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ESTIMATION OF $Pr(X > Y)$ FOR EXPONENTIATED GUMBEL DISTRIBUTION BASED ON LOWER RECORD VALUES

Bahman Tarvirdizade
Department of Statistics
Allameh Tabatabaei University
Tehran, Iran
email: bahtary@gmail.com

Abstract : In this paper, we consider the estimation of $R = Pr(X > Y)$ based on lower record values when X and Y are independently but not identically exponentiated Gumbel distributed random variables. The maximum likelihood, Bayes and empirical Bayes estimators of R are obtained and their properties are studied. Confidence intervals, exact and approximate, as well as the Bayesian credible sets for R are obtained. A simulation study is conducted to investigate and compare the performance of the intervals.

Key words: Bayes estimation; empirical Bayes estimation; exponentiated Gumbel distribution; maximum likelihood estimation; record values; stress-strength reliability
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1. Introduction

The exponentiated gumbel (EG) distribution was introduced by Nadarajah (2005). He illustrated its use for modeling rainfall data from Orland, Florida. The gumbel distribution have very applications in climate modeling include: global warming problems, flood frequency analysis, offshore modeling, rainfall modeling, and wind speed modeling. For other applications, see Kotz and Nadarajah (2000).

The cumulative distribution function (cdf) of the exponentiated gumbel distribution with shape parameter $\alpha > 0$ is given by;

$$F(x) = \left(\exp\left(-e^{-x}\right)\right)^{\alpha}, \qquad \alpha > 0, -\infty < x < \infty. \tag{1}$$

The probability density function (pdf) corresponding to (1) is given by;

$$f(x) = \alpha e^{-x}\left(\exp\left(-e^{-x}\right)\right)^{\alpha}, \qquad \alpha > 0, -\infty < x < \infty. \tag{2}$$

We will denote exponentiated gumbel distribution with shape parameter α by EG(α).

Let $\{X_i,\, i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with an absolutely continuous cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. An observation X_j is called an upper record if its value exceeds all previous observations, i.e. X_j is an upper record if $X_j > X_i$ for every $i < j$. An analogous definition can be given for lower records. These type of data arise in a wide variety of practical situations such as industrial stress testing, meteorology, hydrology, sports, and stock market analysis. Interested readers may refer to the book by Arnold et al. (1998) and the references contained therein.

In this paper, we consider the problem of estimating the stress-strength reliability $Pr(X > Y)$ in the exponentiated gumbel distribution based on lower record values. This problem was considered by Kakade et al. (2008) for ordinary samples from the exponentiated gumbel distribution with shape and scale parameters. Kang et al. (2013) and Abdi (2014) discussed different methods of estimation for the two-parameter exponentiated Gumbel distribution based on record values. The reader is referred to Kotz et al. (2003) for some applications and motivations for the study of the stress-strength reliability $Pr(X>Y)$. In Section 2, we discussed likelihood inference for the stress-strength reliability, while in Section 3 we considered Bayesian inference. A simulation study is described in Section 4.

2. Likelihood inference

Let $X \sim EG(\alpha)$ and $Y \sim EG(\beta)$ be independent random variables. Let $R = \Pr(X>Y)$ be the stress strength reliability. then,

$$R = \Pr(X>Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} \alpha e^{-x}\left(\exp\left(-e^{-x}\right)\right)^{\alpha} \beta e^{-y}\left(\exp\left(-e^{-y}\right)\right)^{\beta} dydx = \frac{\alpha}{\alpha+\beta}.$$

Our interest is in estimating R based on lower record values on both variables. Let $\underset{\sim}{r} = (r_1, ..., r_n)$ be a set of lower records from $EG(\alpha)$ and let $\underset{\sim}{s} = (s_1, ..., s_m)$ be an independent set of lower records from $EG(\beta)$. The likelihood functions are given by (Ahsanullah, 2004);

$$L(\alpha|\underset{\sim}{r}) = f(r_n)\prod_{i=1}^{n-1}\left(\frac{f(r_i)}{F(r_i)}\right), \qquad -\infty < r_n < ... < r_1 < \infty$$

