) $\xi(\mu_1\mu_2) - \xi(\mu_1)\xi(\mu_2) \in \mathfrak{P}$, vi) $\xi(\mu_1\mu_2) - \xi(\mu_2)\,\xi(\mu_1) \in \mathfrak{P}$, vii) $\xi(\mu_1)\,\xi(\mu_2) - [\mu_1,\mu_2] \in \mathfrak{P}$, viii) $\xi(\mu_1)\,\xi(\mu_2) - \mu_1 o \mu_2 \in \mathfrak{P}$, for all $\mu_1,\mu_2 \in \mathcal{L}$.

Keywords: *Commutativity, Homoderivation, Prime ideal.*

[a] 0009-0007-3142-8797 [b] 0000-0002-4346-2331

1. Introduction

Throughout, let $\mathcal{L}$ denote an associative ring and write $Z(\mathcal{L})$ for its center. A proper ideal $\mathfrak{P}$ of $\mathcal{L}$ is called prime if for any $\mu_1,\mu_2 \in \mathcal{L}$, the condition $\mu_1\mathcal{L}\mu_2 \subseteq \mathfrak{P}$ implies $\mu_1 \in \mathfrak{P}$ or $\mu_2 \in \mathfrak{P}$. The ring $\mathcal{L}$ itself is prime if and only if the zero ideal (0) is a prime ideal. For any $\mu_1,\mu_2 \in \mathcal{L}$, the symbol $[\mu_1,\mu_2]$ signifies the Lie commutator $\mu_1\mu_2 - \mu_2\mu_1$ and also the symbol $\mu_1 o \mu_2$ stands for the Jordan product $\mu_1\mu_2 + \mu_2\mu_1$.

An additive mapping $\delta: \mathcal{L} \to \mathcal{L}$ is called a derivation if $\delta(\mu_1\mu_2) = \delta(\mu_1)\mu_2 + \mu_1\delta(\mu_2)$ holds for all $\mu_1,\mu_2 \in \mathcal{L}$. The study of derivations in a prime ring was initiated by E. C. Posner in [1]. Over the last several years, a number of authors studied commutativity theorems for prime rings admitting automorphisms or derivations on appropriate subsets of $\mathcal{L}$.

In 2000, El Sofy [2] defined a homoderivation on $\mathcal{L}$ as an additive mapping $\xi: \mathcal{L} \to \mathcal{L}$ satisfying $\xi(\mu_1\mu_2) = \xi(\mu_1)\xi(\mu_2) + \xi(\mu_1)\mu_2 + \mu_1\xi(\mu_2)$ for all $\mu_1,\mu_2 \in \mathcal{L}$. An example of such mapping is to let $\xi(\mu_1) = f(\mu_1) - \mu_1$, for all $\mu_1,\mu_2 \in \mathcal{L}$ where f is an endomorphism of $\mathcal{L}$. If $S \subseteq \mathcal{L}$, then a mapping $\xi: \mathcal{L} \to \mathcal{L}$ preserves S if $f(S) \subseteq S$. A mapping $\xi: \mathcal{L} \to \mathcal{L}$ is zero-power valued on S if ξ preserves S and if, for each $a \in S$, there exists a positive integer $n(a) > 1$ such that $\xi^{n(a)} = 0$.

In [3], Daif and Bell proved that $\mathcal{L}$ is semiprime ring, I is a nonzero ideal of $\mathcal{L}$ and δ is a derivation of $\mathcal{L}$ such that $\delta([\mu_1,\mu_2]) = \pm[\mu_1,\mu_2]$ for all $\mu_1,\mu_2 \in I$, then $\mathcal{L}$ contains a nonzero central ideal. In addition, Hongan [4] extended this theorem in the following manner: Let $\mathcal{L}$ be a 2-torsion free semiprime ring and I a nonzero ideal of $\mathcal{L}$ and δ a derivation of $\mathcal{L}$. If $\delta([\mu_1,\mu_2]) \pm [\mu_1,\mu_2] \in Z$, for all $\mu_1,\mu_2 \in I$, then $I \subseteq Z$.

