Conformal anti-invariant submersions from cosymplectic manifolds

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Abstract

We introduce conformal anti-invariant submersions from cosymplectic manifolds onto Riemannian manifolds. We give an example of a conformal anti-invariant submersion such that characteristic vector field ξ is vertical. We investigate the geometry of foliations which are arisen from the definition of a conformal submersion and show that the total manifold has certain product structures. Moreover, we examine necessary and sufficient conditions for a conformal anti-invariant submersion to be totally geodesic and check the harmonicity of such submersions.

1. Introduction

Immersions and submersions, which are special tools in differential geometry, also play a fundamental role in Riemannian geometry, especially when the involved manifolds carry an additional structure (such as contact, Hermitian and product structure). In particular, Riemannian submersions (which we always assume to have connected fibers) are fundamentally important in several areas of Riemannian geometry. For instance, it is a classical and important problem in Riemannian geometry to construct Riemannian manifolds with positive or non-negative sectional curvature.

The theory of Riemannian submersions between Riemannian manifold was initiated by O'Neill [23] and Gray [17]. In [33], the Riemannian submersions were considered between almost Hermitian manifolds by Watson under the name of almost Hermitian submersions. In this case, the Riemannian submersion is also an almost complex mapping and consequently the vertical and horizontal distributions are invariant with respect to the almost complex structure of the total manifold of the submersion. The study of anti-invariant Riemannian submersions from almost Hermitian manifolds were initiated by

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Şahin [27]. In this case, the fibres are anti-invariant with respect to the almost complex structure of the total manifold. There are many other notions related to anti-invariant Riemannian submersions, (see also: [16], [19], [26], [28], [29], [30], [31], [32]). Recently, Murathan and Erken [7] extended this notion to the case when the total manifold is cosymplectic manifold. There are some other recent paper which involve other structures such as almost contact [1, 20] and Sasakian [8].

On the other hand, as a generalization of Riemannian submersion, horizontally conformal submersions are defined as follows [6]: Suppose that (M, g_M) and (B, g_B) are Riemannian manifolds and $F : M \longrightarrow B$ is a smooth submersion, then F is called a horizontally conformal submersion, if there is a positive function λ such that

$$\lambda^2 g_M(X, Y) = g_B(F_* X, F_* Y)$$

for every $X, Y \in \Gamma((\ker F_*)^\perp)$. It is obvious that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. We note that horizontally conformal submersions are special horizontally conformal maps which were introduced independently by Fuglede [12] and Ishihara [18]. We also note that a horizontally conformal submersion $F : M \longrightarrow B$ is said to be horizontally homothetic if the gradient of its dilation λ is vertical, i.e.,

(1.1) $\mathcal{H}(grad\lambda) = 0$

at $p \in M$, where $\mathcal{H}$ is the projection on the horizontal space $(\ker F_*)^\perp$. For conformal submersion, see [6], [14], [24].

One can see that Riemannian submersions are very special maps comparing with conformal submersions. Although conformal maps do not preserve distance between points contrary to isometries, they preserve angles between vector fields. This property enables one to transfer certain properties of a manifold to another manifold by deforming such properties.

As a generalization of holomorphic submersions, conformal holomorphic submersions were studied by Gudmundsson and Wood [15]. They obtained necessary and sufficient conditions for conformal holomorphic submersions to be a harmonic morphism, see also [9], [10] and [11] for the harmonicity of conformal holomorphic submersions.

Recently, In [2] we have introduced conformal anti-invariant submersions and [3] conformal semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds and investigated the geometry of such submersions. We showed that the geometry of such submersions are different from anti-invariant Riemannian submersions. In this paper, we consider conformal anti-invariant submersions from cosymplectic manifolds. The paper is organized as follows. In section 2, we present some background about conformal submersions needed for this paper. In section 3, we mention about cosymplectic manifolds. In section 4, we introduce conformal anti-invariant submersions from cosymplectic manifolds. We give an example of a conformal anti-invariant submersion such that characteristic vector field ξ is vertical. We also investigate the geometry of foliations which are arisen from the definition of a conformal submersion and show that the total manifold has certain product structures. In the last section, we obtain necessary and sufficient conditions for a conformal anti-invariant submersion to be totally geodesic and check the harmonicity of such submersions.

2. Conformal Submersions

In this section, we recall the notion of (horizontally) conformal submersions between Riemannian manifolds and give a brief review of basic facts of (horizontally) conformal submersions. Conformal submersions belong to a wide class of conformal maps that we are going to recall their definition, but we will not study such maps in this paper.

2.1. Definition. *([6])* Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds, and let $x \in M$. Then φ is called horizontally weakly conformal or semi conformal at x if either

    (i) $d\varphi_x = 0$, or

    (ii) $d\varphi_x$ maps horizontal space $\mathcal{H}_x = (ker(d\varphi_x))^{\perp}$ conformally onto $T_{\varphi(x)}N$, i.e., $d\varphi_x$ is surjective and there exists a number $\Lambda(x) \neq 0$ such that

$$(2.1) \qquad h(d\varphi_x X, d\varphi_x Y) = \Lambda(x)g(X, Y) \ (X, Y \in \mathcal{H}_x).$$

Note that we can write the last equation more sufficiently as

$$(\varphi^* h)_x \mid_{\mathcal{H}_x \times \mathcal{H}_x} = \Lambda(x)g_x \mid_{\mathcal{H}_x \times \mathcal{H}_x}.$$

A point x is of type (i) in Definition if and only if it is a critical point of φ; we shall call a point of type (ii) a *regular point*. At a critical point, $d\varphi_x$ has rank 0; at a regular point, $d\varphi_x$ has rank n and φ is submersion. The number $\Lambda(x)$ is called the *square dilation* (of φ at x); it is necessarily non-negative; its square root $\lambda(x) = \sqrt{\Lambda(x)}$ is called the dilation (of φ at x). The map φ is called *horizontally weakly conformal* or *semi conformal* (on M) if it is horizontally weakly conformal at every point of M. It is clear that if φ has no critical points, then we call it a *(horizontally)* conformal submersion.

Next, we recall the following definition from [14]. Let $\pi : M \longrightarrow N$ be a submersion. A vector field E on M is said to be projectable if there exists a vector field $\check{E}$ on N, such that $d\pi(E_x) = \check{E}_{\pi(x)}$ for all $x \in M$. In this case E and $\check{E}$ are called $\pi-$ related. A horizontal vector field Y on (M, g) is called basic, if it is projectable. It is well known fact, that is $\check{Z}$ is a vector field on N, then there exists a unique basic vector field Z on M, such that Z and $\check{Z}$ are $\pi-$ related. The vector field Z is called the horizontal lift of $\check{Z}$.

