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Dual-Variable Functions on Time Scale*

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ABSTRACT: In this paper, we study the concept of dual-variable functions parameterized by the product of two arbitrary time scales. Firstly, we give some preliminaries of the dual numbers and dual-variable functions. Secondly, we introduce some properties of the time scales. In the main result, we investigate the limit, derivative, partial differentiation and Cauchy-Riemann equation of the dual-variable functions on time scales.

Keywords: Cauchy-Riemann equation, dual space, dual-variable functions, time scales.

Zaman Skalasında Dual Değişkenli Fonksiyonlar

ÖZET: Bu çalışmada, iki keyfi zaman skalasının çarpımı ile parametrelendirilmiş dual değişkenli fonksiyonlar konusunu inceledik. İlk olarak dual sayıları ve dual değişkenli fonksiyonların bazı özelliklerini verdik. İkinci olarak zaman skalasının bazı özelliklerini tanıttık. Ana sonuçlar kısmında dual değişkenli fonksiyonların limit, türev, kısmi diferensiyel ve Cauchy-Riemann denklemini araştırdık.

Anahtar Kelimeler: Cauchy-Riemann denklemi, dual değişkenli fonksiyonlar, dual uzay, zaman skalası.

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INTRODUCTION

W. K. Clifford first proposed the dual numbers in 1873. A dual number z is an ordered pair of real numbers (x, y) combined with the real unit 1 and the dual unit ε, with $\varepsilon^2 = 0$ and $\varepsilon \neq 0$. It is generally given in the form of $z = x + \varepsilon y$, where $\varepsilon^2 = 0$ and $\varepsilon \neq 0$. The dual numbers therefore constitute the elements of the $\mathbb{D} = \{z = x + \varepsilon y \ | \varepsilon^2 = 0 \text{ and } \varepsilon \neq 0\}$ set, generated by 1 and ε (Yaglom, 1969).

The definitions of the two operations in the $\mathbb{D}$ set are given below:

The addition operation in the set $\mathbb{D}$ is '+' and defined by $z_1 + z_2 = (x_1 + x_2) + \varepsilon(y_1 + y_2)$, while the multiplication operation in the set $\mathbb{D}$ is '.' and defined by $z_1.z_2 = x_1.x_2 + \varepsilon(x_1.y_2 + x_2.y_1)$. The multiplication operation is commutative, associative and distributes over addition. W. K. Clifford showed that since dual numbers do not have any inverse elements, they form an algebra and not a field. Therefore, the divisors of zero in the algebra of dual numbers is $\varepsilon y (y \in \mathbb{R})$. No εy numbers have an inverse in the algebra of dual numbers (Yaglom, 1969). The conjugate of the dual number, $z = x + \varepsilon y$, is represented by $\overline{z}$ and defined by $\overline{z} = x - \varepsilon y$; hence, $z.\overline{z} = x^2$. The division of the dual number "$z_1 = x_1 + \varepsilon y_1$" by the dual number "$z_2 = x_2 + \varepsilon y_2$" becomes $\frac{z_1}{z_2} = \frac{z_1.\overline{z_2}}{z_2.\overline{z_2}} = \frac{x_1}{x_2} + \varepsilon \frac{x_2 y_1 - x_1 y_2}{x_2^{\ 2}}$ where $x_2 \neq 0$. Hence, if $x_2^2 \neq 0$, the division $\frac{z_1}{z_2}$ becomes possible and unambiguous. The modulus of the dual number z is $|z|$ and defined by $|z|^2 = z.\overline{z} = x^2$. In other words, for a dual number "$z = x + \varepsilon y$", $|z|$ is replaced by x to allow the modulus of the dual number to be positive, zero, or negative. The dual plane is the set of all dual number $z \in \mathbb{D}$. The $d(z, z_1)$ is the distance between two points of the dual plane as z and z_1 and defined by $d(z,z_1)=|z_1-z|$, $d^2(z,z_1)=(z_1-z)(\overline{z_1}-\overline{z})$ or $d(z,z_1)=|x_1-x|$. For more details and other algebraic properties, see also: Yaglom, 1969; Onder and Ugurlu, 2013; Study, 1901; Veldkamp, 1976. In 1891, E. Study regarded using associative algebra as an ideal way to describe the group of motions of three-dimensional space. However, up until now, only a few pursued the mathematical study of dual-variable functions. An early attempt was made by Kramer (1930), followed by Ercan and Yüce who, later in 2011, obtained generalized Euler's and De Moivre's formulas for functions with dual quaternion variable. In addition, Messelmi (2013) developed a theory that was inspired by the complex analysis of dual functions in which the notion of holomorphic dual functions was introduced and a general representation of holomorphic dual functions was achieved. The researcher also offered other properties that can be used in the analysis of dual functions. The time scale calculus theory, which is of great importance and use to the unification of discrete and indiscrete analyses and used in the mathematical modelling of several important dynamic processes as well, was developed by Hilger (1990) and Aulbach and Hilger (1990) at an earlier date. The preliminaries for the timescale can be established by referring to Bohner and Peterson (2001). The paper published by Bohner and Guseinov (Bohner and Guseinov, 2006) focused on the complex functions on the products of two time scales. The authors proposed a time scale complex plane, $\mathbb{T}_1+i\mathbb{T}_2$, by taking $\mathbb{T}_1$ and $\mathbb{T}_2$ as time scales and investigated the concept of analyticity for the complex-valued functions of a complex time scale variable and derived a time scale counterpart for the classical Cauchy-Riemann equations. In like manner, in this paper, we introduce the concept of dual-valued functions and investigate the derivation of dual-valued functions on the products of two timescales. Moreover, we offer Cauchy-Riemann equations of the dual-variable functions on timescales.