Let $\mathcal{L}$ be a ring and $\delta: \mathcal{L} \to \mathcal{L}$ a derivation. We say that δ acts as an endomorphism on $\mathcal{L}$ (resp. as an anti-endomorphism) if $\delta(\mu_1\mu_2) = \delta(\mu_1)\delta(\mu_2)$ (resp. $\delta(\mu_1\mu_2) = \delta(\mu_2)\delta(\mu_1)$) for all $\mu_1,\mu_2 \in \mathcal{L}$. Bell and Kappe showed that if a derivation δ of a semiprime ring $\mathcal{L}$ acts as a homomorphism or an anti-homomorphism on some nonzero right ideal of $\mathcal{L}$, then $\delta = 0$ [5]. Ali et al. [6] extended this to Lie ideals: if a derivation δ acts as an endomorphism or an anti-endomorphism on a nonzero Lie ideal U of a prime ring $\mathcal{L}$, then either $\delta = 0$ or $U \subseteq Z(L)$.

Building on the above, we study differential identities associated with a prime ideal without assuming that the ring is prime. This perspective allows us to generalize the earlier results.

Throughout this paper, we perform numerous computations involving the Lie and Jordan products, frequently relying on the following identities:

$$[\mu_1,\mu_2\mu_3] = \mu_2[\mu_1,\mu_3] + [\mu_1,\mu_2]\mu_3$$
$$[\mu_1\mu_2,\mu_3] = [\mu_1,\mu_3]\mu_2 + \mu_1[\mu_2,\mu_3]$$

$$\mu_1 o(\mu_2\mu_3) = (\mu_1 o\mu_2)\mu_3 - \mu_2[\mu_1,\mu_3] = \mu_2(\mu_1 o\mu_3) + [\mu_1,\mu_2]\mu_3$$
$$(\mu_1\mu_2)o\mu_3 = \mu_1(\mu_2 o\mu_3) - [\mu_1,\mu_3]\mu_2 = (\mu_1 o\mu_3)\mu_2 + \mu_1[\mu_2,\mu_3].$$

2. Results

Remark. Let $\mathcal{L}$ be a ring and ξ a nonzero homoderivation of $\mathcal{L}$. For all $\mu_1,\mu_2 \in \mathcal{L}$, the following identity holds:

$$\xi([\mu_1,\mu_2]) = [\xi(\mu_1),\xi(\mu_2)] + [\xi(\mu_1),\mu_2] + [\mu_1,\xi(\mu_2)].$$

Proof. For all $\mu_1,\mu_2 \in \mathcal{L}$, we get

$$\xi([\mu_1,\mu_2]) = \xi(\mu_1\mu_2 - \mu_2\mu_1) = \xi(\mu_1\mu_2) - \xi(\mu_2\mu_1)$$
$$=\xi(\mu_1)\xi(\mu_2) + \xi(\mu_1)\mu_2 + \mu_1\xi(\mu_2) - \xi(\mu_2)\xi(\mu_1) - \xi(\mu_2)\mu_1 - \mu_2\xi(\mu_1)$$
$$= [\xi(\mu_1),\xi(\mu_2)] + [\xi(\mu_1),\mu_2] + [\mu_1,\xi(\mu_2)].$$

Lemma 1. [7, Lemma 1.2] Let $\mathcal{L}$ be a ring, $\mathfrak{P}$ a prime ideal of $\mathcal{L}$. If $[\mu_1,\mu_2] \in \mathfrak{P}$ for every $\mu_1,\mu_2 \in \mathcal{L}$ or $\mu_1 o\mu_2 \in \mathfrak{P}$ for every $\mu_1,\mu_2 \in \mathcal{L}$, then $\mathcal{L}/\mathfrak{P}$ is commutative.