The fundamental tensors of a submersion were introduced in [23]. They play a similar role to that of the second fundamental form of an immersion. More precisely, O'Neill's tensors T and A defined for vector fields E, F on M by

$$(2.2) \qquad A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F$$

$$(2.3) \qquad T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections (see [13]). On the other hand, from (2.2) and (2.3), we have

$$(2.4) \qquad \nabla_V W = T_V W + \hat{\nabla}_V W$$

$$(2.5) \qquad \nabla_V X = \mathcal{H}\nabla_V X + T_V X$$

$$(2.6) \qquad \nabla_X V = A_X V + \mathcal{V}\nabla_X V$$

$$(2.7) \qquad \nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y$$

for $X, Y \in \Gamma((ker F_*)^{\perp})$ and $V, W \in \Gamma(ker F_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If X is basic, then $\mathcal{H}\nabla_V X = A_X V$. It is easily seen that for $x \in M$, $X \in \mathcal{H}_x$ and $V \in \mathcal{V}_x$ the linear operators $T_V, A_X : T_x M \longrightarrow T_x M$ are skew-symmetric, that is

$$g(T_V E, G) = -g(E, T_V G) \text{ and } g(A_X E, G) = -g(E, A_X G)$$

for all $E, G \in T_x M$. We also see that the restriction of T to the vertical distribution $T \mid_{ker\pi_* \times ker\pi_*}$ is exactly the second fundamental form of the fibres of π. Since T_V skew-symmetric we get: π has totally geodesic fibres if and only if $T \equiv 0$. For the special case when π is horizontally conformal we have the following:

2.2. Proposition. *([14])* Let $\pi : (M^m, g) \longrightarrow (N^n, h)$ be a horizontally conformal submersion with dilation λ and X, Y be horizontal vectors, then

$$(2.8) \qquad A_X Y = \frac{1}{2}\{\mathcal{V}[X, Y] - \lambda^2 g(X, Y)grad_{\mathcal{V}}(\frac{1}{\lambda^2})\}.$$

We see that the skew-symmetric part of $A \mid_{(ker\pi_*)^\perp \times (ker\pi_*)^\perp}$ measures the obstruction integrability of the horizontal distribution $ker\pi_*$.

We recall the notion of harmonic maps between Riemannian manifolds. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\pi : M \longrightarrow N$ is a smooth map between them. Then the differential of π_* of π can be viewed a section of the bundle $Hom(TM, \pi^{-1}TN) \longrightarrow M$, where $\pi^{-1}TN$ is the pullback bundle which has fibres $(\pi^{-1}TN)_p = T_{\pi(p)}N$, $p \in M$. $Hom(TM, \pi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of π is given by

$$(2.9) \qquad (\nabla\pi_*)(X, Y) = \nabla_X^\pi \pi_* Y - \pi_*(\nabla_X^M Y)$$

for $X, Y \in \Gamma(TM)$, where ∇^π is the pullback connection. It is known that the second fundamental form is symmetric. We recall the following lemma from [6].

2.3. Lemma. *Suppose that $\pi : M \longrightarrow N$ is a horizontally conformal submersion. Then, for any horizontal vector fields X, Y and vertical vector fields V, W we have*

(i) $(\nabla\pi_*)(X, Y) = X(\ln \lambda)\pi_* Y + Y(\ln \lambda)\pi_* X - g(X, Y)\pi_*(grad \ln \lambda)$;
(ii) $(\nabla\pi_*)(V, W) = -\pi_*(T_V W)$;
(iii) $(\nabla\pi_*)(X, V) = -\pi_*(\nabla_X^M V) = -\pi_*(A_X V)$.

Let g be a Riemannian metric tensor on the manifold $M = M_1 \times M_2$ and assume that the canonical foliations D_{M_1} and D_{M_2} intersect perpendicularly everywhere. Then g is the metric tensor of a usual product of Riemannian manifolds if and only if D_{M_1} and D_{M_2} are totally geodesic foliations [25].

3. Cosymplectic Manifolds

A $(2m + 1)$-dimensional C^∞-manifold M said to have an almost contact structure if there exist on M a tensor field φ of type (1,1), a vector field ξ and 1-form η satisfying:

$$(3.1) \qquad \varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta o \varphi = 0, \quad \eta(\xi) = 1.$$

There always exists a Riemannian metric g on an almost contact manifold M satisfying the following conditions

$$(3.2) \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

where X, Y are vector fields on M.

An almost contact structure (φ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $M \times \mathbb{R}$ is given by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}),$$

where f is a C^∞-function on $M \times \mathbb{R}$ has no torsion i.e., J is integrable. The condition for normality in terms of φ, ξ and η is $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$ on M, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ. Finally, the fundamental two-form Φ is defined $\Phi(X, Y) = g(X, \varphi Y)$.

An almost contact metric structure (φ, ξ, η, g) is said to be cosymplectic, if it is normal and both Φ and η are closed ([4], [5], [21]), and the structure equation of a cosymplectic manifold is given by

$$(3.3) \qquad (\nabla_X \varphi)Y = 0$$

for any X, Y tangent to M, where ∇ denotes the Riemannian connection of the metric g on M. Moreover, for cosymplectic manifold

$$(3.4) \qquad \nabla_X \xi = 0.$$

The canonical example of cosymplectic manifold is given by the product $B^{2n} \times \mathbb{R}$ Kaehler manifold $B^{2n}(J, g)$ with the $\mathbb{R}$ real line. Now we will introduce a well known coymplectic manifold example of $\mathbb{R}^{2n+1}$.

3.1. Example. ([22], [7]) We consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x_i, y_i, z)(i = 1, ..., n)$ and its usual contact form
$$\eta = dz.$$
The characteristic vector field ξ is given by $\frac{\partial}{\partial z}$ and its Riemannian metric g and tensor field φ are given by

$$g = \sum_{i=1}^{n}((dx_i)^2 + (dy_i)^2) + (dz)^2, \quad \varphi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i = 1, ..., n.$$

This gives a cosymplectic manifold on $\mathbb{R}^{2n+1}$. The vector fields $e_i = \frac{\partial}{\partial y_i}$, $e_{n+i} = \frac{\partial}{\partial x_i}$, ξ form a φ-basis for the cosymplectic structure. On the other hand, it can be shown that $\mathbb{R}^{2n+1}(\varphi, \xi, \eta, g)$ is a cosymplectic manifold.

4. Conformal anti-invariant submersions from cosymplectic manifolds

In this section, we define conformal anti-invariant submersions from cosymplectic manifolds onto Riemannian manifolds and investigate the integrability of distributions and obtain a necessary and sufficient condition for such submersions to be totally geodesic map. We also investigate the harmonicity of such submersions.

4.1. Definition. Let $M(\varphi, \xi, \eta, g_M)$ be a cosymplectic manifold and (N, g_N) be a Riemannian manifold. A horizontally conformal submersion $F : M \longrightarrow N$ is called a conformal anti-invariant submersion if $ker F_*$ is anti-invariant with respect to φ, i.e., $\varphi(ker F_*) \subset (ker F_*)^\perp$.