MATERIAL AND METHOD

Here, we will give some preliminaries of dual-variable functions.

Definition 2.1. Ω is a dual subset of the dual plane $\mathbb{D}$ if there exists a subset $O \subset \mathbb{R}$ such that $\Omega = Ox\mathbb{R}$. O is called the generator of Ω**.**

Definition 2.2. A dual-variable function is a mapping from a subset $\Omega \subset \mathbb{D}$ to $\mathbb{D}$.

Definition 2.3. A dual-variable function f defined on subset $\Omega \subset \mathbb{D}$ is called homogeneous dual functions if $f\left(real\left(z\right)\right) \in \mathbb{R}$. Let Ω be an open subset of $\mathbb{D}$ and the function $f : \Omega \to \mathbb{D}$, $z_0 = x_0 + \varepsilon y_0 \to f\left(z_0\right)$ be a dual function.

Definition 2.4. The dual-variable function f is continuous at z_0 if $\lim\limits_{z \to z_0} f\left(z\right) = f\left(z_0\right)$.

Definition 2.5. The dual-variable function is continuous in $\Omega \subset \mathbb{D}$ if it is continuous at every point of Ω**.**

Definition 2.6. The dual-variable function f is said to be differentiable at $z_0 = x_0 + \varepsilon y_0$, if the limit below exists $\frac{df}{dz}\left(z_0\right) = \lim\limits_{z \to z_0} \frac{f\left(z\right) - f\left(z_0\right)}{z - z_0}$ is called the derivative of f at the point z_0, if f is differentiable for all points in a neighborhood of the point z_0 then f is called holomorphic at z_0.

Definition 2.7. The dual-variable function f is holomorphic in $\Omega \subset \mathbb{D}$ if it is holomorphic at every point of Ω**.**

The definition of derivative in dual sense has to be treated with a little more care that its real companion.

Lemma 2.1. Suppose f and g are differentiable at $z\in\mathbb{D}$ and that $c\in\mathbb{D}, n\in\mathbb{Z}$ and it is differentiable at $g(z)$. Then following equations are satisfied.

1) $\dfrac{d(f+cg)}{dz}=\dfrac{df}{dz}+c\dfrac{dg}{dz}$ 2) $\dfrac{d(f.g)}{dz}=\dfrac{df}{dz}g+f\dfrac{dg}{dz}$

3) $\dfrac{d\left(\dfrac{f}{g}\right)}{dz}=\dfrac{\dfrac{df}{dz}g-f\dfrac{dg}{dz}}{g^2},\quad g\neq 0$ 4) $\dfrac{d(hog)}{dz}=\dfrac{dh}{dz}(g)\dfrac{dg}{dz}$

Theorem 2.1. Let f be a dual function in $\Omega\subset\mathbb{D}$ which can be written in terms of its real and dual parts as $f(z)=\varphi(x,y)+\varepsilon\psi(x,y)$. f is holomorphic in $\Omega\subset\mathbb{D}$ if and only if the derivative of f satisfies $\dfrac{df}{dz}=\dfrac{\partial f}{\partial x}=\dfrac{\partial\varphi}{\partial x}+\dfrac{\partial\psi}{\partial x}\varepsilon$.