Theorem 1. Let $\mathcal{L}$ be a ring, $\mathfrak{P}$ a prime ideal of $\mathcal{L}$ and ξ a nonzero homoderivation of $\mathcal{L}$. If $\xi([\mu_1,\mu_2]) \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$ then $\xi(\mathcal{L}) \subseteq \mathfrak{P}$ or $\mathcal{L}/\mathfrak{P}$ is commutative.

Proof. By the hypothesis, we have

$$\xi([\mu_1,\mu_2]) \in \mathfrak{P} \quad (1)$$

Writing $\mu_2\mu_1$ instead of μ_2 in (1), we have

$\xi([\mu_1,\mu_2])\xi(\mu_1) + \xi([\mu_1,\mu_2])\mu_1 + [\mu_1,\mu_2]\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$.

Using the hypothesis, we get

$[\mu_1,\mu_2]\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$. (2)

Substituting $\mu_2 r$ for μ_2, $r \in \mathcal{L}$ in this relation and using this, we arrive at

$[\mu_1,\mu_2]r\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1,\mu_2,r \in \mathcal{L}$ (3)

and so

$[\mu_1,\mu_2]\mathcal{L}\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$.

Since $\mathfrak{P}$ is a prime ideal, the last relation implies that

$[\mu_1,\mu_2] \in \mathfrak{P}$ or $\xi(\mu_1) \in \mathfrak{P}$ for any $\mu_2 \in \mathcal{L}$.

Let us set $K_1 = \{\mu_1 \in \mathcal{L} | \xi(\mu_1) \in \mathfrak{P}\}$ and $K_2 = \{\mu_1 \in \mathcal{L} | [\mu_1,\mu_2] \in \mathfrak{P}$ for all $\mu_2 \in \mathcal{L}\}$. Clearly each of K_1 and K_2 is additive subgroup of $\mathcal{L}$ such that $\mathcal{L} = K_1 \cup K_2$. But, a group can not be the set-theoretic union of its two proper subgroups. This follows that either $K_1 = \mathcal{L}$ or $K_2 = \mathcal{L}$. In the former case, $\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1 \in \mathcal{L}$. That is $\xi(\mathcal{L}) \subseteq \mathfrak{P}$. In the latter case, we have $[\mu_1,\mu_2] \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$. By Lemma 1, we get $\mathcal{L}/\mathfrak{P}$ commutative. This completes the proof.

Theorem 2. Let $\mathcal{L}$ be a ring, $\mathfrak{P}$ a prime ideal of $\mathcal{L}$ and ξ a nonzero homoderivation of $\mathcal{L}$. If $\xi(\mu_1 o\mu_2) \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$ then $\xi(\mathcal{L}) \subseteq \mathfrak{P}$ or $\mathcal{L}/\mathfrak{P}$ is commutative.

Proof. By the hypothesis, we have

$\xi(\mu_1 o\mu_2) \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$. (4)

Replacing μ_2 by $\mu_2\mu_1$ in (4) and using this, we arrive at

$$\xi(\mu_1 o\mu_2)\xi(\mu_1) + \xi(\mu_1 o\mu_2)\mu_1 + (\mu_1 o\mu_2)\xi(\mu_1) \in \mathfrak{P}$$

Using the hypothesis, we have

$(\mu_1 o\mu_2)\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$. (5)

Writing $\mu_2 r$ for μ_2, $r \in \mathcal{L}$ in this relation and using it, we obtain

$[\mu_1,\mu_2]r\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$.

This expression is same as (3) in the proof of Theorem 1. Employing the same arguments used in the proof of Theorem 1 yields the desired conclusion.

Theorem 3. Let $\mathcal{L}$ be a ring, $\mathfrak{P}$ a prime ideal of $\mathcal{L}$, $\mathcal{L}/\mathfrak{P}$ 2-torsion free ring and ξ a nonzero homoderivation of $\mathcal{L}$. If $\xi([\mu_1,\mu_2]) - [\mu_1,\mu_2] \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$ then $\xi(\mathcal{L}) \subseteq \mathfrak{P}$ or $\mathcal{L}/\mathfrak{P}$ is commutative.