$$L(\beta|\underset{\sim}{s}) = g(s_m)\prod_{i=1}^{m-1}\left(\frac{g(s_i)}{G(s_i)}\right), \qquad -\infty < s_m < ... < s_1 < \infty \tag{3}$$

where f and F are the pdf and cdf of $X \sim EG(\alpha)$ respectively and g and G are the pdf and cdf of $Y \sim EG(\beta)$ respectively. Substituting f, F, g and G in the likelihood functions and using Equation (3), we obtain

$$L(\alpha|\underset{\sim}{r}) = \alpha^n\left(\exp\left(-e^{-r_n}\right)\right)^{\alpha}\prod_{i=1}^{n} e^{-r_i},$$

$$L(\beta|\underset{\sim}{s}) = \beta^m\left(\exp\left(-e^{-s_m}\right)\right)^{\beta}\prod_{i=1}^{m} e^{-s_i}. \tag{4}$$

It can be shown that the maximum likelihood estimators (MLE) of α and β based on the lower record values are

$$\hat{\alpha} = ne^{r_n}, \qquad \hat{\beta} = me^{s_m}. \tag{5}$$

Therefore the MLE of R is given by $\hat{R} = \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}$. To study the distribution of $\hat{R}$ we need the distributions of $\hat{\alpha}$ and $\hat{\beta}$. Consider first $\hat{\alpha} = ne^{r_n}$, the pdf of R_n is given by (Ahsanullah, 2004);

$$f_{R_n}(r_n) = \frac{1}{(n-1)!} f(r_n)[-\ln F(r_n)]^{n-1}$$

$$= \frac{1}{(n-1)!}\left(\alpha e^{-r_n}\right)^n\left(\exp\left(-e^{-r_n}\right)\right)^{\alpha}, \quad -\infty < r_n < \infty. \tag{6}$$

Consequently, the pdf of $Z_1 = \hat{\alpha}$ is given by;

$$f_{Z_1}(z_1) = \frac{(n\alpha)^n}{(n-1)!{z_1}^{n+1}} e^{-\frac{n\alpha}{z_1}}, \qquad z_1 > 0. \tag{7}$$

This is recognized as the inverted gamma distribution, i.e., $Z_1 \sim IGamma(n, n\alpha)$. Similarly, the pdf of $Z_2 = \hat{\beta}$ is given by;

$$f_{Z_2}(z_2) = \frac{(m\beta)^m}{(m-1)!{z_2}^{m+1}} e^{-\frac{m\beta}{z_2}}, \qquad z_2 > 0. \tag{8}$$

Thus $Z_2 \sim IGamma(m, m\beta)$. Therefore we can find the pdf of $\hat{R} = \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}} = \frac{Z_1}{Z_1+Z_2} = \frac{1}{1+\frac{Z_2}{Z_1}}$. Consider $\frac{Z_2}{Z_1}$. Note that, by the properties of the inverted gamma distribution and its relation with the gamma distribution we have $\frac{n\alpha}{Z_1} \sim Gamma(n,1)$ and $\frac{m\beta}{Z_2} \sim Gamma(m,1)$. Hence $\frac{2n\alpha}{Z_1} \sim \chi^2_{2n}$ and $\frac{2m\beta}{Z_2} \sim \chi^2_{2m}$. Note that, by the independence of two random quantities we have $\frac{(2n\alpha/2nZ_1)}{(2m\beta/2mZ_2)} = \frac{\alpha Z_2}{\beta Z_1} \sim F_{(2n,2m)}$. hence, $\frac{Z_2}{Z_1} = \frac{\beta}{\alpha} F_{(2n,2m)}$, has a scaled F distribution. It follows that the distribution of $\hat{R}$ is that of $\frac{1}{1+\frac{\beta}{\alpha}F_{(2n,2m)}}$ which can be obtained using simple transformation techniques. This fact can be used to construct the following $(1-a)\%$ confidence interval for R;

$$\left(\left(1 + \frac{z_2}{z_1 F_{a/2,2n,2m}}\right)^{-1}, \left(1 + \frac{z_2}{z_1 F_{1-a/2,2n,2m}}\right)^{-1} \right). \tag{9}$$

