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Fractional Hermite-Hadamard Inequalities for Different Convex Classes Based on ψ-Hilfer-Atangana-Baleanu Operators

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Abstract

Hermite-Hadamard type inequalities play a central role in the study of convexity and its generalizations, providing a fundamental tool for both theoretical analysis and applications. In recent years, fractional integral operators have been widely employed to establish new versions of these inequalities. Among them, the ψ-Hilfer-Atangana-Baleanu (ψ-HAB) fractional integral operators have attracted attention as a powerful extension of the Atangana-Baleanu and ABK operators, offering a flexible framework to explore generalized convexity classes.

In this paper, with the help of the identities proved by Kermausor et al. in (Kermausuor et al. 2026), we obtain some new integral inequalities of Hermite-Hadamard type for the quasi-convex functions and the P-functions, respectively.

Keywords: Hermite-Hadamard inequalities, quasi-convex function, P-function, ψ-Hilfer-Atangana-Baleanu fractional integral operators.

1. Introduction

The classical Hermite–Hadamard inequality, established for convex functions, has played a fundamental role in the development of mathematical analysis, optimization theory, and integral inequalities.

If $G:[a,b]\to\mathbb{R}$ is a convex function, then the inequality

$$G\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b G(x)dx\le\frac{G(a)+G(b)}{2}$$

provides sharp bounds connecting the midpoint value and the integral mean of G. Over the past few decades, this classical result has contributed to the literature in terms of obtaining more general inequalities, particularly through the definition of new convexity classes and the introduction of fractional integral operators (Azzouz et al. 2025), (Butt et al. 2025), (Coşkun et al. 2024), (Çelik et al. 2021), (Hezenci et al. 2023), (Munir et al. 2025), (Pachpatte 2005), (Set and Çelik 2018), (Set et al. 2018), (Xi et al. 2012), (Yalçın el al. 2025).

In the framework of convex analysis, several generalizations of classical convexity have been introduced to obtain more general versions of integral inequalities such as the Hermite–Hadamard inequality. Among them, the notions of quasi-convexity (Ion 2007) and P-functions (Pearce and Rubinov 1999) play a crucial role, as they allow the extension of classical results to wider functional classes beyond standard convexity.

In what follows, we will give two different classes of convexity, namely quasi-convex functions and P-functions, which will play a key role in our main results.

Definition 1.1 (Ion 2007) Let $G:I\subseteq\mathbb{R}\to\mathbb{R}$ for all $\lambda\in[0,1]$ and all $x,y\in I$, if the following inequality

$$G(\lambda x+(1-\lambda)y)\le\max\{G(x),G(y)\}$$

holds, then G is called a quasi-convex function on I.

Definition 1.2 (Pearce and Rubinov 1999) *A function* $G:I\subseteq\mathbb{R}\to\mathbb{R}$ *is P-function or that* G *belongs to the class of* $P(I)$, *if it is nonnegative and, for all*

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$x, y \in I$ and $\lambda \in [0,1]$, satisfies the following inequality;

$$G(\lambda x + (1-\lambda) y) \le G(x) + G(y). \qquad (1.1)$$

Parallel to these developments, fractional calculus has emerged as a powerful tool in inequality theory. Starting from the Riemann–Liouville and Caputo definitions, numerous fractional integral operators have been proposed to model memory effects in physics, engineering, and biological systems (Kilbas et al. 2006). Among them, the Atangana–Baleanu (AB) and Atangana–Baleanu in the Caputo sense (ABK) operators have attracted considerable interest due to their kernel involving the Mittag-Leffler function, offering non-local and non-singular behavior (Atangana and Baleanu 2016). These operators have not only refined classical bounds but also expanded the Hermite–Hadamard type inequalities to include nonlocal fractional structures and generalized convexity definitions (Akdemir et al. 2021), (Avcı-Ardıç et al. 2023), (Budak et al. 2022), (Sarıkaya et al. 2013).