Proof. By the hypothesis, we have

$$\xi([\mu_1,\mu_2]) - [\mu_1,\mu_2] \in \mathfrak{P} \tag{6}$$

Replacing $\mu_2\mu_1$ instead of μ_2 in (6), we have

$$\xi([\mu_1,\mu_2])\xi(\mu_1) + \xi([\mu_1,\mu_2])\mu_1 + [\mu_1,\mu_2]\xi(\mu_1) - [\mu_1,\mu_2]\mu_1 \in \mathfrak{P} \text{ for all } \mu_1,\mu_2 \in \mathcal{L}.$$

Using the hypothesis, we get

$$\xi([\mu_1,\mu_2])\xi(\mu_1) + [\mu_1,\mu_2]\xi(\mu_1) \in \mathfrak{P} \text{ for all } \mu_1,\mu_2 \in \mathcal{L}.$$

By adding and subtracting $[\mu_1,\mu_2]\xi(\mu_1)$ from the last expression, we obtain that

$$\xi([\mu_1,\mu_2])\xi(\mu_1) + [\mu_1,\mu_2]\xi(\mu_1) + [\mu_1,\mu_2]\xi(\mu_1) - [\mu_1,\mu_2]\xi(\mu_1) \in \mathfrak{P} \text{ for all } \mu_1,\mu_2 \in \mathcal{L}.$$

Using the hypothesis again, we have

$$2[\mu_1,\mu_2]\xi(\mu_1) \in \mathfrak{P} \text{ for all } \mu_1,\mu_2 \in \mathcal{L}.$$

Since $\mathcal{L}/\mathfrak{P}$ 2-torsion free, we get

$$[\mu_1,\mu_2]\xi(\mu_1) \in \mathfrak{P} \text{ for all } \mu_1,\mu_2 \in \mathcal{L}.$$

This expression is same as (2) in the proof of Theorem 1. Employing the same arguments used in the proof of Theorem 1 yields the desired conclusion.

Theorem 4. Let $\mathcal{L}$ be a ring, $\mathfrak{P}$ a prime ideal of $\mathcal{L}$, $\mathcal{L}/\mathfrak{P}$ 2-torsion free ring and ξ a nonzero homoderivation of $\mathcal{L}$. If $\xi(\mu_1 o \mu_2) - \mu_1 o \mu_2 \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$ then $\xi(\mathcal{L}) \subseteq \mathfrak{P}$ or $\mathcal{L}/\mathfrak{P}$ is commutative.

Proof. By the hypothesis, we have

$$\xi(\mu_1 o \mu_2) - \mu_1 o \mu_2 \in \mathfrak{P} \text{ for all } \mu_1,\mu_2 \in \mathcal{L}. \tag{7}$$

Substituting μ_2 by $\mu_2\mu_1$ in (7) and using this, we arrive at

$$\xi(\mu_1 o \mu_2)\xi(\mu_1) + \xi(\mu_1 o \mu_2)\mu_1 + (\mu_1 o \mu_2)\xi(\mu_1) - (\mu_1 o \mu_2)\mu_1 \in \mathfrak{P}$$

Using the hypothesis, we have

$$\xi(\mu_1 o \mu_2)\xi(\mu_1) + (\mu_1 o \mu_2)\xi(\mu_1) \in \mathfrak{P} \text{ for all } \mu_1,\mu_2 \in \mathcal{L}.$$

After the necessary adjustments, using the hypothesis

$$2(\mu_1 o \mu_2)\xi(\mu_1) \in \mathfrak{P} \text{ for all } \mu_1,\mu_2 \in \mathcal{L}.$$

Since $\mathcal{L}/\mathfrak{P}$ 2-torsion free, we get

$(\mu_1 o \mu_2)\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$.

This expression is same as (5) in the proof of Theorem 2. When we continue with a similar approach, the desired result is achieved.