Records are rare in practice (Arnold et al., 1998) and sample sizes are often very small, however, intervals based on the asymptotic normality of MLEs can be of interest in cases when the number of records is sufficiently large. This is because of their optimal asymptotic properties under very general conditions (Lehmann, 1999). Note that $(\hat{\alpha} - \alpha) \xrightarrow{d} N(0, I^{-1}(\alpha))$ as $n \to \infty$, where $(\xrightarrow{d})$ denotes convergence in distribution and $I^{-1}(\alpha)$ is the asymptotic variance given by the reciprocal of the Fisher information; $\left[-E\left(\frac{\partial^2 \ln L(\alpha|\underset{\sim}{r})}{\partial\alpha^2}\right)\right]^{-1} = \frac{\alpha^2}{n}$. Similarly, $(\hat{\beta} - \beta) \xrightarrow{d} N(0, I^{-1}(\beta))$ as $m \to \infty$, where $I^{-1}(\beta) = \frac{\beta^2}{m}$. Let $n \to \infty$ and $m \to \infty$ such that $m/n \to p$ where $0 < p < 1$, it follows that $\sqrt{n}(\hat{\beta} - \beta) \to N(0, \beta^2/p)$. Since $R = \frac{\alpha}{\alpha+\beta} = h(\alpha,\beta)$ and $\hat{R} = \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}} = h(\hat{\alpha},\hat{\beta})$, we have;

$$\sqrt{n}(\hat{R} - R) = \sqrt{n}\left(h(\hat{\alpha},\hat{\beta}) - h(\alpha,\beta)\right) \xrightarrow{d} N(0,\eta^2)$$

where $\eta^2 = \left(\frac{\partial h(\alpha,\beta)}{\partial\alpha}\right)^2 \alpha^2 + \left(\frac{\partial h(\alpha,\beta)}{\partial\beta}\right)^2 \beta^2/p$. A $(1-a)\%$ approximate confidence interval for R based on this asymptotic result is given by $(\hat{R} - z_{1-a/2}\hat{\eta}/\sqrt{n}, \hat{R} + z_{1-a/2}\hat{\eta}/\sqrt{n})$, where $\hat{\eta}$ is obtained by substituting m/n for p and the MLEs $\hat{\alpha}$ and $\hat{\beta}$ in the asymptotic standard deviation η. In these calculations we assumed that m is the smaller sample in size and n is the larger. However, if this is not the case then the formula for the asymptotic variance in the asymptotic interval should be modified accordingly.

3. Bayesian inference

Consider the likelihood functions of α and β based on the two sets of lower record values from the exponentiated gumbel distribution mentioned in previous section. We rewrite them as,

$$L(\alpha|\underset{\sim}{r}) = \alpha^n u_1(\underset{\sim}{r}) e^{-\alpha v_1(r_n)} \quad \textit{and} \quad L(\beta|\underset{\sim}{s}) = \beta^m u_2(\underset{\sim}{s}) e^{-\beta v_2(s_m)}, \tag{10}$$

where $u_1(\underset{\sim}{r}) = \prod_{i=1}^{n} e^{-r_i}$, $u_2(\underset{\sim}{s}) = \prod_{i=1}^{m} e^{-s_i}$, $v_1(r_n) = e^{-r_n}$ and $v_2(s_m) = e^{-s_m}$. These suggest that the conjugate family of prior distributions for α and β is the Gamma family of probability distributions;

$$\pi(\alpha) = \frac{\theta_1^{\gamma_1}\alpha^{\gamma_1-1}e^{-\theta_1\alpha}}{\Gamma(\gamma_1)}, \ \alpha > 0 \qquad \textit{and} \qquad \pi(\beta) = \frac{\theta_2^{\gamma_2}\beta^{\gamma_2-1}e^{-\theta_2\beta}}{\Gamma(\gamma_2)}, \ \beta > 0 \tag{11}$$