Now, we assume that $F : M(\varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant submersion from a cosymplectic manifold $M(\varphi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). First of all, from Definition 4.1, we have $\varphi(ker F_*)^\perp \cap ker F_* \neq 0$. We denote the complementary orthogonal distribution to $\varphi(ker F_*)$ in $(ker F_*)^\perp$ by μ. Then we have

$$(4.1) \quad (ker F_*)^\perp = \varphi(ker F_*) \oplus \mu.$$

It is easy to see that μ is an invariant distribution of $(ker F_*)^\perp$, under the endomorphism φ. Thus, for $X \in \Gamma((ker F_*)^\perp)$, we have

$$(4.2) \quad \varphi X = \mathcal{B}X + \mathcal{C}X,$$

where $\mathcal{B}X \in \Gamma(ker F_*)$ and $\mathcal{C}X \in \Gamma(\mu)$. On the other hand, since $F_*((ker F_*)^\perp) = TN$ and F is a conformal submersion, using (4.2) we derive $\frac{1}{\lambda^2} g_N(F_* \varphi V, F_* \mathcal{C}X) = 0$ for any $X \in \Gamma((ker F_*)^\perp)$ and $V \in \Gamma(ker F_*)$, which implies that

$$(4.3) \quad TN = F_*(\varphi ker F_*) \oplus F_*(\mu).$$

The proof of the following result is exactly same the proof of Theorem 1([7]), therefore we omit its proof.

4.2. Theorem. *Let $M(\varphi, \xi, \eta, g_M)$ be a cosymplectic manifold of dimension $2m + 1$ and (N, g_N) be a Riemannian manifold of dimension n. Let $F : M(\varphi, \xi, \eta, g_M) \longrightarrow N$ be a conformal anti-invariant submersion such that $\varphi ker F_* = (ker F_*)^\perp$. Then the characteristic vector field ξ is vertical and $m = n$.*

4.3. Remark. In this paper, we suppose that the characteristic vector field ξ is vertical.

Now we give the following examples.

4.4. Example. Every anti-invariant Riemannian submersion from cosymplectic manifold onto Riemannian manifold is a conformal anti-invariant submersion from cosymplectic manifold onto Riemannian manifold with $\lambda = I$, where I is the identity function [7].

4.5. Example. $\mathbb{R}^5$ has got a cosymplectic structure as in Example 1.
Consider the following submersion given by

$$F : \quad \begin{array}{c} \mathbb{R}^5 \\ (x_1, x_2, y_1, y_2, z) \end{array} \quad \longrightarrow \quad \begin{array}{c} \mathbb{R}^2 \\ (\frac{e^{x_1} \sin y_2}{\sqrt{2}}, \frac{e^{x_1} \cos y_2}{\sqrt{2}}). \end{array}$$

Then it follows that

$$ker F_* = span\{Z_1 = \partial x_2, \ Z_2 = \partial y_1, Z_3 = \xi = \partial z\}$$

and

$$(ker F_*)^\perp = span\{H_1 = \frac{e^{x_1} \sin y_2}{\sqrt{2}} \partial x_1 + \frac{e^{x_1} \cos y_2}{\sqrt{2}} \partial y_2,$$

$$H_2 = \frac{e^{x_1} \cos y_2}{\sqrt{2}} \partial x_1 - \frac{e^{x_1} \sin y_2}{\sqrt{2}} \partial y_2\}.$$

Then by direct computations $\varphi Z_1 = \sqrt{2}(e^{-x_1} \sin y_2 H_1 + e^{-x_1} \cos y_2 H_2)$,
$\varphi Z_2 = \sqrt{2}(-e^{-x_1} \cos y_2 H_1 + e^{-x_1} \sin y_2 H_2)$ and $\varphi Z_3 = 0$ imply that $\varphi(ker F_*) = (ker F_*)^\perp$.
Also by direct computations, we get

$$F_* H_1 = \frac{(e^{x_1})^2}{2} \partial z_1, \quad F_* H_2 = \frac{(e^{x_1})^2}{2} \partial z_2.$$

Hence, we have

$$g_2(F_* H_1, F_* H_1) = \frac{(e^{x_1})^2}{2} g_1(H_1, H_1), \quad g_2(F_* H_2, F_* H_2) = \frac{(e^{x_1})^2}{2} g_1(H_2, H_2),$$

where g_1 and g_2 denote the standard metrics (inner products) of $\mathbb{R}^5$ and $\mathbb{R}^2$. Thus F is a conformal anti-invariant submersion with $\lambda = \frac{e^{x_1}}{\sqrt{2}}$.

For any $X \in \Gamma((ker F_*)^\perp)$ and $U \in \Gamma(ker F_*)$, from (3.1) and (4.2) we easily have

$$\mathcal{B}\mathcal{C}X = 0, \quad \eta(\mathcal{B}X) = 0, \quad \mathcal{C}^2 X = -X - \varphi \mathcal{B}X,$$

$$\mathcal{C}^3 X + \mathcal{C}X = 0, \quad \mathcal{C}\varphi U = 0, \quad \mathcal{B}\varphi U = -U + \eta(U)\xi.$$

4.6. Lemma. *Let F be a conformal anti-invariant submersion from a cosymplectic manifold $M(\varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

(4.4) $g_M(\mathcal{C}Y, \varphi V) = 0,$

(4.5) $g_M(\nabla_X \mathcal{C}Y, \varphi V) = -g_M(\mathcal{C}Y, \varphi A_X V),$

(4.6) $T_V \xi = 0,$

(4.7) $A_X \xi = 0$

for any $X, Y \in \Gamma((ker F_*)^\perp)$ and $V \in \Gamma(ker F_*)$.

Proof. For $Y \in \Gamma((ker F_*)^\perp)$ and $V \in \Gamma(ker F_*)$, using (3.2), we have

$$g_M(\mathcal{C}Y, \varphi V) = g_M(\varphi Y - \mathcal{B}Y, \varphi V) = g_M(\varphi Y, \varphi V) = g_M(Y, V) - \eta(Y)\eta(V) = g_M(Y, V) = 0,$$

due to $\mathcal{B}Y$, $\xi \in \Gamma(kerF_*)$ and $\varphi V \in \Gamma((kerF_*)^\perp)$. Differentiating (4.4) with respect to X and using (2.6), we get

$$
\begin{aligned}
g_M(\nabla_X \mathcal{C}Y, \varphi V) &= -g_M(\mathcal{C}Y, \nabla_X \varphi V) \\
&= -g_M(\mathcal{C}Y, (\nabla_X \varphi)V) - g_M(\mathcal{C}Y, \varphi(\nabla_X V)) \\
&= -g_M(\mathcal{C}Y, \varphi(\nabla_X V)) \\
&= -g_M(\mathcal{C}Y, \varphi A_X V) - g_M(\mathcal{C}Y, \varphi \mathcal{V} \nabla_X V) \\
&= -g_M(\mathcal{C}Y, \varphi A_X V)
\end{aligned}
$$

due to $\varphi \mathcal{V} \nabla_X V \in \Gamma(\varphi ker F_*)$. By virtue of (2.5) and (3.4), we get (4.6). Using (2.6) and (3.4) we obtain (4.7). Our assertion is complete. $\qquad \square$

4.7. Theorem. *Let $M(\varphi, \xi, \eta, g_M)$ be a cosymplectic manifold of dimension $2m+1$ and (N, g_N) be a Riemannian manifold of dimension n. Let $F : M(\varphi, \xi, \eta, g_M) \longrightarrow N$ be a conformal anti-invariant submersion. Then the fibres are not proper totally umbilical.*

Proof. It is very similar to the proof of Theorem 4.10 ([7]), so we omit it. $\qquad \square$

We now study the integrability of the distribution $(kerF_*)^\perp$ and then we investigate the geometry of leaves of $kerF_*$ and $(kerF_*)^\perp$. We note that it is known that the distribution $kerF_*$ is integrable.