Theorem 5. Let $\mathcal{L}$ be a ring, $\mathfrak{P}$ a prime ideal of $\mathcal{L}$ and ξ a nonzero homoderivation which is zero-power valued of $\mathcal{L}$. If $\xi(\mu_1\mu_2) - \xi(\mu_1)\xi(\mu_2) \in \mathfrak{P}$ for all $\mu_1,\mu_2 \in \mathcal{L}$ then $\xi(\mathcal{L}) \subseteq \mathfrak{P}$ or $\mathcal{L}/\mathfrak{P}$ is commutative.

Proof. For all $\mu_1,\mu_2 \in \mathcal{L}$, we have

$$\xi(\mu_1\mu_2) - \xi(\mu_1)\xi(\mu_2) \in \mathfrak{P} \tag{8}$$

Replacing μ_2 by $\mu_2\mu_1$ in this relation, we get

$$\xi(\mu_1\mu_2\mu_1) - \xi(\mu_1)\xi(\mu_2\mu_1) \in \mathfrak{P}, \text{ for all } \mu_1,\mu_2 \in \mathcal{L}.$$

Since ξ is a homoderivation, we have

$$\xi(\mu_1)\xi(\mu_2\mu_1) + \xi(\mu_1)\mu_2\mu_1 + \mu_1\xi(\mu_2\mu_1) - \xi(\mu_1)\xi(\mu_2\mu_1) \text{ for all } \mu_1,\mu_2 \in \mathcal{L}.$$

and so

$\xi(\mu_1)\mu_2\mu_1 + \mu_1\xi(\mu_2\mu_1) \in \mathfrak{P}$ for all $\mu_1, \mu_2 \in \mathcal{L}$.

Again since ξ is a homoderivation, we get

$\xi(\mu_1)\mu_2\mu_1 + \mu_1\xi(\mu_2)\xi(\mu_1) + \mu_1\xi(\mu_2)\mu_1 + \mu_1\mu_2\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1, \mu_2 \in \mathcal{L}$.

If this expression is rearranged, we obtain that

$$\big(\xi(\mu_1)\mu_2 + \mu_1\xi(\mu_2)\big)\mu_1 + \mu_1\xi(\mu_2)\xi(\mu_1) + \mu_1\mu_2\xi(\mu_1) \in \mathfrak{P} \text{ for all } \mu_1, \mu_2 \in \mathcal{L}. \tag{9}$$

On the other hand, from the definition of homoderivation ξ, we get

$\xi(\mu_1\mu_2) - \xi(\mu_1)\xi(\mu_2) = \xi(\mu_1)\mu_2 + \mu_1\xi(\mu_2)$ for all $\mu_1, \mu_2 \in \mathcal{L}$.

Now by the hypothesis, we obtain that

$$\xi(\mu_1)\mu_2 + \mu_1\xi(\mu_2) \in \mathfrak{P} \text{ for all } \mu_1, \mu_2 \in \mathcal{L}. \tag{10}$$

Using (10) in expression (9), we get

$\mu_1\xi(\mu_2)\xi(\mu_1) + \mu_1\mu_2\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1, \mu_2 \in \mathcal{L}$.

and so

$$\mu_1(\xi(\mu_2) + \mu_2)\xi(\mu_1) \in \mathfrak{P} \text{ for all } \mu_1, \mu_2 \in \mathcal{L}. \tag{11}$$

Since ξ is zero-power valued on $\mathcal{L}$, there exist an integer $n > 1$ such that $\xi^n(\mu_2) = 0$ for all $\mu_2 \in \mathcal{L}$.
Replacing μ_2 by $\mu_2 - \xi(\mu_2) + \xi^2(\mu_2) + \cdots + (-1)^{n-1}\xi^{n-1}(\mu_2)$ in this expression, we find

$$\mu_1\mu_2\xi(\mu_1) \in \mathfrak{P} \text{ for all } \mu_1, \mu_2 \in \mathcal{L}. \tag{12}$$