where γ_1, θ_1, γ_2 and θ_2 are the parameters of the prior distributions of α and β respectively. It can be shown that $(\alpha|\underset{\sim}{r}) \sim Gamma(n+\gamma_1, v_1(r_n)+\theta_1)$ and $(\beta|\underset{\sim}{s}) \sim Gamma(m+\gamma_2, v_2(s_m)+\theta_2)$. It follows that $2(v_1(r_n)+\theta_1)(\alpha|\underset{\sim}{r}) \sim \chi^2_{2(n+\gamma_1)}$ and $2(v_2(s_m)+\theta_2)(\beta|\underset{\sim}{s}) \sim \chi^2_{2(m+\gamma_2)}$. It follows that $\pi(R|\underset{\sim}{r},\underset{\sim}{s})$, the posterior distribution of R is equal to that of $(1+AW)^{-1}$, where $W \sim F_{2(m+\gamma_2),2(n+\gamma_1)}$ and $A = \frac{(m+\gamma_2)(v_1(r_n)+\theta_1)}{(n+\gamma_1)(v_2(s_m)+\theta_2)}$. The Bayes estimator under squared error loss is the mean of this posterior distribution which may be approximated. Kang et al. (2013) discussed it by using Lindley's approximation technique. A Bayesian $(1-a)\%$ confidence interval for R is given by;

$$\left((AF_{1-a/2,2(m+\gamma_2),2(n+\gamma_1)}+1)^{-1}, (AF_{a/2,2(m+\gamma_2),2(n+\gamma_1)}+1)^{-1}\right). \tag{12}$$

The case of a noninformative prior can be treated similarly. We consider Jeffereys prior that say, $\pi(\alpha) \propto \sqrt{|I(\alpha)|}$. This suggest that prior densitys for α and β are proportional to $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ respectively. Using direct arguments one can show that $(\alpha|\underset{\sim}{r}) \sim Gamma(n, v_1(r_n))$ and $(\beta|\underset{\sim}{s}) \sim Gamma(m, v_2(s_m))$. It follows that the posterior distribution of R is equal to that of $(1+\frac{mv_1(r_n)}{nv_2(s_m)}W)^{-1}$ where $W \sim F_{2m,2n}$. Therefore a Bayesian $(1-a)\%$ confidence interval for R is given by;

$$\left(\left(\frac{mv_1(r_n)}{nv_2(s_m)}F_{1-a/2,2m,2n}+1\right)^{-1}, \left(\frac{mv_1(r_n)}{nv_2(s_m)}F_{a/2,2m,2n}+1\right)^{-1}\right). \tag{13}$$

Now consider the case when the parameters of prior distributions are themselves unknown. We consider the conjugate prior distributions for α and β above when the parameters θ_1 and θ_2 are unknown. In the empirical Bayes model, we must estimate them. In order to, we calculate the marginal distribution of lower records, with densitys

$$m(\underset{\sim}{r}|\theta_1) = \int f_{\underset{\sim}{R}}(\underset{\sim}{r}|\alpha)\pi(\alpha|\theta_1)d\alpha, \qquad -\infty < r_n < ... < r_1 < \infty,$$

$$m(\underset{\sim}{s}|\theta_2) = \int f_{\underset{\sim}{S}}(\underset{\sim}{s}|\beta)\pi(\beta|\theta_2)d\beta, \qquad -\infty < s_m < ... < s_1 < \infty.$$

Using Equations (10) and (11), we obtain

$$m(\underset{\sim}{r}|\theta_1) = \frac{\Gamma(n+\gamma_1)u_1(\underset{\sim}{r})\theta_1^{\gamma_1}}{\Gamma(\gamma_1)(\theta_1+v_1(r_n))^{n+\gamma_1}},$$

$$m(\underset{\sim}{s}|\theta_2) = \frac{\Gamma(m+\gamma_2)u_2(\underset{\sim}{s})\theta_2^{\gamma_2}}{\Gamma(\gamma_2)(\theta_2+v_2(s_m))^{m+\gamma_2}}. \tag{14}$$

It can be shown that the maximum likelihood estimators (MLE) of θ_1 and θ_2 based on the marginal distributions (14) are

$$\hat{\theta}_1 = \frac{\gamma_1 v_1(r_n)}{n}, \qquad \hat{\theta}_2 = \frac{\gamma_2 v_2(s_m)}{m}. \tag{15}$$