Lemma 2.2. Let f be a dual-variable function in $\Omega\subset\mathbb{D}$ which can be written in terms of its real and dual parts as $f=\varphi+\varepsilon\psi$ and suppose that the partial derivatives of f exist. Then

1) f is holomorphic in $\Omega\subset\mathbb{D}$ if and only if its partial derivatives satisfy

$$\varepsilon\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y},$$

2) f is holomorphic in $\Omega\subset\mathbb{D}$ if and only if the following formula holds

$$\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\varphi}{\partial y}=0.$$

The other properties of the dual-variable functions can be obtain from the reference (Messelmi, 2013). Now, we will give some preliminaries of time scale.

Definition 2.8. Let $\mathbb{T}$ be a timescale. The forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by $\sigma(t)=\inf\{s\in\mathbb{T}:s\ \rangle t,\forall t\in\mathbb{T}\}$ and the backward jump operator

$g:\mathbb{T}\to\mathbb{T}$ is defined by $g(t)=\sup\{s\in\mathbb{T}:s\langle t,\ \forall t\in\mathbb{T}\}$.

Definition 2.9. The sets $\mathbb{T}^k$ and $\mathbb{T}_k$ are derived from the time scale $\mathbb{T}$. If $\mathbb{T}$ has a left- scattered maximum t_1, then $\mathbb{T}^k=\mathbb{T}-\{t_1\}$, otherwise $\mathbb{T}^k=\mathbb{T}$. If $\mathbb{T}$ has right- scattered minimum t_2, then $\mathbb{T}_k=\mathbb{T}-\{t_2\}$, otherwise $\mathbb{T}_k=\mathbb{T}$.

Definition 2.10. Let $\mathbb{T}$ be an arbitrary time scale and $t\in\mathbb{T}^k$. The delta-derivative of f is given by $f^{\Delta}(t)=\lim\limits_{s\to t}\dfrac{f(\sigma(t))-f(s)}{\sigma(t)-s}$.

Theorem 2.2. Assume that $f,g:\mathbb{T}\to\mathbb{R}$ are delta differentiable at $t\in\mathbb{T}^k$. Then for $a,b\in\mathbb{R}$, the equations satisfy following equations:

1) $(af+bg)^{\Delta}(t)=af^{\Delta}(t)+bg^{\Delta}(t)$

2) $(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g(\sigma(t))$

3) $\left(\dfrac{f}{g}\right)^{\Delta}(t)=\dfrac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}\qquad g\neq 0.$

RESULTS AND DISCUSSION

For given time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, let us set

$$\mathbb{T}_1+\varepsilon\mathbb{T}_2:\{z=x+\varepsilon y:x\in\mathbb{T}_1,y\in\mathbb{T}_2\},$$

where $\varepsilon\neq 0$ and $\varepsilon^2=0$ is the dual unit. The set $\mathbb{T}_1+\varepsilon\mathbb{T}_2$ is called the timescale dual plane. Any function $f:\mathbb{T}_1+\varepsilon\mathbb{T}_2\to\mathbb{D}$ can be represented in the form

$$f(z)=\varphi(x,y)+\varepsilon\psi(x,y)\quad\text{for}\quad z=x+\varepsilon y\in\mathbb{T}_1+\varepsilon\mathbb{T}_2,$$

where $\varphi:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$ is the real part of f and $\psi:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$ is the dual part of f. Let σ_1 and σ_2 be the forward jump operators for $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. For

$z=x+\varepsilon y\in\mathbb{T}_1+\varepsilon\mathbb{T}_2$, let us set $z^{\sigma_1}=\sigma_1(t)+\varepsilon y$ and $z^{\sigma_2}=x+\varepsilon\sigma_2(y)$. Let ρ_1 and ρ_2 be the backward jump operators for $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. For $z=x+\varepsilon y\in\mathbb{T}_1+\varepsilon\mathbb{T}_2$ let us set $z^{\rho_1}=\rho_1(t)+\varepsilon y$ and $z^{\rho_2}=x+\varepsilon\rho_2(y)$.