Left multiplication by μ_3, $\mu_3 \in \mathcal{L}$ in (12), we have

$$\mu_3\mu_1\mu_2\xi(\mu_1) \in \mathfrak{P} \text{ for all } \mu_1, \mu_2, \mu_3 \in \mathcal{L}. \tag{13}$$

Replacing μ_2 by $\mu_3\mu_2$ in (12), we get

$$\mu_1\mu_3\mu_2\xi(\mu_1) \in \mathfrak{P} \text{ for all } \mu_1, \mu_2, \mu_3 \in \mathcal{L}. \tag{14}$$

Combining (13) and (14), we get

$[\mu_1, \mu_3]\mu_2\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1, \mu_2, \mu_3 \in \mathcal{L}$.

which implies that

$[\mu_1, \mu_3]\mathcal{L}\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1, \mu_3 \in \mathcal{L}$.

This expression is same as (3) in the Theorem 1. When we continue with a similar approach, the desired result is achieved.

Theorem 6. Let $\mathcal{L}$ be a ring, $\mathfrak{P}$ a prime ideal of $\mathcal{L}$ and ξ a nonzero homoderivation which is zero-power valued of $\mathcal{L}$. If $\xi(\mu_1\mu_2) - \xi(\mu_2)\xi(\mu_1) \in \mathfrak{P}$ for all $\mu_1, \mu_2 \in \mathcal{L}$ then $\xi(\mathcal{L}) \subseteq \mathfrak{P}$ or $\mathcal{L}/\mathfrak{P}$ is commutative.

Proof. By the hypothesis, we have

$$\xi(\mu_1\mu_2) - \xi(\mu_2)\xi(\mu_1) \in \mathfrak{P} \text{ for all } \mu_1, \mu_2 \in \mathcal{L}. \tag{15}$$

Replacing μ_2 by $\mu_1\mu_2$ in this relation, we get

$\xi(\mu_1\mu_1\mu_2) - \xi(\mu_1\mu_2)\xi(\mu_1) \in \mathfrak{P}$, for all $\mu_1, \mu_2 \in \mathcal{L}$.

Since ξ is a homoderivation, we get

$\xi(\mu_1)\xi(\mu_1\mu_2) + \xi(\mu_1)\mu_1\mu_2 + \mu_1\xi(\mu_1\mu_2) - \xi(\mu_1)\xi(\mu_2)\xi(\mu_1) - \xi(\mu_1)\mu_2\xi(\mu_1) - \mu_1\xi(\mu_2)\xi(\mu_1) \in \mathfrak{P}$

and using the hypothesis, we find that

$$\xi(\mu_1)\mu_1\mu_2 - \xi(\mu_1)\mu_2\xi(\mu_1) \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2 \in \mathcal{L}. \tag{16}$$

Writing μ_2 by $\mu_2 t$, $t \in \mathcal{L}$ in (16), we get

$$\xi(\mu_1)\mu_1\mu_2 t - \xi(\mu_1)\mu_2 t\xi(\mu_1) \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2 \in \mathcal{L}. \tag{17}$$

By adding and subtracting the term $\xi(\mu_1)\mu_2\xi(\mu_1)t$ from the last expression, we obtain

$$\xi(\mu_1)\mu_1\mu_2t - \xi(\mu_1)\mu_2t\xi(\mu_1) - \xi(\mu_1)\mu_2\xi(\mu_1)t + \xi(\mu_1)\mu_2\xi(\mu_1)t$$

$$(\xi(\mu_1)\mu_1\mu_2 - \xi(\mu_1)\mu_2\xi(\mu_1))t - \xi(\mu_1)\mu_2t\xi(\mu_1) + \xi(\mu_1)\mu_2\xi(\mu_1)t$$

Using (16) in last expression, we get

$$\xi(\mu_1)\mu_2\xi(\mu_1)t - \xi(\mu_1)\mu_2t\xi(\mu_1) \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2 \in \mathcal{L}. \quad (18)$$

And so

$$\xi(\mu_1)\mu_2[\xi(\mu_1), t] \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2 \in \mathcal{L}.$$

that is

$$\xi(\mu_1)\mathcal{L}[\xi(\mu_1), t] \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2 \in \mathcal{L}.$$