With substitution $\hat{\theta}_1$ and $\hat{\theta}_2$ for θ_1 and θ_2 in prior distributions and using similar arguments above, one can show that $(\alpha|\underset{\sim}{r},\hat{\theta}_1) \sim Gamma(n+\gamma_1, (1+\frac{\gamma_1}{n})v_1(r_n))$ and $(\beta|\underset{\sim}{s},\hat{\theta}_2) \sim Gamma(m+\gamma_2, (1+\frac{\gamma_2}{m})v_2(s_m))$. It follows that $2((1+\frac{\gamma_1}{n})v_1(r_n))(\alpha|\underset{\sim}{r},\hat{\theta}_1) \sim \chi^2_{2(n+\gamma_1)}$ and $2((1+\frac{\gamma_2}{m})v_2(s_m))(\beta|\underset{\sim}{s},\hat{\theta}_2) \sim \chi^2_{2(m+\gamma_2)}$. It follows that $\pi(R|\underset{\sim}{r},\hat{\theta}_1,\underset{\sim}{s},\hat{\theta}_2)$, the empirical posterior distribution of R is equal to that of

$(1+A'W)^{-1}$, where $W \sim F_{2(m+\gamma_2),2(n+\gamma_1)}$ and $A' = \frac{mv_1(r_n)}{nv_2(s_m)}$. A Bayesian $(1-a)\%$ confidence interval for R is given by;

$$\left((A'F_{1-a/2,2(m+\gamma_2),2(n+\gamma_1)}+1)^{-1},(A'F_{a/2,2(m+\gamma_2),2(n+\gamma_1)}+1)^{-1}\right) \tag{16}$$

The construction of highest posterior density (HPD) regions requires finding the set $C=\{\theta : \pi(\theta| \underset{\sim}{r}, \underset{\sim}{s}) \geq k_a\}$, where k_a is the largest constant such that $\Pr(\theta \in C) \geq 1-a$. This often requires numerical optimization techniques. Chen and Shao (1999) presented a simple Monte Carlo technique to approximate the HPD region.

4. A simulation study

In this section, a simulation study is conducted to investigate and compare the performance of the confidence intervals presented in this paper and some bootstrap intervals. There are several bootstrap based intervals discussed in the literature (Efron and Tibshirani, 1993). It is important here to note that all inference procedures in this paper depend only on the smallest records, r_n and s_m. Therefore we shall use the parametric bootstrap based on the marginal distribution of R_n as given in Equation (6). In follows we describe the bootstrapping procedure.

1. Calculate $\hat{\alpha}$, $\hat{\beta}$ and $\hat{R}$, the maximum likelihood estimators of α, β and R based on r_n and s_m.
2. Generate r_n^* from the distribution given in Equation (6) with α replaced by $\hat{\alpha}$ and generate s_m^* similarly.
3. Calculate $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{R}^*$ using the r_n^* and s_m^* obtained in step 2.
4. Repeat steps 2 and 3, B times to obtain $\hat{R}_1^*,...,\hat{R}_B^*$.

Then we can calculate the following bootstrap intervals;

Normal Interval: The simplest $(1-a)$ bootstrap interval is the Normal interval

$$(\hat{R}-z_{1-a/2}\,\hat{se}_{boot},\hat{R}+z_{1-a/2}\,\hat{se}_{boot})$$

where $\hat{se}_{boot}$ is the bootstrap estimate of the standard error based on $\hat{R}_1^*,...,\hat{R}_B^*$.

Basic Pivotal Interval: The $(1-a)$ bootstrap basic pivotal confidence interval is

$$(2\hat{R}-\hat{r}^*_{(1-a/2)B},2\hat{R}-\hat{r}^*_{(a/2)B})$$

where $\hat{r}^*_\beta$ is the β quantile of $\hat{R}_1^*,...,\hat{R}_B^*$.

Percentile Interval: The $(1-a)$ bootstrap percentile interval is defined by

$$(\hat{r}^*_{(a/2)B},\hat{r}^*_{(1-a/2)B})$$

that is, just use the $a/2$ and $1-a/2$ quantiles of the bootstrap sample.