4.8. Theorem. *Let F be a conformal anti-invariant submersion from a cosymplectic manifold $M(\varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other;*

(a) *$(kerF_*)^\perp$ is integrable,*

(b) $\dfrac{1}{\lambda^2} g_N(\nabla_Y^F F_* \mathcal{C}X - \nabla_X^F F_* \mathcal{C}Y, F_* \varphi V) = g_M(A_X \mathcal{B}Y - A_Y \mathcal{B}X - \mathcal{C}Y(\ln \lambda)X$
$$+ \mathcal{C}X(\ln \lambda)Y - 2g_M(\mathcal{C}X, Y)\mathcal{H}grad \ln \lambda, \varphi V)$$

for $X, Y \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$.

Proof. For $Y \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$, we see from Definition 4.1, $\varphi V \in \Gamma((kerF_*)^\perp)$ and $\varphi Y \in \Gamma(kerF_* \oplus \mu)$. For $X, Y \in \Gamma((kerF_*)^\perp)$ and $\xi \in \Gamma(kerF_*)$, from (3.4) we have $g_M([X,Y], \xi) = g_M(\nabla_X Y, \xi) - g_M(\nabla_Y X, \xi) = -g_M(X, \nabla_Y \xi) + g_M(Y, \nabla_X \xi) = 0$. Thus this case is trivial. Thus using (3.2) and (3.3), we note that for $X \in \Gamma((kerF_*)^\perp)$,

$$
\begin{aligned}
g_M([X,Y], V) &= g_M(\nabla_X Y, V) - g_M(\nabla_Y X, V) \\
&= g_M(\nabla_X \varphi Y, \varphi V) - g_M(\nabla_Y \varphi X, \varphi V).
\end{aligned}
$$

Then from (4.2), we get

$$
\begin{aligned}
g_M([X,Y], V) &= g_M(\nabla_X \mathcal{B}Y, \varphi V) + g_M(\nabla_X \mathcal{C}Y, \varphi V) - g_M(\nabla_Y \mathcal{B}X, \varphi V) \\
&\quad - g_M(\nabla_Y \mathcal{C}X, \varphi V).
\end{aligned}
$$

Since F is a conformal submersion, using (2.6) we arrive at

$$
\begin{aligned}
g_M([X,Y], V) &= g_M(A_X \mathcal{B}Y - A_Y \mathcal{B}X, \varphi V) + \frac{1}{\lambda^2} g_N(F_*(\nabla_X \mathcal{C}Y), F_* \varphi V) \\
&\quad - \frac{1}{\lambda^2} g_N(F_*(\nabla_Y \mathcal{C}X), F_* \varphi V).
\end{aligned}
$$

Thus, from (2.9) and Lemma 2.3 we derive

$$g_M([X,Y],V) = g_M(A_X \mathcal{B}Y - A_Y \mathcal{B}X, \varphi V) - g_M(\mathcal{H}grad \ln \lambda, X)g_M(\mathcal{C}Y, \varphi V)$$
$$-g_M(\mathcal{H}grad \ln \lambda, \mathcal{C}Y)g_M(X, \varphi V) + g_M(X, \mathcal{C}Y)g_M(\mathcal{H}grad \ln \lambda, \varphi V)$$
$$+ \frac{1}{\lambda^2}g_N(\nabla_X^F F_* \mathcal{C}Y, F_* \varphi V) + g_M(\mathcal{H}grad \ln \lambda, Y)g_M(\mathcal{C}X, \varphi V)$$
$$+g_M(\mathcal{H}grad \ln \lambda, \mathcal{C}X)g_M(Y, \varphi V) - g_M(Y, \mathcal{C}X)g_M(\mathcal{H}grad \ln \lambda, \varphi V)$$
$$- \frac{1}{\lambda^2}g_N(\nabla_Y^F F_* \mathcal{C}X, F_* \varphi V).$$

Moreover, using (4.4), we obtain

(4.8) $\quad g_M([X,Y],V) = g_M(A_X \mathcal{B}Y - A_Y \mathcal{B}X - \mathcal{C}Y(\ln \lambda)X + \mathcal{C}X(\ln \lambda)Y$

$$-2g_M(\mathcal{C}X, Y)\mathcal{H}grad \ln \lambda, \varphi V) - \frac{1}{\lambda^2}g_N(\nabla_Y^F F_* \mathcal{C}X - \nabla_X^F F_* \mathcal{C}Y, F_* \varphi V)$$

which proves $(a) \Leftrightarrow (b)$. $\hfill\square$

From Theorem 4.8 we deduce the following which shows that a conformal anti-invariant submersion with integrable $(ker F_*)^\perp$ turns out to be a horizontally homothetic submersion.

4.9. Theorem. *Let F be a conformal anti-invariant submersion from a cosymplectic manifold $M(\varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then any two conditions below imply the third;*

(i) *$(ker F_*)^\perp$ is integrable*

(ii) *F is a horizontally homothetic submersion*

(iii) *$g_N(\nabla_Y^F F_* \mathcal{C}X - \nabla_X^F F_* \mathcal{C}Y, F_* \varphi V) = \lambda^2 g_M(A_X \mathcal{B}Y - A_Y \mathcal{B}X, \varphi V)$*

for $X, Y \in \Gamma((ker F_*)^\perp)$ and $V \in \Gamma(ker F_*)$.

Proof. From (4.8), we have

(4.9) $\quad g_M([X,Y],V) = g_M(A_X \mathcal{B}Y - A_Y \mathcal{B}X - \mathcal{C}Y(\ln \lambda)X + \mathcal{C}X(\ln \lambda)Y$

$$-2g_M(\mathcal{C}X, Y)\mathcal{H}grad \ln \lambda, \varphi V) - \frac{1}{\lambda^2}g_N(\nabla_Y^F F_* \mathcal{C}X - \nabla_X^F F_* \mathcal{C}Y, F_* \varphi V)$$

for $X, Y \in \Gamma((ker F_*)^\perp)$ and $V \in \Gamma(ker F_*)$. Now, if we have (i) and (iii), then we arrive at

(4.10) $\quad - g_M(\mathcal{H}grad \ln \lambda, \mathcal{C}Y)g_M(X, \varphi V) + g_M(\mathcal{H}grad \ln \lambda, \mathcal{C}X)g_M(Y, \varphi V)$
$$- 2g_M(\mathcal{C}X, Y)g_M(\mathcal{H}grad \ln \lambda, \varphi V) = 0.$$

If we take $V = \xi$ in (4.10) for $\xi \in \Gamma(ker F_*)$, using (3.1), we get

$$- g_M(\mathcal{H}grad \ln \lambda, \mathcal{C}Y)g_M(X, \varphi \xi) + g_M(\mathcal{H}grad \ln \lambda, \mathcal{C}X)g_M(Y, \varphi \xi)$$
$$- 2g_M(\mathcal{C}X, Y)g_M(\mathcal{H}grad \ln \lambda, \varphi \xi) = 0.$$