In light of primeness of $\mathfrak{P}$, the last relation implies that $\xi(\mu_1) \in \mathfrak{P}$ or $[\xi(\mu_1), t] \in \mathfrak{P}$, for all $\mu_1, t \in \mathcal{L}$. Let us set $K_1 = \{\mu_1 \in \mathcal{L} | \xi(\mu_1) \in \mathfrak{P}\}$and $K_2 = \{\mu_1 \in \mathcal{L} | [\xi(\mu_1), t] \in \mathfrak{P} \text{ for all t} \in \mathcal{L}\}$. Clearly each of $K_1$and K_2 is additive subgroup of $\mathcal{L}$ such that $\mathcal{L} = K_1 \cup K_2$. But, a group can not be the set-theoretic union of its two proper subgroups. This follows that either $\mathcal{L} = K_1$ or $\mathcal{L} = K_2$. Suppose that $\mathcal{L} = K_1$, then $\xi(\mu_1) \in \mathfrak{P}$, for all $\mu_1 \in \mathcal{L}$, that is $\xi(\mathcal{L}) \subseteq \mathfrak{P}$. In the second case, we have $[\xi(\mu_1), t] \in \mathfrak{P}$ for all $\mu_1 \in \mathcal{L}$. If we put t=$\xi(\mu_2)$ in last relation, we find that

$$\xi(\mu_1)\xi(\mu_2) - \xi(\mu_2)\xi(\mu_1) \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2 \in \mathcal{L}. \quad (19)$$

Now by the hypothesis

$$\xi(\mu_1)\xi(\mu_2) - \xi(\mu_1\mu_2) + \xi(\mu_1\mu_2) - \xi(\mu_2)\xi(\mu_1) \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2 \in \mathcal{L}.$$

Using the (15), we obtain

$$\xi(\mu_1)\xi(\mu_2) - \xi(\mu_1\mu_2) \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2 \in \mathcal{L}.$$

This result is the same as the hypothesis of Theorem 5. By following the method in the proof of Theorem 5, we arrive at the targeted result.

Theorem 7. Let $\mathcal{L}$ be a ring, $\mathfrak{P}$ a prime ideal of $\mathcal{L}$ and ξ a nonzero homoderivation which is zero-power valued of $\mathcal{L}$. If $\xi(\mu_1)\xi(\mu_2) - [\mu_1, \mu_2] \in \mathfrak{P}$ for all $\mu_1, \mu_2 \in \mathcal{L}$ then $\xi(\mathcal{L}) \subseteq \mathfrak{P}$.

Proof. By the hypothesis, we have

$$\xi(\mu_1)\xi(\mu_2) - [\mu_1, \mu_2] \in \mathfrak{P} \text{ for all } \mu_1, \mu_2 \in \mathcal{L}. \quad (20)$$

Writing $\mu_1\mu_3$ instead of μ_1 in (21), we have

$$\xi(\mu_1\mu_3)\xi(\mu_2) - [\mu_1\mu_3, \mu_2] \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2, \mu_3 \in \mathcal{L}.$$

Since ξ is a homoderivation, we have

$$\xi(\mu_1)\xi(\mu_3)\xi(\mu_2) + \xi(\mu_1)\mu_3\xi(\mu_2) + \mu_1\xi(\mu_3)\xi(\mu_2) - \mu_1[\mu_3, \mu_2] - [\mu_1, \mu_2]\mu_3 \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2, \mu_3 \in \mathcal{L}.$$

Using (20), we get

$$\xi(\mu_1)\xi(\mu_3)\xi(\mu_2) + \xi(\mu_1)\mu_3\xi(\mu_2) - [\mu_1, \mu_2]\mu_3 \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2, \mu_3 \in \mathcal{L}. \quad (21)$$