Interested readers may refer to DiCiccio and Efron (1996) and the references contained therein to observe more details.

In the simulation design we used all combinations of $n=5,10,15$ and $m=5,10,15$. We used $\alpha=1$ and $R=0.1,0.25,0.5$. The value of β is determined by the choice of α and R. The confidence level taken is $(1-a)=0.90$ and 0.95. For each combination of the simulation indices we generated 5000 samples of lower records from the distributions of X and Y. For each generated pair of samples we calculated the following intervals;

1) ML: The interval based on the MLE given in Equation (9).
2) Bayes: The interval based on the Bayes estimator given in Equation (12).
3) J.B: The interval based on the Bayes estimator given in Equation (13).

4) E.B: The interval based on the empirical Bayes estimator given in Equation (16).
5) Norm: The normal interval.
6) Basic: The basic pivotal interval.
7) Perc: The percentile interval.

The empirical coverage probability and expected lengths of intervals are obtained by using the 5000 replications. In the interval based on the Bayes estimators we used $\gamma_1=2$ and $\gamma_2=5$ and $\theta_1=\theta_2=2$. For bootstrap intervals we used 1000 bootstrap samples. The results of our simulations are given in Tables 1 and 2.

5. Conclusion and discussion

Based on simulation results, it appears that the length of the intervals is maximized when $R=0.5$ and gets shorter and shorter as we move away to the extremes. Increasing the sample size on either variable also results in shorter intervals. The performance of the both basic pivotal interval and percentile interval is similar in terms of expected length but in terms of coverage rate percentile interval has the better performance. The percentile interval appears to be the best among bootstrap intervals. The interval based on the MLE appears to perform almost as well as the percentile interval specially for small to moderate sample sizes. The interval based on the Bayes estimator given in Equation (12) has the low coverage rate for small values of R since it is dependent on θ_1 and θ_2 values. Furthermore, the interval based on the Bayes estimator given in Equation (13) has the more expected length rather than the interval based on the empirical Bayes estimator. It appears that the interval based on the empirical Bayes estimator simultaneously has the short expected length and very good coverage rate in comparison with the other intervals. Hence, we recommend to use this confidence interval in all.

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TABLE 1. Expected lengths and coverage rates (in parentheses) of the confidence intervals with (1-α)=0.95