Theorem 3.1. Suppose f and g are differentiable at $z\in\mathbb{T}_1+\varepsilon\mathbb{T}_2$ and that $c\in\mathbb{D},\ n\in\mathbb{Z}$ and

1) $(f+cg)^{\Delta}(z)=f^{\Delta}(z)+cg^{\Delta}(z)$

2) $(fg)^{\Delta}(z)=f^{\Delta}(z)g(z)+f(\sigma(z))g^{\Delta}(z)=f(z)g^{\Delta}(z)+f^{\Delta}(z)g(\sigma(z))$

3) $\left(\dfrac{f}{g}\right)^{\Delta}(z)=\dfrac{f^{\Delta}(z)g(z)-f(z)g^{\Delta}(z)}{g(z)g(\sigma(z))}\qquad g\neq 0$

Proof:

1) $$(f+cg)^{\Delta}(z)=\lim_{z\to z_0}\frac{(f+cg)(\sigma(z))-(f+cg)(z_0)}{\sigma(z)-z_0}$$

$$=\lim_{z\to z_0}\frac{\left[f(\sigma(z))-f(z_0)\right]+c.\left[g(\sigma(z))-g(z_0)\right]}{\sigma(z)-z_0}$$

$$=\lim_{z\to z_0}\frac{\left[f(\sigma(z))-f(z_0)\right]}{\sigma(z)-z_0}+\lim_{z\to z_0}\frac{\left[g(\sigma(z))-g(z_0)\right]}{\sigma(z)-z_0}$$

$$=f^{\Delta}(z)+cg^{\Delta}(z)$$

2) $$(f.g)^{\Delta}(z)=\lim_{z_0\to z}\frac{f.g(\sigma(z))-f.g(z_0)}{\sigma(z)-z_0}$$

$$=\lim_{z_0\to z}\frac{f.g(\sigma(z))-f.g(z_0)+f(\sigma(z)).g(z_0)-f(\sigma(z)).g(z_0)}{\sigma(z)-z_0}$$

$$=f(\sigma(z)).\lim_{z_0\to z}\frac{g(\sigma(z))-g(z_0)}{\sigma(z)-z_0}+g(z).\lim_{z_0\to z}\frac{f(\sigma(z))-f(z_0)}{\sigma(z)-z_0}$$

$$=f(\sigma(z)).g^{\Delta}(z)+g(z).f^{\Delta}(z)$$

3) The proof is obvious from (2).

Definition 3.1. We say that a dual-valued function $f:\mathbb{T}_1+\varepsilon\mathbb{T}_2\to\mathbb{D}$ is delta differentiable at point $z_0=x_0+\varepsilon y_0\in\mathbb{T}_1^k+\varepsilon\mathbb{T}_2^k$ if there exist a dual number $A=A_1+\varepsilon A_2$ (depending in general on z_0) such that

$$f(z_0)-f(z)=A(z_0-z)+\alpha(z_0-z) \tag{1}$$

$$f(z_0^{\sigma_1})-f(z)=A(z_0^{\sigma_1}-z)+\beta(z_0^{\sigma_1}-z) \tag{2}$$

$$f(z_0^{\sigma_2})-f(z)=A(z_0^{\sigma_2}-z)+\gamma(z_0^{\sigma_2}-z) \tag{3}$$

for all $z\in U_\delta(z_0)$, where $U_\delta(z_0)$ is a $\delta-$neighborhood of z_0 in $\mathbb{T}_1+\varepsilon\mathbb{T}_2$, $\alpha=\alpha(z_0,z)$, $\beta=\beta(z_0,z)$ and $\gamma=\gamma(z_0,z)$ are defined for $z\in U_\delta(z_0)$, they are equal to zero at $z_0=z$ and $\lim\limits_{z\to z_0}\alpha(z_0,z)=\lim\limits_{z\to z_0}\beta(z_0,z)=\lim\limits_{z\to z_0}\gamma(z_0,z)=0$.

Then the number A is called the delta derivative of f at z_0, and is denoted by $f^\Delta(z_0)$.