Replacing μ_2 by μ_1 in (21), we obtain

$$\xi(\mu_1)(\xi(\mu_3) + \mu_3)\xi(\mu_1) \in \mathfrak{P}, \text{ for all } \mu_1, \mu_3 \in \mathcal{L}. \quad (22)$$

Since ξ is zero-power valued on $\mathcal{L}$, there exist an integer $n > 1$ such that $\xi^n(\mu_3) = 0$ for all $\mu_3 \in \mathcal{L}$.
Replacing μ_3 by $\mu_3 - \xi(\mu_3) + \xi^2(\mu_3) + \cdots + (-1)^{n-1}\xi^{n-1}(\mu_3)$ in this expression, we find

$$\xi(\mu_1)\mu_3\xi(\mu_1) \in \mathfrak{P}, \text{ for all } \mu_1, \mu_3 \in \mathcal{L}.$$

which implies that

$$\xi(\mu_1)\mathcal{L}\xi(\mu_1) \in \mathfrak{P}, \text{ for all } \mu_1 \in \mathcal{L}.$$

Since $\mathfrak{P}$ is a prime ideal, we arrive at the conclusion $\xi(\mathcal{L}) \subseteq \mathfrak{P}$. With this result, the proof is complete.

Theorem 8. Let $\mathcal{L}$ be a ring, $\mathfrak{P}$ a prime ideal of $\mathcal{L}$ and ξ a nonzero homoderivation which is zero-power valued of $\mathcal{L}$. If $\xi(\mu_1)\xi(\mu_2) - \mu_1 o \mu_2 \in \mathfrak{P}$ for all $\mu_1, \mu_2 \in \mathcal{L}$ then $\xi(\mathcal{L}) \subseteq \mathfrak{P}$.

Proof. By the hypothesis, we have

$$\xi(\mu_1)\xi(\mu_2) - \mu_1 o \mu_2 \in \mathfrak{P} \text{ for all } \mu_1, \mu_2 \in \mathcal{L}. \quad (23)$$

Writing $\mu_1\mu_3$ instead of μ_1 in (23), we have

$$\xi(\mu_1\mu_3)\xi(\mu_2) - (\mu_1\mu_3) o \mu_2 \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2, \mu_3 \in \mathcal{L}.$$

If this expression is expanded, we get

$$\xi(\mu_1)\xi(\mu_3)\xi(\mu_2) + \xi(\mu_1)\mu_3\xi(\mu_2) + \mu_1\xi(\mu_3)\xi(\mu_2) - \mu_1(\mu_3 o \mu_2) + [\mu_1, \mu_2]\mu_3 \in \mathfrak{P}$$

Using (23), we obtain

$$\xi(\mu_1)\xi(\mu_3)\xi(\mu_2) + \xi(\mu_1)\mu_3\xi(\mu_2) + [\mu_1, \mu_2]\mu_3 \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2, \mu_3 \in \mathcal{L}.$$

Taking μ_2 by μ_1 in last expression, we get

$$\xi(\mu_1)(\xi(\mu_3) + \mu_3)\xi(\mu_1) \in \mathfrak{P}, \text{ for all } \mu_1, \mu_2, \mu_3 \in \mathcal{L}.$$

Last expression is same as (22) in the proof of Theorem 7. By proceeding in parallel with the arguments applied in Theorem 7, we obtain the desired result.

Example: Suppose the ring $\mathcal{L} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{R}\}$. Define maps $\xi: \mathcal{L} \to \mathcal{L}$ as follows: $\xi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Let $\mathfrak{P} = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | a, b, d \in \mathbb{R}\}$. It is obvious that $\mathfrak{P}$ is not a prime ideal. Then it is easy to verify that ξ is a homoderivation of $\mathcal{L}$ and $\mathfrak{P}$ is a nonzero right ideal of $\mathcal{L}$ and $\xi([\mu_1, \mu_2]) \in \mathfrak{P}$ for all $\mu_1, \mu_2 \in \mathcal{L}$. However, $\mathcal{L}/\mathfrak{P}$ is not commutative.

Conflict of Interest

There are no conflicts of interest in this work.

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