n	m	R	ML	$Bayes$	$J.B$	$E.B$	$Norm$	$Basic$	$Perc$
5	5	0/1	0.273(0.955)	0.224(0.887)	0.263(0.993)	0.187(0.991)	0.279(0.933)	0.271(0.807)	0.271(0.951)
5	5	0/25	0.451(0.948)	0.378(0.995)	0.471(0.993)	0.360(0.989)	0.461(0.902)	0.449(0.801)	0.449(0.944)
5	5	0/5	0.539(0.956)	0.448(0.996)	0.576(0.995)	0.462(0.993)	0.552(0.909)	0.539(0.808)	0.539(0.955)
5	10	0/1	0.220(0.949)	0.263(0.944)	0.213(0.991)	0.175(0.990)	0.256(0.955)	0.249(0.835)	0.249(0.933)
5	10	0/25	0.390(0.947)	0.342(0.994)	0.399(0.990)	0.336(0.988)	0.425(0.931)	0.417(0.851)	0.417(0.935)
5	10	0/5	0.480(0.944)	0.426(0.991)	0.507(0.989)	0.431(0.986)	0.488(0.856)	0.479(0.823)	0.479(0.934)
5	15	0/1	0.202(0.959)	0.230(0.993)	0.196(0.996)	0.166(0.995)	0.247(0.976)	0.240(0.861)	0.240(0.941)
5	15	0/25	0.366(0.947)	0.324(0.996)	0.372(0.991)	0.321(0.989)	0.408(0.931)	0.401(0.850)	0.401(0.927)
5	15	0/5	0.461(0.948)	0.412(0.996)	0.481(0.990)	0.414(0.987)	0.466(0.909)	0.458(0.852)	0.458(0.911)
10	5	0/1	0.240(0.955)	0.286(0.500)	0.229(0.993)	0.160(0.991)	0.214(0.925)	0.209(0.847)	0.209(0.944)
10	5	0/25	0.403(0.947)	0.334(0.990)	0.417(0.991)	0.312(0.989)	0.380(0.893)	0.373(0.829)	0.373(0.932)
10	5	0/5	0.482(0.956)	0.395(0.996)	0.507(0.994)	0.404(0.991)	0.489(0.907)	0.479(0.826)	0.479(0.909)
10	10	0/1	0.177(0.950)	0.225(0.840)	0.172(0.990)	0.144(0.990)	0.180(0.937)	0.176(0.858)	0.176(0.949)
10	10	0/25	0.326(0.950)	0.295(0.993)	0.332(0.991)	0.285(0.989)	0.329(0.926)	0.325(0.866)	0.325(0.951)
10	10	0/5	0.406(0.955)	0.365(0.997)	0.422(0.994)	0.369(0.992)	0.411(0.923)	0.406(0.874)	0.406(0.953)
10	15	0/1	0.157(0.950)	0.194(0.926)	0.153(0.991)	0.135(0.991)	0.167(0.947)	0.165(0.878)	0.165(0.948)
10	15	0/25	0.296(0.947)	0.274(0.991)	0.300(0.989)	0.264(0.988)	0.308(0.932)	0.305(0.874)	0.305(0.949)
10	15	0/5	0.375(0.946)	0.347(0.995)	0.388(0.989)	0.349(0.989)	0.379(0.919)	0.375(0.887)	0.375(0.945)
15	5	0/1	0.220(0.957)	0.269(0.251)	0.216(0.996)	0.148(0.993)	0.186(0.899)	0.182(0.822)	0.182(0.931)
15	5	0/25	0.387(0.950)	0.314(0.991)	0.395(0.991)	0.291(0.986)	0.354(0.899)	0.348(0.833)	0.348(0.938)
15	5	0/5	0.460(0.948)	0.370(0.996)	0.481(0.990)	0.377(0.988)	0.465(0.901)	0.457(0.839)	0.457(0.930)
15	10	0/1	0.159(0.952)	0.207(0.707)	0.157(0.992)	0.131(0.991)	0.153(0.931)	0.151(0.869)	0.151(0.947)
15	10	0/25	0.300(0.952)	0.271(0.993)	0.305(0.992)	0.261(0.990)	0.294(0.923)	0.291(0.878)	0.291(0.949)
15	10	0/5	0.375(0.949)	0.336(0.997)	0.388(0.994)	0.339(0.993)	0.378(0.915)	0.375(0.867)	0.375(0.953)
15	15	0/1	0.139(0.955)	0.176(0.891)	0.136(0.995)	0.121(0.995)	0.140(0.936)	0.138(0.875)	0.138(0.954)
15	15	0/25	0.268(0.953)	0.249(0.994)	0.270(0.993)	0.243(0.992)	0.269(0.937)	0.267(0.898)	0.267(0.954)
15	15	0/5	0.340(0.955)	0.315(0.998)	0.349(0.995)	0.317(0.993)	0.342(0.929)	0.339(0.899)	0.339(0.954)

TABLE 2. Expected lengths and coverage rates (in parentheses) of the confidence intervals with (1-α)=0.90