Theorem 3.2. Let the function $f:\mathbb{T}_1+\varepsilon\mathbb{T}_2\to\mathbb{D}$ have the form $f(z)=\varphi(x,y)+\varepsilon\psi(x,y)$ for $z=x+\varepsilon y\in\mathbb{T}_1+\varepsilon\mathbb{T}_2$. Also for the function f which is delta differentiable at the point $z_0=x_0+\varepsilon y_0\in\mathbb{T}_1^k+\varepsilon\mathbb{T}_2^k$ has the functions $\varphi(x,y)$ and $\psi(x,y)$ be completely delta differentiable at the point $z_0=x_0+\varepsilon y_0$ and satisfy the Cauchy-Riemann equations $\dfrac{\Delta\varphi}{\Delta_1 x}=\dfrac{\Delta\psi}{\Delta_2 y}$ and $\dfrac{\Delta\varphi}{\Delta_2 y}=0$ at $z_0=x_0+\varepsilon y_0$. If these equations are satisfied, then $f^\Delta(z_0)$ can be represented as $f^\Delta(z_0)=\dfrac{\Delta\varphi}{\Delta_1 x}+\varepsilon\dfrac{\Delta\psi}{\Delta_1 x}$.

First Method for Proof :

Assume that f is $\Delta-$differentiable at $z_0=x_0+\varepsilon y_0$ with $f^\Delta(z_0)=A$. Then

Eq. (1), Eq. (2) and Eq. (3) are satisfied. Let be $f(z)=\varphi(x,y)+\varepsilon\psi(x,y)$, $A=A_1+\varepsilon A_2$, $\alpha=\alpha_1+\varepsilon\alpha_2$, $\beta=\beta_1+\varepsilon\beta_2$, $\gamma=\gamma_1+\varepsilon\gamma_2$. If we replacement the above equations on the Eq. (1) we obtain following equations by multiplication of dual numbers.

$$f(z_0)-f(z)=A(z_0-z)+\alpha(z_0-z)$$

$$[\varphi(x_0,y_0)+\varepsilon\psi(x_0,y_0)]-[\varphi(x,y)+\varepsilon\psi(x,y)]=(A_1+\varepsilon A_2)(x_0+\varepsilon y_0-x-\varepsilon y)$$
$$+(\alpha_1+\varepsilon\alpha_2)(x_0+\varepsilon y_0-x-\varepsilon y)$$

$$[\varphi(x_0,y_0)-\varphi(x,y)]+\varepsilon[\psi(x_0,y_0)-\psi(x,y)]=(A_1+\varepsilon A_2)[(x_0-x)+\varepsilon(y_0-y)]$$
$$+(\alpha_1+\varepsilon\alpha_2)[(x_0-x)+\varepsilon(y_0-y)]$$

$$[\varphi(x_0,y_0)-\varphi(x,y)]+\varepsilon[\psi(x_0,y_0)-\psi(x,y)]=A_1(x_0-x)+\alpha_1(x_0-x)$$
$$+\varepsilon[A_2(x_0-x)+A_1(y_0-y)+\alpha_2(x_0-x)+\alpha_1(y_0-y)]$$

We get from above equations, solving the real and dual parts of both sides in each of these equations

$$\varphi(x_0,y_0)-\varphi(x,y)=A_1(x_0-x)+\alpha_1(x_0-x)$$

$$\psi(x_0,y_0)-\psi(x,y)=A_2(x_0-x)+A_1(y_0-y)+\alpha_2(x_0-x)+\alpha_1(y_0-y)$$

Hence, taking into account that $\alpha_{1,2}\to 0$, $\beta_{1,2}\to 0$, $\gamma_{1,2}\to 0$ as $x\to x_0$ and $y\to y_0$, we get that the functions φ and ψ are completely delta differentiable

$$A_1=\frac{\partial\varphi(x_0,y_0)}{\Delta_1 x},\qquad A_2=\frac{\partial\psi(x_0,y_0)}{\Delta_1 x},\qquad A_1=\frac{\partial\psi(x_0,y_0)}{\Delta_2 y}.$$

Thus we can obtain $\dfrac{\Delta\varphi}{\Delta_1 x}=\dfrac{\Delta\psi}{\Delta_2 y}$. Similarly, if we calculate the Eq. (2) and Eq. (3), we get some results.