Now, taking $Y = \varphi V$ in (4.10) for $V \in \Gamma(ker F_*)$, using (3.2) and (4.4), we get

$$g_M(\mathcal{H}grad \ln \lambda, \mathcal{C}X)g_M(\varphi V, \varphi V) = 0.$$

Hence λ is a constant on $\Gamma(\mu)$. On the other hand, taking $Y = \mathcal{C}X$ in (4.10) for $X \in \Gamma(\mu)$ and using (4.4) we derive

$$- g_M(\mathcal{H}grad \ln \lambda, \mathcal{C}^2 X)g_M(X, \varphi V) + g_M(\mathcal{H}grad \ln \lambda, \mathcal{C}X)g_M(\mathcal{C}X, \varphi V)$$
$$- 2g_M(\mathcal{C}X, \mathcal{C}X)g_M(\mathcal{H}grad \ln \lambda, \varphi V) = 0,$$

thus, we arrive at

$$2g_M(\mathcal{C}X, \mathcal{C}X)g_M(\mathcal{H}grad \ln \lambda, \varphi V) = 0.$$

From above equation, λ is a constant on $\Gamma(\varphi ker F_*)$. Similarly, one can obtain the other assertions. $\qquad\square$

4.10. Remark. If $\varphi(ker F_*) = (ker F_*)^\perp$ then we get $\mathcal{C} = 0$, and moreover (4.3) implies that $TN = F_*(\varphi ker F_*)$.

Hence we have the following result.

4.11. Corollary. *Let F be a conformal anti-invariant submersion from a cosymplectic manifold $M(\varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with $\varphi(ker F_*) = (ker F_*)^\perp$. Then the following assertions are equivalent to each other;*

    (i) $(ker F_*)^\perp$ *is integrable.*
   (ii) $(\nabla F_*)(X, \varphi Y) = (\nabla F_*)(Y, \varphi X)$
  (iii) $A_X \varphi Y = A_Y \varphi X$

for $X, Y \in \Gamma((ker F_*)^\perp)$.

For the geometry of leaves of the horizontal distribution, we have the following theorem.

4.12. Theorem. *Let F be a conformal anti-invariant submersion from a cosymplectic manifold $M(\varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other;*

    (i) $(ker F_*)^\perp$ *defines a totally geodesic foliation on M.*
   (ii) $-\dfrac{1}{\lambda^2} g_N(\nabla_X^F F_* \mathcal{C}Y, F_* \varphi V) = g_M(A_X BY - \mathcal{C}Y(\ln \lambda)X, \varphi V)$

$$+ g_M(\mathcal{H}grad\,\ln \lambda, \varphi V)g_M(X, \mathcal{C}Y)$$

for $X, Y \in \Gamma((ker F_*)^\perp)$ and $V \in \Gamma(ker F_*)$.

Proof. From (3.2) and (3.3) we obtain

$$g_M(\nabla_X Y, V) = g_M(\nabla_X \varphi Y, \varphi V)$$

for $X, Y \in \Gamma((ker F_*)^\perp)$ and $V \in \Gamma(ker F_*)$. If we take $V = \xi$ in above equation for $\xi \in \Gamma(ker F_*)$, then from (3.4) we have $g_M(\nabla_X Y, \xi) = -g_M(Y, \nabla_X \xi) = 0$. Thus this case is trivial. Now, Using (2.6), (2.7), (4.1) and (4.2), we get

$$g_M(\nabla_X Y, V) = g_M(A_X BY, \varphi V) + g_M(\mathcal{H}\nabla_X \mathcal{C}Y, \varphi V).$$

Since F is a conformal submersion, using (2.9) and Lemma (2.3) we arrive at

$$g_M(\nabla_X Y, V) = g_M(A_X BY, \varphi V) - \frac{1}{\lambda^2} g_M(\mathcal{H}grad\,\ln \lambda, X)g_N(F_* \mathcal{C}Y, F_* \varphi V)$$

$$- \frac{1}{\lambda^2} g_M(\mathcal{H}grad\,\ln \lambda, \mathcal{C}Y)g_N(F_* X, F_* \varphi V)$$

$$+ \frac{1}{\lambda^2} g_M(X, \mathcal{C}Y)g_N(F_*(\mathcal{H}grad\,\ln \lambda), F_* \varphi V)$$

$$+ \frac{1}{\lambda^2} g_N(\nabla_X^F F_* \mathcal{C}Y, F_* \varphi V).$$

Moreover, using Definiton 4.1 and (4.4) we obtain

(4.11) $g_M(\nabla_X Y, V) = g_M(A_X BY - \mathcal{C}Y(\ln \lambda)X, \varphi V) + g_M(\mathcal{H}grad\,\ln \lambda, \varphi V)g_M(X, \mathcal{C}Y)$

$$+ \frac{1}{\lambda^2} g_N(\nabla_X^F F_* \mathcal{C}Y, F_* \varphi V)$$

which proves $(i) \Leftrightarrow (ii)$. $\qquad\square$

From Theorem 4.12, we also deduce the following characterization.

4.13. Theorem. *Let F be a conformal anti-invariant submersion from a cosymplectic manifold $M(\varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then any two conditions below imply the third;*

   (i) *$(ker F_*)^\perp$ defines a totally geodesic foliation on M.*
   (ii) *F is a horizontally homothetic submersion.*
   (iii) *$g_N(\nabla_X^F F_* \mathcal{C}Y, F_*\varphi V) = \lambda^2 g_M(A_X \varphi V, \mathcal{B}Y)$*

for $X, Y \in \Gamma((ker F_*)^\perp)$ and $V \in \Gamma(ker F_*)$.

Proof. For $X, Y \in \Gamma((ker F_*)^\perp)$ and $V \in \Gamma(ker F_*)$, from (4.11), we have

$$(4.12) \quad g_M(\nabla_X Y, V) = g_M(A_X \mathcal{B}Y - \mathcal{C}Y(\ln \lambda)X, \varphi V) + g_M(\mathcal{H}grad \ln \lambda, \varphi V)g_M(X, \mathcal{C}Y)$$
$$+ \frac{1}{\lambda^2} g_N(\nabla_X^F F_* \mathcal{C}Y, F_*\varphi V).$$

Now, if we have (i) and (iii), then we obtain

$$(4.13) \quad -g_M(\mathcal{H}grad \ln \lambda, \mathcal{C}Y)g_M(X, \varphi V) + g_M(\mathcal{H}grad \ln \lambda, \varphi V)g_M(X, \mathcal{C}Y) = 0.$$

If we take $V = \xi$ in (4.13) for $\xi \in \Gamma(ker F_*)$, using (3.1), we get

$$-g_M(\mathcal{H}grad \ln \lambda, \mathcal{C}Y)g_M(X, \varphi \xi) + g_M(\mathcal{H}grad \ln \lambda, \varphi \xi)g_M(X, \mathcal{C}Y) = 0.$$

Now, taking $X = \mathcal{C}Y$ in (4.13) and using (4.4), we get

$$g_M(\mathcal{H}grad \ln \lambda, \varphi V)g_M(\mathcal{C}Y, \mathcal{C}Y) = 0.$$

Thus, λ is a constant on $\Gamma(\varphi ker F_*)$. On the other hand, taking $X = \varphi V$ in (4.13) and using (4.4) we derive

$$g_M(\mathcal{H}grad \ln \lambda, \mathcal{C}Y)g_M(\varphi V, \varphi V) = 0.$$

From above equation, λ is a constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. □

In particular, if $\varphi(ker F_*) = (ker F_*)^\perp$ then we have the following result.