n	m	R	ML	$Bayes$	$J.B$	$E.B$	$Norm$	$Basic$	$Perc$
5	5	0/1	0.223(0.903)	0.295(0.608)	0.213(0.975)	0.154(0.971)	0.235(0.903)	0.222(0.787)	0.222(0.901)
5	5	0/25	0.381(0.894)	0.364(0.920)	0.398(0.976)	0.303(0.973)	0.387(0.853)	0.381(0.766)	0.381(0.893)
5	5	0/5	0.462(0.914)	0.393(0.990)	0.497(0.972)	0.395(0.969)	0.464(0.845)	0.462(0.776)	0.462(0.915)
5	10	0/1	0.182(0.892)	0.229(0.919)	0.176(0.969)	0.145(0.967)	0.215(0.924)	0.203(0.809)	0.203(0.891)
5	10	0/25	0.331(0.905)	0.328(0.952)	0.339(0.971)	0.284(0.970)	0.357(0.892)	0.351(0.809)	0.351(0.892)
5	10	0/5	0.410(0.890)	0.367(0.983)	0.435(0.971)	0.367(0.966)	0.409(0.840)	0.409(0.778)	0.409(0.881)
5	15	0/1	0.169(0.913)	0.206(0.934)	0.163(0.977)	0.139(0.973)	0.207(0.947)	0.195(0.836)	0.195(0.900)
5	15	0/25	0.311(0.897)	0.308(0.970)	0.316(0.969)	0.272(0.965)	0.342(0.887)	0.336(0.811)	0.336(0.871)
5	15	0/5	0.393(0.909)	0.353(0.981)	0.411(0.959)	0.352(0.959)	0.391(0.860)	0.391(0.814)	0.391(0.891)
10	5	0/1	0.195(0.902)	0.239(0.405)	0.184(0.963)	0.131(0.960)	0.179(0.891)	0.173(0.826)	0.173(0.903)
10	5	0/25	0.338(0.895)	0.315(0.948)	0.348(0.961)	0.262(0.959)	0.319(0.850)	0.316(0.800)	0.316(0.887)
10	5	0/5	0.411(0.912)	0.344(0.985)	0.435(0.960)	0.343(0.957)	0.410(0.842)	0.410(0.783)	0.410(0.890)
10	10	0/1	0.146(0.894)	0.180(0.722)	0.141(0.961)	0.119(0.961)	0.151(0.905)	0.146(0.827)	0.146(0.950)
10	10	0/25	0.279(0.899)	0.275(0.980)	0.279(0.965)	0.239(0.963)	0.276(0.876)	0.273(0.822)	0.273(0.898)
10	10	0/5	0.345(0.904)	0.313(0.987)	0.360(0.971)	0.313(0.969)	0.345(0.867)	0.345(0.824)	0.345(0.906)
10	15	0/1	0.131(0.900)	0.159(0.784)	0.127(0.670)	0.112(0.967)	0.140(0.915)	0.136(0.850)	0.136(0.897)
10	15	0/25	0.250(0.897)	0.254(0.989)	0.253(0.967)	0.226(0.965)	0.258(0.882)	0.255(0.837)	0.255(0.899)
10	15	0/5	0.318(0.901)	0.296(0.989)	0.330(0.974)	0.296(0.971)	0.318(0.875)	0.318(0.841)	0.318(0.897)
15	5	0/1	0.178(0.907)	0.218(0.183)	0.173(0.961)	0.121(0.957)	0.156(0.860)	0.152(0.801)	0.152(0.890)
15	5	0/25	0.323(0.902)	0.293(0.970)	0.329(0.968)	0.243(0.964)	0.297(0.847)	0.296(0.793)	0.296(0.887)
15	5	0/5	0.392(0.893)	0.321(0.981)	0.411(0.959)	0.319(0.955)	0.391(0.849)	0.391(0.791)	0.391(0.884)
15	10	0/1	0.131(0.909)	0.179(0.605)	0.129(0.973)	0.108(0.970)	0.129(0.890)	0.126(0.841)	0.126(0.907)
15	10	0/25	0.251(0.902)	0.250(0.985)	0.256(0.977)	0.219(0.975)	0.247(0.874)	0.245(0.835)	0.245(0.898)
15	10	0/5	0.318(0.893)	0.287(0.988)	0.330(0.971)	0.287(0.967)	0.317(0.858)	0.318(0.824)	0.318(0.892)
15	15	0/1	0.115(0.898)	0.137(0.724)	0.113(0.967)	0.101(0.963)	0.118(0.899)	0.115(0.841)	0.115(0.900)
15	15	0/25	0.225(0.907)	0.228(0.988)	0.227(0.974)	0.204(0.972)	0.226(0.894)	0.224(0.861)	0.224(0.906)
15	15	0/5	0.287(0.903)	0.268(0.991)	0.296(0.977)	0.268(0.974)	0.287(0.876)	0.287(0.856)	0.287(0.902)