Second Method for Proof:

We can proof the Cauchy-Riemann equation with the definition of derivation in

(Messelmi, 2013). Let take $z_0^{\sigma_1} = \sigma_1(x_0) + \varepsilon y_0$ and $z^{\sigma_2} = x_0 + \varepsilon\sigma_2(y_0) \in \mathbb{T}_1 \times \mathbb{T}_2$. If we take derivation of dual variable function f at the point $z_0^{\sigma_1}$, we get the following equations:

$$\frac{df}{\Delta_1 z}(z_0^{\sigma_1}) = \lim_{z \to z_0^{\sigma_1}} \frac{f(z) - f(z_0^{\sigma_1})}{z - z_0^{\sigma_1}}$$

$$= \lim_{z \to z_0^{\sigma_1}} \frac{f(x+\varepsilon y) - f(\sigma_1(x_0) + \varepsilon y_0)}{(x+\varepsilon y) - (\sigma_1(x_0) + \varepsilon y_0)}$$

$$= \lim_{z \to z_0^{\sigma_1}} \frac{[\varphi(x,y) + \varepsilon\psi(x,y)] - [\varphi(\sigma_1(x_0), y_0) + \varepsilon\psi((\sigma_1(x_0), y_0)]}{(x+\varepsilon y) - (\sigma_1(x_0) + \varepsilon y_0)}$$

$$= \lim_{z \to z_0^{\sigma_1}} \frac{\varphi(x,y) - \varphi(\sigma_1(x_0), y_0)}{(x - \sigma_1(x_0))} + \varepsilon \lim_{z \to z_0^{\sigma_1}} \frac{\psi(x,y) - \psi(\sigma_1(x_0), y_0)}{(x - \sigma_1(x_0))}$$
$$-\varepsilon \lim_{z \to z_0^{\sigma_1}} \frac{[\varphi(x,y) - \varphi(\sigma_1(x_0), y_0)](y - yo)}{(x - \sigma_1(x_0))^2}$$

$$= \frac{\partial\varphi}{\Delta_1 x} + \varepsilon \frac{\partial\psi}{\Delta_1 x} - \varepsilon \lim_{z \to z_0^{\sigma_1}} \frac{[\varphi(x,y) - \varphi(\sigma_1(x_0), y_0)]}{(x - \sigma_1(x_0))} \frac{(y - yo)}{x - \sigma_1(x_0)} = \frac{\partial\varphi}{\Delta_1 x} + \varepsilon \frac{\partial\psi}{\Delta_1 x}$$

Here, if the limit exist, thus following rule must satisfied

$$\lim_{z \to z_0^{\sigma_1}} \frac{\varphi(x,y) - \varphi(\sigma_1(x_0), y_0)}{(x - \sigma_1(x_0))} = \frac{\partial\varphi}{\Delta_1 x} = 0$$

On the other hand if we take the derivation of at $z_0^{\sigma_2}$, we get

$$\frac{df}{\Delta_2 z}(z_0^{\sigma_2}) = \lim_{z \to z_0^{\sigma_2}} \frac{f(z) - f(z_0^{\sigma_2})}{z - z_0^{\sigma_2}}$$

$$= \lim_{z \to z_0^{\sigma_2}} \frac{f(x+\varepsilon y) - f(x_0 + \varepsilon\sigma_2(y_0))}{(x+\varepsilon y) - (x_0 + \varepsilon\sigma_2(y_0))}$$

$$= \lim_{z \to z_0^{\sigma_2}} \frac{[\varphi(x,y) + \varepsilon\psi(x,y)] - [\varphi(x_0, \sigma_2(y_0)) + \varepsilon\psi(x_0, \sigma_2(y_0))]}{(x - x_0) + \varepsilon(y - \sigma_2(y_0))}$$

$$= \lim_{z \to z_0^{\sigma_2}} \frac{[\varphi(x,y) - \varphi(x_0, \sigma_2(y_0))](x - x_0) + \varepsilon[\psi(x,y) - \psi(x_0, \sigma_2(y_0))](x - x_0)}{(x - x_0)^2}$$
$$-\varepsilon \frac{[\varphi(x,y) - \varphi(x_0, \sigma_2(y_0))](y - \sigma_2(y_0))}{(x - x_0)^2}$$

$$= \lim_{z\to z_0^{\sigma 2}} \frac{\varphi(x,y)-\varphi(x_0,\sigma_2(y_0))}{y-\sigma_2(y_0)} . \frac{y-\sigma_2(y_0)}{(x-x_0)}$$

$$+\varepsilon \lim_{z\to z_0^{\sigma 2}} \frac{\psi(x,y)-\psi(x_0,\sigma_2(y_0))}{y-\sigma_2(y_0)} . \frac{y-\sigma_2(y_0)}{x-x_0}$$

$$-\varepsilon \lim_{z\to z_0^{\sigma 2}} \frac{\varphi(x,y)-\varphi(x_0,\sigma_2(y_0))}{y-\sigma_2(y_0)} . \frac{(y-\sigma_2(y_0))^2}{(x-x_0)^2}$$