More recently, the ψ-Hilfer and ψ-Hilfer–Atangana–Baleanu (ψ-HAB) fractional operators have been introduced as a flexible framework unifying several earlier approaches. In particular, Kermausuor et al. established key identities that enable the derivation of new integral inequalities via ψ-HAB operators (Kermausuor et al. 2026).

Definition 1.3 Let $d_1, d_2 \in \mathbb{R}$ with $d_1 < d_2$ and let ψ be a strictly increasing and positive function on $[d_1, d_2]$, having a continuous derivative ψ' on (d_1, d_2). The left-sided and right-sided ψ-Hilfer-Atangana-Baleanu (HAB) fractional integral operators of a real-valued function G of order $\omega \in (0,1)$, are defined as

$${}^{HAB}_{\psi}I^{\omega}_{d_1^+}G(r) = \frac{1-\omega}{B(\omega)}G(r) + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_{d_1}^{r}(\psi(r)-\psi(z))^{\omega-1}\psi'(z)G(z)dz,\ r > d_1,$$

and

$${}^{HAB}_{\psi}I^{\omega}_{d_2^-}G(r) = \frac{1-\omega}{B(\omega)}G(r) + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_{r}^{d_2}(\psi(z)-\psi(r))^{\omega-1}\psi'(z)G(z)dz,\ r < d_2,$$

where $B(\omega) > 0$ is a normalization function and satisfies the property $B(0) = B(1) = 1$.

Remark 1.1 *If $\psi(t) = t$ in Definition 1.3, we have the Atangana-Baleanu fracitonal integral operators, and if $\psi(t) = \dfrac{t^{\rho}}{\rho}$, $\rho > 0$, then we have the ABK-fractional integral operators.*

Lemma 1.1 (Kermausuor et al. 2026) *Let $\delta, \varepsilon \in \mathbb{R}$ with $\delta < \varepsilon$ and $G : [\delta, \varepsilon] \to \mathbb{R}$ be a function. If G is differentiable and $G' \in L_1([\delta, \varepsilon])$, then the equality*

$$\left(\frac{(\varepsilon-\delta)^{\omega} + (1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)[G(\delta) + G(\varepsilon)]$$

$$-\left\lfloor {}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(G\circ\psi)(\psi^{-1}(\varepsilon)) + {}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^-}(G\circ\psi)(\psi^{-1}(\delta))\right\rfloor$$

$$= \frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\int_0^1\left[(1-r)^{\omega} - r^{\omega}\right]G'(r\delta + (1-r)\varepsilon)dr$$

holds for all $\omega > 0$.

Lemma 1.2 (Kermausuor et al. 2026) *Let $\delta, \varepsilon \in \mathbb{R}$ with $\delta < \varepsilon$ and $G : [\delta, \varepsilon] \to \mathbb{R}$ be a function. If G is differentiable and $G' \in L_1([\delta, \varepsilon])$, then the equality*

$$\left[{}^{HAB}_{\psi}I^{\omega}_{\left(\psi^{-1}(\frac{\delta+\varepsilon}{2})\right)^+}(G\circ\psi)(\psi^{-1}(\varepsilon)) + {}^{HAB}_{\psi}I^{\omega}_{\left(\psi^{-1}(\frac{\delta+\varepsilon}{2})\right)^-}(G\circ\psi)(\psi^{-1}(\delta))\right]$$

$$-\left(\frac{(\varepsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}G\left(\tfrac{\varepsilon+\delta}{2}\right) + \frac{1-\omega}{B(\omega)}[G(\delta) + G(\varepsilon)]\right)$$

$$= \frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\int_0^{1/2} r^{\omega}G'(r\delta + (1-r)\varepsilon)dr - \int_{1/2}^{1}(1-r)^{\omega}G'(r\delta + (1-r)\varepsilon)dr\right]$$

holds for all $\omega > 0$.

The main aim of this paper is to establish new integral inequalities using ψ-Hilfer-Atangana-Baleanu integral operators for quasi-convex and P-function. In this paper, integration techniques, quasi-convex function and P-function definitions were used and new integral inequalities that produce different bounds were proved.