4.14. Corollary. *Let F be a conformal anti-invariant submersion from a cosymplectic manifold $M(\varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with $\varphi(ker F_*) = (ker F_*)^\perp$. Then the following assertions are equivalent to each other;*

   (i) *$(ker F_*)^\perp$ defines a totally geodesic foliation on M.*
   (ii) *$A_X \varphi Y = 0$*
   (iii) *$(\nabla F_*)(X, \varphi Y) = 0$*

for $X, Y \in \Gamma((ker F_*)^\perp)$.

In the sequel we are going to investigate the geometry of leaves of the distribution $ker F_*$.

4.15. Theorem. *Let F be a conformal anti-invariant submersion from a cosymplectic manifold $M(\varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other;*

   (i) *$ker F_*$ defines a totally geodesic foliation on M.*
   (ii) *$-\dfrac{1}{\lambda^2} g_N(\nabla_{\varphi W}^F F_*\varphi V, F_*\varphi \mathcal{C}X) = g_M(T_V \varphi W, \mathcal{B}X)$*
$$+ g_M(\mathcal{H}grad ln\lambda, \mathcal{C}X)g_M(V, W)$$

for $V, W \in \Gamma(ker F_*)$ and $X \in \Gamma((ker F_*)^\perp)$.

Proof. For $X \in \Gamma((kerF_*)^\perp)$ and $V, W \in \Gamma(kerF_*)$, $g_M(W, \xi) = 0$ implies that from (3.4) $g_M(\nabla_V W, \xi) = -g_M(W, \nabla_V \xi) = 0$. If we take $W = \xi$ for $\xi \in \Gamma(kerF_*)$, then from (3.4) we have $g_M(\nabla_V \xi, X) = 0$. Thus this case is trivial. Thus using (3.2) and (3.3) we have

$$g_M(\nabla_V W, X) = g_M(\varphi\nabla_V W, \varphi X) + \eta(\nabla_V W)\eta(X)$$
$$= g_M(\nabla_V \varphi W, \varphi X).$$

Using (2.4) and (4.2) we have

$$g_M(\nabla_V W, X) = g_M(T_V \varphi W, \mathcal{B}X) + g_M(\mathcal{H}\nabla_V \varphi W, \mathcal{C}X).$$

Since ∇ is torsion free and $[V, \varphi W] \in \Gamma(kerF_*)$ we obtain

$$g_M(\nabla_V W, X) = g_M(T_V \varphi W, \mathcal{B}X) + g_M(\mathcal{H}\nabla_{\varphi W} V, \mathcal{C}X).$$

Using (2.7) and (3.3) we have

$$g_M(\nabla_V W, X) = g_M(T_V \varphi W, \mathcal{B}X) + g_M(\varphi\nabla_{\varphi W} V, \varphi\mathcal{C}X)$$
$$= g_M(T_V \varphi W, \mathcal{B}X) + g_M(\nabla_{\varphi W}\varphi V, \varphi\mathcal{C}X),$$

here we have used that μ is invariant. Since F is a conformal submersion, using (2.9) and Lemma 2.3 (i) we obtain

$$g_M(\nabla_V W, X) = g_M(T_V \varphi W, \mathcal{B}X) - \frac{1}{\lambda^2} g_M(\mathcal{H}grad \ln \lambda, \varphi W) g_N(F_* \varphi V, F_* \varphi\mathcal{C}X)$$
$$- \frac{1}{\lambda^2} g_M(\mathcal{H}grad \ln \lambda, \varphi V) g_N(F_* \varphi W, F_* \varphi\mathcal{C}X)$$
$$+ g_M(\varphi W, \varphi V)\frac{1}{\lambda^2} g_N(F_*(\mathcal{H}grad \ln \lambda), F_* \varphi\mathcal{C}X)$$
$$+ \frac{1}{\lambda^2} g_N(\nabla_{\varphi W}^F F_* \varphi V, F_* \varphi\mathcal{C}X).$$

Moreover, using Definition 4.1 and (4.4), we obtain

$$g_M(\nabla_V W, X) = g_M(T_V \varphi W, \mathcal{B}X) + g_M(\varphi W, \varphi V)g_M(\mathcal{H}grad \ln \lambda, \varphi\mathcal{C}X)$$
$$+ \frac{1}{\lambda^2} g_N(\nabla_{\varphi W}^F F_* \varphi V, F_* \varphi\mathcal{C}X)$$

which proves $(i) \Leftrightarrow (ii)$. $\square$

From Theorem 4.15, we deduce the following result.

4.16. Theorem. *Let F be a conformal anti-invariant submersion from a cosymplectic manifold $M(\varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then any two conditions below imply the third;*

(i) *$kerF_*$ defines a totally geodesic foliation on M.*
(ii) *λ is a constant on $\Gamma(\mu)$.*
(iii) *$g_N(\nabla_{\varphi W}^F F_* \varphi V, F_* \varphi\mathcal{C}X) = \lambda^2 g_M(T_V \mathcal{B}X, \varphi W)$*

for $V, W \in \Gamma(kerF_*)$ and $X \in \Gamma((kerF_*)^\perp)$.

Proof. For $V, W \in \Gamma(kerF_*)$ and $X \in \Gamma((kerF_*)^\perp)$, from Theorem (4.15) we have

$$g_M(\nabla_V W, X) = g_M(T_V \varphi W, \mathcal{B}X) + g_M(\varphi W, \varphi V)g_M(\mathcal{H}grad \ln \lambda, \varphi\mathcal{C}X)$$
$$+ \frac{1}{\lambda^2} g_N(\nabla_{\varphi W}^F F_* \varphi V, F_* \varphi\mathcal{C}X).$$

Now, if we have (i) and (iii), then we obtain

$$g_M(\varphi W, \varphi V)g_M(\mathcal{H}gradln\lambda, \varphi\mathcal{C}X) = 0.$$

If we take $W = \xi$ in above equation for $\xi \in \Gamma(kerF_*)$, using (3.1), we get

$$g_M(\varphi\xi, \varphi V)g_M(\mathcal{H}gradln\lambda, \varphi\mathcal{C}X) = 0.$$

Hence, λ is a constant on $\Gamma(\varphi\mu)$. Similarly, one can obtain the other assertions. $\qquad\square$

In particular, if $\varphi(kerF_*) = (kerF_*)^\perp$ then we have the following result.

4.17. Corollary. *Let F be a conformal anti-invariant submersion from a cosymplectic manifold $M(\varphi,\xi,\eta,g_M)$ onto a Riemannian manifold (N,g_N) with $(kerF_*)^\perp = \varphi(kerF_*)$. Then the following assertions are equivalent to each other;*

    (i) $kerF_*$ *defines a totally geodesic foliation on M.*
    (ii) $T_V\varphi W = 0$

for $V,W \in \Gamma(kerF_*)$ and $X \in \Gamma((kerF_*)^\perp)$.

Using [25], Theorem 4.12 and Theorem 4.15 we will give the following decomposition theorem for conformal anti-invariant submersions.