If the first limit exist, so the following limit must be zero.

$$\lim_{z\to z_0^{\sigma 2}} \frac{\varphi(x,y)-\varphi(x_0,\sigma_2(y_0))}{y-\sigma_2(y_0)} = \frac{\partial\varphi}{\Delta_2 y} = 0$$

If the second limit exist, so the following limit must be zero.

$$\lim_{z\to z_0^{\sigma 2}} \frac{\psi(x,y)-\psi(x_0,\sigma_2(y_0))}{y-\sigma_2(y_0)} = \frac{\partial\psi}{\Delta_2 y} = 0$$

If the third limit exist, then we have

$$\lim_{z\to z_0^{\sigma 2}} \frac{\varphi(x,y)-\varphi(x_0,\sigma_2(y_0))}{y-\sigma_2(y_0)} = \frac{\partial\varphi}{\Delta_2 y} = 0 .$$

Thus the result of above equation is $\frac{\partial\varphi}{\Delta_2 y} = \frac{\partial\psi}{\Delta_2 y} = 0$. Because of above results, we get Cauchy-Riemann equation depend on x variable as

$$\frac{df}{\Delta z} = \frac{df}{\Delta_1 z} = \frac{\partial f}{\Delta_1 x} = \frac{\partial\varphi}{\Delta_1 x} + \varepsilon \frac{\partial\psi}{\Delta_1 x} \qquad \left(\frac{\partial\varphi}{\Delta_1 x} = 0\right)$$

Lemma 3.1. Let f be a dual function in $\Omega \subset \mathbb{D}$, which can be written in terms of its real and dual parts as $f = \varphi + \varepsilon\psi$ and suppose that the partial derivatives of f exist on product of two time scales. Then

1) f is holomorphic in $\Omega \subset \mathbb{D}$ if and only if its partial derivatives on time scales satisfy $\varepsilon \frac{\partial f}{\Delta_1 x} = \frac{\partial f}{\Delta_2 y}$.

2) f is holomorphic in $\Omega \subset \mathbb{D}$ if and only if the following formula holds

$$\frac{\partial\varphi}{\Delta_1 x} = \frac{\partial\psi}{\Delta_2 y}, \qquad \frac{\partial\varphi}{\Delta_2 y} = 0$$

Proof: We can assert that the total differential on time scales of the function f can be written as

$$df = (\frac{\partial \varphi}{\Delta_1 x} + \frac{\partial \psi}{\Delta_1 x}\varepsilon)\ d(x + \varepsilon y)$$

$$\frac{\partial f}{\Delta_1 x}dx + \frac{\partial f}{\Delta_2 y}dy = (\frac{\partial \varphi}{\Delta_1 x} + \frac{\partial \psi}{\Delta_1 x}\varepsilon)\ (dx + \varepsilon dy)$$

$$\frac{\partial f}{\Delta_1 x}dx + \frac{\partial f}{\Delta_2 y}dy = (\frac{\partial \varphi}{\Delta_1 x} + \frac{\partial \psi}{\Delta_1 x}\varepsilon)dx + \varepsilon\frac{\partial \varphi}{\Delta_1 x}dy$$

$$\frac{\partial f}{\Delta_1 x} = \frac{\partial \varphi}{\Delta_1 x} + \varepsilon\frac{\partial \psi}{\Delta_1 x} \tag{4}$$

$$\frac{\partial f}{\Delta_2 y} = \varepsilon\frac{\partial \varphi}{\Delta_1 x}. \tag{5}$$

If we multiply Equation (4) with ε

$$\varepsilon.\frac{\partial f}{\Delta_1 x} = \varepsilon.(\frac{\partial \varphi}{\Delta_1 x} + \varepsilon\frac{\partial \psi}{\Delta_1 x})$$

$$\varepsilon.\frac{\partial f}{\Delta_1 x} = \varepsilon.\frac{\partial \varphi}{\Delta_1 x}$$

Combination with the second equation in Eq. (5) gives $\varepsilon\frac{\partial \varphi}{\Delta_1 x} = \frac{\partial f}{\Delta_2 y}$.