2. *Main Results*

In this part, we obtained some fractional integral inequalities for quasi-convex functions and P-functions by using ψ-Hilfer-Atangana-Baleanu fractional integral operators as follows.

Theorem 2.1 *Let $G : [\delta, \varepsilon] \to \mathbb{R}$ be a differentiable function on (δ, ε), $G' \in L_1([\delta, \varepsilon])$, and $\delta < \varepsilon$.*

If $|\mathsf{G}'|$ *is a quasi-convex function, then the inequality*

$$\left|\left(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)[\mathsf{G}(\delta)+\mathsf{G}(\varepsilon)] - \left[{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathsf{G}\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^{-}}(\mathsf{G}\circ\psi)(\psi^{-1}(\delta))\right]\right|$$

$$\leq\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{2}{\omega+1}\left(1-\frac{1}{2^{\omega}}\right)\right)^{1-\frac{1}{q}}\left(\frac{2}{\omega+1}\left(1-\frac{1}{2^{\omega}}\right)\max\{|\mathsf{G}'(\delta)|^{q},|\mathsf{G}'(\varepsilon)|^{q}\}\right)^{\frac{1}{q}}$$

holds for $\omega\in(0,1)$ and $q\geq 1$.

Proof. Using Lemma 1.1 and property of absolute value, we get

$$\left|\left(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)[\mathsf{G}(\delta)+\mathsf{G}(\varepsilon)] - \left[{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathsf{G}\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^{-}}(\mathsf{G}\circ\psi)(\psi^{-1}(\delta))\right]\right|$$

$$=\left|\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\int_{0}^{1}\left[(1-r)^{\omega}-r^{\omega}\right]\mathsf{G}'(r\delta+(1-r)\varepsilon)dr\right|$$

$$\leq\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\int_{0}^{1}\left|(1-r)^{\omega}-r^{\omega}\right|\left|\mathsf{G}'(r\delta+(1-r)\varepsilon)\right|dr.$$

By applying power mean inequality, we get

$$\left|\left(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)[\mathsf{G}(\delta)+\mathsf{G}(\varepsilon)] - \left[{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathsf{G}\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^{-}}(\mathsf{G}\circ\psi)(\psi^{-1}(\delta))\right]\right|$$

$$\leq\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\int_{0}^{1}\left|(1-r)^{\omega}-r^{\omega}\right|dr\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}\left|(1-r)^{\omega}-r^{\omega}\right|\left|\mathsf{G}'(r\delta+(1-r)\varepsilon)\right|^{q}dr\right)^{\frac{1}{q}}.$$

By using quasi-convexity of $|\mathsf{G}'|$, we obtain

$$\left|\left(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)[\mathsf{G}(\delta)+\mathsf{G}(\varepsilon)] - \left[{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathsf{G}\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^{-}}(\mathsf{G}\circ\psi)(\psi^{-1}(\delta))\right]\right|$$

$$\leq\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\int_{0}^{1}\left|(1-r)^{\omega}-r^{\omega}\right|dr\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}\left|(1-r)^{\omega}-r^{\omega}\right|\max\{|\mathsf{G}'(\delta)|^{q},|\mathsf{G}'(\varepsilon)|^{q}\}dr\right)^{\frac{1}{q}}$$

$$=\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{2}{\omega+1}\left(1-\frac{1}{2^{\omega}}\right)\right)^{1-\frac{1}{q}}\left(\frac{2}{\omega+1}\left(1-\frac{1}{2^{\omega}}\right)\max\{|\mathsf{G}'(\delta)|^{q},|\mathsf{G}'(\varepsilon)|^{q}\}\right)^{\frac{1}{q}}.$$

Here, we used the facts that

$$\int_{0}^{1}\left|(1-r)^{\omega}-r^{\omega}\right|dr=\frac{2}{\omega+1}\left(1-\frac{1}{2^{\omega}}\right).$$

This completes the proof.