4.18. Theorem. *Let $F : M(\varphi,g_M,\xi,\eta) \longrightarrow (N,g_N)$ be a conformal anti-invariant submersion, where $M(\varphi,\xi,\eta,g_M)$ is a cosymplectic manifold and (N,g_N) is a Riemannian manifold. Then the total space M is a locally product manifold of the leaves of $(kerF_*)^\perp$, and $kerF_*$ i.e., $M = M_{(kerF_*)^\perp} \times M_{kerF_*}$, if*

$$-\frac{1}{\lambda^2}g_N(\nabla_X^F F_*\mathcal{C}Y, F_*\varphi V) = g_M(A_X BY - \mathcal{C}Y(\ln\lambda)X, \varphi V)$$
$$+ g_M(\mathcal{H}grad\ln\lambda, \varphi V)g_M(X, \mathcal{C}Y)$$

and

$$-\frac{1}{\lambda^2}g_N(\nabla_{\varphi W}^F F_*\varphi V, F_*\varphi\mathcal{C}X) = g_M(T_V\varphi W, \mathcal{B}X)$$
$$+ g_M(\mathcal{H}gradln\lambda, \mathcal{C}X)g_M(V, W)$$

for $V,W \in \Gamma(kerF_*)$ and $X,Y \in \Gamma((kerF_*)^\perp)$, where $M_{(kerF_*)^\perp}$ and $M_{(kerF_*)}$ are integral manifolds of the distributions $(kerF_*)^\perp$ and $(kerF_*)$. Conversely, if M is a locally product manifold of the form $M_{(kerF_*)^\perp} \times M_{(\ker F_*)}$ then we have

$$\frac{1}{\lambda^2}g_N(\nabla_X^F F_*\mathcal{C}Y, F_*\varphi V) = \mathcal{C}Y(\ln\lambda)g_M(X, \varphi V)$$
$$- g_M(\mathcal{H}grad\ln\lambda, \varphi V)g_M(X, \mathcal{C}Y)$$

and

$$-\frac{1}{\lambda^2}g_N(\nabla_{\varphi W}^F F_*\varphi V, F_*\varphi\mathcal{C}X) = g_M(\mathcal{H}gradln\lambda, \mathcal{C}X)g_M(V, W).$$

From Corollary 4.14 and Corollary 4.17, we have the following theorem.

4.19. Theorem. *Let $F : M(\varphi,\xi,\eta,g_M) \longrightarrow (N,g_N)$ be a conformal anti-invariant submersion with $\varphi kerF_* = (kerF_*)^\perp$, where $M(\varphi,\xi,\eta,g_M)$ is a cosymplectic manifold and (N,g_N) is a Riemannian manifold. Then the total space M is a locally product manifold if $T_V\varphi W = 0$ and $A_X\varphi Y = 0$ for $X,Y \in \Gamma((kerF_*)^\perp)$ and $V,W \in \Gamma(kerF_*)$.*

5. Totally Geodesicness and Harmonicity of The conformal anti-invariant submersions

In this section, we shall examine the totally geodesicness and harmonicity of a conformal anti-invariant submersion. First we give a necessary and sufficient condition for a conformal anti-invariant submersion to be totally geodesic map. Recall that a smooth map F between two Riemannian manifolds is called totally geodesic if $\nabla F_* = 0$ [6].

5.1. Theorem. *Let $F : M(\varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant submersion, where $M(\varphi, \xi, \eta, g_M)$ is a cosymplectic manifold and (N, g_N) is a Riemannian manifold. Then F is a totally geodesic map if and only if*

$$-\nabla_X^F F_* Y_2 = F_*(\varphi(A_X \varphi Y_1 + \mathcal{V}\nabla_X \mathcal{B}Y_2 + A_X \mathcal{C}Y_2)$$

(5.1)
$$+ \mathcal{C}(\mathcal{H}\nabla_X \varphi Y_1 + A_X \mathcal{B}Y_2 + \mathcal{H}\nabla_X \mathcal{C}Y_2))$$

and

(5.2) $\quad T_U \varphi Y_1 + \hat{\nabla}_U \mathcal{B}Y_2 + T_U \mathcal{C}Y_2 = 0, \ T_U \mathcal{B}Y_2 + \mathcal{H}\nabla_U \varphi Y_1 + \mathcal{H}\nabla_U \mathcal{C}Y_2 \in \Gamma(\varphi ker F_*)$

for any $X \in \Gamma((ker F_*)^\perp)$, $U \in \Gamma(ker F_*)$ and $Y = Y_1 + Y_2 \in \Gamma(TM)$, where $Y_1 \in \Gamma(ker F_*)$ and $Y_2 \in \Gamma((ker F_*)^\perp)$.

Proof. By virtue of (2.9) and (3.3) we have

$$(\nabla F_*)(X, Y) = \nabla_X^F F_* Y - F_*(\nabla_X Y)$$
$$= \nabla_X^F F_* Y + F_*(\varphi \nabla_X \varphi Y)$$

for any $X \in \Gamma((ker F_*)^\perp)$ and $Y \in \Gamma(TM)$. Then from (2.6), (2.7) and (4.2) we get

$$(\nabla F_*)(X, Y) = \nabla_X^F F_* Y + F_*(\varphi A_X \varphi Y_1 + \mathcal{B}\mathcal{H}\nabla_X \varphi Y_1 + \mathcal{C}\mathcal{H}\nabla_X \varphi Y_1 + \mathcal{B}A_X \mathcal{B}Y_2$$
$$+ \mathcal{C}A_X \mathcal{B}Y_2 + \varphi \mathcal{V}\nabla_X \mathcal{B}Y_2 + \varphi A_X \mathcal{C}Y_2 + \mathcal{B}\mathcal{H}\nabla_X \mathcal{C}Y_2 + \mathcal{C}\mathcal{H}\nabla_X \mathcal{C}Y_2)$$

for any $Y = Y_1 + Y_2 \in \Gamma(TM)$, where $Y_1 \in \Gamma(ker F_*)$ and $Y_2 \in \Gamma((ker F_*)^\perp)$. Thus taking into account the vertical parts, we find

$$(\nabla F_*)(X, Y) = \nabla_X^F F_* Y_2 + F_*(\varphi(A_X \varphi Y_1 + \mathcal{V}\nabla_X \mathcal{B}Y_2 + A_X \mathcal{C}Y_2)$$
$$+ \mathcal{C}(\mathcal{H}\nabla_X \varphi Y_1 + A_X \mathcal{B}Y_2 + \mathcal{H}\nabla_X \mathcal{C}Y_2)).$$

Thus $(\nabla F_*)(X, Y) = 0$ if and only if the equation (5.1) is satisfied.
Now, for any $U \in \Gamma(ker F_*)$ and $Y \in \Gamma(TM)$, from (2.9) and (3.3) we have

$$(\nabla F_*)(U, Y) = \nabla_U^F F_* Y - F_*(\nabla_U Y)$$
$$= F_*(\varphi \nabla_U \varphi Y).$$