Theorem 3.3. The function f is holomorphic in the open subset $\Omega \subset \mathbb{D}$, if and only if there exist a pair of real functions φ and k, such that $\varphi \in C^{\sigma_1}(P_x(\Omega)).\frac{d\varphi}{\Delta_1 x}$ is delta1-(differentiable) in $P_x(\Omega)$ and k is delta1-differentiable in $P_x(\Omega)$, where P_x is the first projection , so that the function f can be written explicitly

$$f(z) = \varphi(x) + (\frac{d\varphi}{\Delta_1 x}y + k(x))\varepsilon, \qquad \forall z \in \Omega.. \tag{6}$$

Proof:

Since f is holomorphic in Ω, we find $\frac{\partial\varphi}{\Delta_1 x}=\frac{\partial\psi}{\Delta_2 y}$ and $\frac{\partial\varphi}{\Delta_2 y}=0$. It follows that $\varphi(x,y)=\varphi(x)$. Hence $\frac{\partial\psi}{\Delta_2 y}=\frac{\partial\varphi}{\Delta_1 x}=0$. So we find $\psi(x,y)=\frac{d\varphi}{\Delta_1 x}y+k(x)$. Thus if we put the $\psi(x,y)$ in the equation $f(z)=\varphi(x,y)+\varepsilon\psi(x,y)$, then we have $f(z)=\varphi(x)+(\frac{d\varphi}{\Delta_1 x}y+k(x))\varepsilon,\ \forall z\in\Omega.$.

Remark 3.1. If, in particular, f is an homogeneous function, the Eq.(6) gives $k\equiv 0$. Thus $f(z)=\varphi(x)+\varepsilon\frac{d\varphi}{\Delta_1 x}y$.

Definition 3.2. A dual function $z=\lambda(t)=\varphi(t)+\varepsilon\psi(t),\ t\in[a,b]\in\mathbb{T}$ where $\varphi:[a,b]\to\mathbb{T}_1$ and $\psi:[a,b]\to\mathbb{T}_2$ are continuous functions, is defined a dual curve on the timescale plane $\mathbb{T}_1+\varepsilon\mathbb{T}_2$.

Example1: If the function $f(\mathrm{z})=$ constant on $\mathbb{T}_1+\varepsilon\mathbb{T}_2$, then $\varphi(x,y)=$ constant and $\psi(x,y)=0$ satisfy the Cauchy-Riemann equations of the dual-variable functions $\frac{\partial\varphi}{\Delta_1 x}=0=\frac{\partial\psi}{\Delta_2 x}$ and $\frac{\partial\varphi}{\Delta_2 y}=0.$

Example2: The function $f(z)=z=x+\varepsilon y=\varphi(x,y)+\varepsilon\psi(x,y)$

on $\mathbb{T}_1+\varepsilon\mathbb{T}_2$, then the functions $\varphi(x,y)=x$ and $\psi(x,y)=y$ functions satisfy the Cauchy-Riemann equations of the dual-variable functions $\frac{\partial\varphi}{\Delta_1 x}=1=\frac{\partial\psi}{\Delta_2 y}$ *and* $\frac{\partial\varphi}{\Delta_2 y}=0.$ Thus $f^{\Delta}(z_0)=\frac{\partial\varphi}{\Delta_1 x}+\varepsilon\frac{\partial\psi}{\Delta_1 x}=1+\varepsilon 0.$ Therefore the derivation of $f(z)=z$ becomes $f^{\Delta}(z_0)=1$, here $z_0\in\mathbb{T}_1+\varepsilon\mathbb{T}_2$.

Example3: Consider the function $f(z)=x^2+\varepsilon 2xy$ *on* $\mathbb{T}_1+\varepsilon\mathbb{T}_2$. Hence

$\varphi(x,y)=x^2$, $\psi(x,y)=2xy$ and $\dfrac{\partial\varphi}{\Delta_1 x}=x+\sigma_1(x)$, $\dfrac{\partial\psi}{\Delta_2 y}=2x$, $\dfrac{\partial\varphi}{\Delta_2 y}=0$. Because of the Cauchy-Riemann equations of the dual-variable functions, the conditions $x+\sigma_1(x)=2x$ and $\dfrac{\partial\varphi}{\Delta_2 y}=0$ are satisfied.

CONCLUSION

In current paper, we research some differential properties of dual-variable functions on time scales. In the literature, up to now, there is not any working about this concept. This research is a guideline for future work.

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