Corollary 2.1 *In Theorem 2.1, if we choose* $q=1$, *we have*

$$\left|\left(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)[\mathsf{G}(\delta)+\mathsf{G}(\varepsilon)] - \left[{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathsf{G}\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^{-}}(\mathsf{G}\circ\psi)(\psi^{-1}(\delta))\right]\right|$$

$$\leq\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\frac{2}{\omega+1}\left(1-\frac{1}{2^{\omega}}\right)\max\{|\mathsf{G}'(\delta)|,|\mathsf{G}'(\varepsilon)|\}.$$

Theorem 2.2 *Let* $\mathsf{G}:[\delta,\varepsilon]\rightarrow\mathsf{R}$ *be a differentiable function on* (δ,ε), $\mathsf{G}'\in L_{1}([\delta,\varepsilon])$, *and* $\delta<\varepsilon$. *If* $|\mathsf{G}'|$ *is a* $P-$*function, then the inequality*

$$\left|\left(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)[\mathsf{G}(\delta)+\mathsf{G}(\varepsilon)] - \left[{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathsf{G}\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^{-}}(\mathsf{G}\circ\psi)(\psi^{-1}(\delta))\right]\right|$$

$$\leq\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{2}{\omega+1}\left(1-\frac{1}{2^{\omega}}\right)\right)^{1-\frac{1}{q}}\left(\frac{2}{\omega+1}\left(1-\frac{1}{2^{\omega}}\right)(|\mathsf{G}'(\delta)|^{q}+|\mathsf{G}'(\varepsilon)|^{q})\right)^{\frac{1}{q}}$$

holds for $\omega\in(0,1)$ and $q\geq 1$.

Proof. By using Lemma 1.1, we have

$$\left|\left(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)[\mathsf{G}(\delta)+\mathsf{G}(\varepsilon)] - \left[{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathsf{G}\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^{-}}(\mathsf{G}\circ\psi)(\psi^{-1}(\delta))\right]\right|$$

$$\leq \frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\int_0^1\left|(1-r)^{\omega}-r^{\omega}\right|\left|G'(r\delta+(1-r)\varepsilon)\right|dr.$$

By applying power mean inequality, we get

$$|(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)})[G(\delta)+G(\varepsilon)]$$

$$-\left[{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(G\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^-}(G\circ\psi)(\psi^{-1}(\delta))\right]|$$

$$\leq \frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\int_0^1\left|(1-r)^{\omega}-r^{\omega}\right|dr\right)^{1-\frac{1}{q}}\left(\int_0^1\left|(1-r)^{\omega}-r^{\omega}\right|\left|G'(r\delta+(1-r)\varepsilon)\right|^q dr\right)^{\frac{1}{q}}.$$

Since $|G'|$ is a $P-$function, we obtain

$$|(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)})[G(\delta)+G(\varepsilon)]$$

$$-\left[{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(G\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^-}(G\circ\psi)(\psi^{-1}(\delta))\right]|$$

$$\leq \frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\int_0^1\left|(1-r)^{\omega}-r^{\omega}\right|dr\right)^{1-\frac{1}{q}}\left(\int_0^1\left|(1-r)^{\omega}-r^{\omega}\right|(|G'(\delta)|^q+|G'(\varepsilon)|^q)dr\right)^{\frac{1}{q}}$$

$$=\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{2}{\omega+1}\left(1-\frac{1}{2^{\omega}}\right)\right)^{1-\frac{1}{q}}\left(\frac{2}{\omega+1}\left(1-\frac{1}{2^{\omega}}\right)(|G'(\delta)|^q+|G'(\varepsilon)|^q)\right)^{\frac{1}{q}}$$

and the proof is completed.