Then from (2.4), (2.5) and (4.2) we get

$$(\nabla F_*)(U, Y) = F_*(\varphi T_U \varphi Y_1 + \mathcal{B}\mathcal{H}\nabla_U \varphi Y_1 + \mathcal{C}\mathcal{H}\nabla_U \varphi Y_1 + \mathcal{B}T_U \mathcal{B}Y_2$$
$$+ \mathcal{C}T_U \mathcal{B}Y_2 + \varphi \hat{\nabla}_U \mathcal{B}Y_2 + \varphi T_U \mathcal{C}Y_2 + \mathcal{B}\mathcal{H}\nabla_U \mathcal{C}Y_2 + \mathcal{C}\mathcal{H}\nabla_U \mathcal{C}Y_2)$$

for any $Y = Y_1 + Y_2 \in \Gamma(TM)$, where $Y_1 \in \Gamma(ker F_*)$ and $Y_2 \in \Gamma((ker F_*)^\perp)$. Thus taking into account the vertical parts, we find

$$(\nabla F_*)(U, Y) = F_*(\varphi(T_U \varphi Y_1 + \hat{\nabla}_U \mathcal{B}Y_2 + T_U \mathcal{C}Y_2)$$
$$+ \mathcal{C}(T_U \mathcal{B}Y_2 + \mathcal{H}\nabla_U \varphi Y_1 + \mathcal{H}\nabla_U \mathcal{C}Y_2)).$$

Thus $(\nabla F_*)(U, Y) = 0$ if and only if the equation (5.2) is satisfied. Hence proof is complete. $\qquad \square$

The following corollary comes from [4, Lemma 4.5.1, page 119], therefore we omit its proof.

5.2. Corollary. *Let F be a conformal anti-invariant submersion from a cosymplectic manifold $M(\varphi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). F totally geodesic map if and only if*

(a) *$T_U \varphi V = 0$ and $\mathcal{H}\nabla_U \varphi V \in \Gamma(\varphi ker F_*)$,*
(b) *F is a horizontally homotetic map,*
(c) *$A_Z \varphi V = 0$ and $\mathcal{H}\nabla_Z \varphi V \in \Gamma(\varphi ker F)$*

for $X, Y, Z \in \Gamma((ker F_*)^\perp)$ and $U, V \in \Gamma(ker F_*)$.

Now, we examine the harmonicity of the conformal anti-invariant submersions. We know that a smooth map F is harmonic if and only if $trace(\nabla \pi_*) = 0$ [6]. First, we give the following Lemma.

5.3. Theorem. *Let* $F : M^{2(m+r)+1}(\varphi, \xi, \eta, g_M) \longrightarrow (N^{m+2r}, g_N)$ *be a conformal anti-invariant submersion with* $\dim(\mu) = 2r$ *and* $\dim(ker F_*) = m + 1$, *where* $M(\varphi, \xi, \eta, g_M)$ *is a cosymplectic manifold and* (N, g_N) *is a Riemannian manifold. Then the tension field τ of F is*

$$(5.3) \qquad \tau(F) = -m F_*(\mu^{ker F_*}) + (2 - m - 2r) F_*(grad \ln \lambda)$$

where $\mu^{ker F_*}$ *is the mean curvature vector field of the distribution of* $ker F_*$.

Proof. Let $\{e_1, ..., e_m, \varphi e_1, ..., \varphi e_m, \xi, \mu_1, ..., \mu_r, \varphi \mu_1, ..., \varphi \mu_r\}$ be orthonormal basis of $\Gamma(TM)$ such that $\{e_1, ..., e_m, \xi\}$ be orthonormal basis of $\Gamma(ker F_*)$, $\{\varphi e_1, ..., \varphi e_m\}$ be orthonormal basis of $\Gamma(\varphi ker F_*)$ and $\{\mu_1, ..., \mu_r, \varphi \mu_1, ..., \varphi \mu_r\}$ be orthonormal basis of $\Gamma(\mu)$. Then the trace of second fundamental form (restriction to $ker F_* \times ker F_*$) is given by

$$trace^{ker F_* \nabla F_*} = \sum_{i=1}^{m} (\nabla F_*)(e_i, e_i) + (\nabla F_*)(\xi, \xi).$$

Then using (2.9) and (4.6) we obtain

$$(5.4) \qquad trace^{ker F_* \nabla F_*} = -m F_*(\mu^{ker F_*}) - F_*(T_\xi \xi).$$
$$= -m F_*(\mu^{ker F_*}).$$

In a similar way, we have

$$trace^{(ker F_*)^\perp} \nabla F_* = \sum_{i=1}^{m} (\nabla F_*)(\varphi e_i, \varphi e_i) + \sum_{i=1}^{2r} (\nabla F_*)(\mu_i, \mu_i).$$

Using Lemma 2.3 we arrive at

$$trace^{(ker F_*)^\perp} \nabla F_* = \sum_{i=1}^{m} 2 g_M(grad \ln \lambda, \varphi e_i) F_* \varphi e_i - m F_*(grad \ln \lambda)$$
$$+ \sum_{i=1}^{2r} 2 g_M(grad \ln \lambda, \mu_i) F_* \mu_i - 2r F_*(grad \ln \lambda).$$

Since F is a conformal anti-invariant submersion, for $p \in M$ and $1 \le i \le m$, $1 \le h \le r$ $\{\frac{1}{\lambda(p)} F_{*p}(\varphi e_i), \frac{1}{\lambda(p)} F_{*p}(\mu_h)\}$ is an orthonormal basis of $T_{F(p)} N$ we derive

$$trace^{(ker F_*)^\perp} \nabla F_* = \sum_{i=1}^{m} 2 g_N(F_*(grad \ln \lambda), \frac{1}{\lambda} F_* \varphi e_i) \frac{1}{\lambda} F_* \varphi e_i - m F_*(grad \ln \lambda)$$
$$+ \sum_{i=1}^{2r} 2 g_N(F_*(grad \ln \lambda), \frac{1}{\lambda} F_* \mu_i) \frac{1}{\lambda} F_* \mu_i - 2r F_*(grad \ln \lambda)$$
$$(5.5) \qquad = (2 - m - 2r) F_*(grad \ln \lambda).$$

Then proof follows from (5.4) and (5.5). $\qquad\qquad\qquad\qquad\qquad\qquad \square$

From Theorem 5.3 we deduce that:

5.4. Theorem. *Let* $F : M^{2(m+r)+1}(\varphi, \xi, \eta, g_M) \longrightarrow (N^{(m+2r)}, g_N)$ *be a conformal anti-invariant submersion, where* $M(\varphi, \xi, \eta, g_M)$ *is a cosymplectic manifold and* (N, g_N) *is a Riemannian manifold. Then any two conditions below imply the third:*

(i) *F is harmonic*

(ii) *The fibres are minimal*

(iii) *F is a horizontally homothetic map.*

Proof. From (5.3), we have

$$\tau(F) = -mF_*(\mu^{kerF_*}) + (2 - m - 2r)F_*(grad \ln \lambda).$$

Now, if we have (i) and (iii) then the fibres are minimal. Similarly, one can obtain the other assertions. □

We also have the following result.

5.5. Corollary. *Let $F : M^{2(m+r)+1}(\varphi, \xi, \eta, g_M) \longrightarrow (N^{(m+2r)}, g_N)$ be a conformal anti-invariant submersion, where $M^{(2(m+r)+1)}(\varphi, \xi, \eta, g_M)$ is a cosymplectic manifold and $(N^{(m+2r)}, g_N)$ is a Riemannian manifold. F is harmonic if and only if the fibres are minimal and F is a horizontally homothetic map.*

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