Corollary 2.2 *In Theorem 2.2, if we choose* $q=1$, *we have*

$$|(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)})[G(\delta)+G(\varepsilon)]$$

$$-\left[{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(G\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^-}(G\circ\psi)(\psi^{-1}(\delta))\right]|$$

$$\leq \frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\frac{2}{\omega+1}\left(1-\frac{1}{2^{\omega}}\right)(|G'(\delta)|+|G'(\varepsilon)|).$$

Theorem 2.3 *Let* $G:[\delta,\varepsilon]\to R$ *be a differentiable function on* (δ,ε), $G'\in L_1([\delta,\varepsilon])$, *and* $\delta<\varepsilon$. *If* $|G'|$ *is a quasi-convex function, then the inequality*

$$|(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)})[G(\delta)+G(\varepsilon)]$$

$$-\left[{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(G\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^-}(G\circ\psi)(\psi^{-1}(\delta))\right]|$$

$$\leq \frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\frac{1}{2^{\omega}(\omega+1)}\left(\max\{|G'(\delta)|^q,|G'(\varepsilon)|^q\}\right)^{\frac{1}{q}}$$

holds for $\omega\in(0,1)$ and $q\geq 1$.

Proof. Utilizing Lemma 1.2, the power-mean inequality and quasi-convexity of $|G'|$, we have

$$\left|\left[{}^{HAB}_{\psi}I^{\omega}_{\left(\psi^{-1}(\frac{\delta+\varepsilon}{2})\right)^+}(G\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\left(\psi^{-1}(\frac{\delta+\varepsilon}{2})\right)^-}(G\circ\psi)(\psi^{-1}(\delta))\right]\right.$$

$$\left.-(\frac{(\varepsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}G(\tfrac{\varepsilon+\delta}{2})+\frac{1-\omega}{B(\omega)}[G(\delta)+G(\varepsilon)])\right|$$

$$\leq \frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\left(\int_0^{1/2}r^{\omega}dr\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}r^{\omega}\max\{|G'(\delta)|^q,|G'(\varepsilon)|^q\}dr\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{1/2}^1(1-r)^{\omega}dr\right)^{1-\frac{1}{q}}\left(\int_{1/2}^1(1-r)^{\omega}\max\{|G'(\delta)|^q,|G'(\varepsilon)|^q\}dr\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\left(\frac{1}{2^{\omega+1}(\omega+1)}\right)^{1-\frac{1}{q}}\left(\frac{1}{2^{\omega+1}(\omega+1)}\max\{|G'(\delta)|^q,|G'(\varepsilon)|^q\}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{2^{\omega+1}(\omega+1)}\right)^{1-\frac{1}{q}}\left(\frac{1}{2^{\omega+1}(\omega+1)}\max\{|G'(\delta)|^q,|G'(\varepsilon)|^q\}\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\frac{1}{2^{\omega}(\omega+1)}\left(\max\{|G'(\delta)|^q,|G'(\varepsilon)|^q\}\right)^{\frac{1}{q}}.$$

Substituting

$$\int_0^{1/2}r^{\omega}dr=\int_{1/2}^1(1-r)^{\omega}dr=\frac{1}{2^{\omega+1}(\omega+1)}$$

into the above inequality and simplifying lead to the required inequality. The proof of Theorem 2.3 is complete.

Theorem 2.4 *Let* $G:[\delta,\varepsilon]\to R$ *be a differentiable function on* (δ,ε), $G^{'}\in L_1([\delta,\varepsilon])$, *and* $\delta<\varepsilon$. *If* $|G^{'}|$ *is a* $P-$ *function, then the inequality*

$$|(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)})[G(\delta)+G(\varepsilon)]$$

$$-\left\lfloor {}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(G\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^{-}}(G\circ\psi)(\psi^{-1}(\delta))\right\rfloor|$$

$$\leq\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\frac{1}{2^{\omega}(\omega+1)}\left(G^{'}(\delta)|^{q}+|G^{'}(\varepsilon)|^{q}\right)^{\frac{1}{q}}$$

holds for $\omega\in(0,1)$ and $q\geq1$.

Proof. Utilizing Lemma 1.2, the power-mean inequality and $P-$function of $|G^{'}|$, we have

$$\left\|\left[{}^{HAB}_{\psi}I^{\omega}_{\left(\psi^{-1}(\frac{\delta+\varepsilon}{2})\right)^{+}}(G\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\left(\psi^{-1}(\frac{\delta+\varepsilon}{2})\right)^{-}}(G\circ\psi)(\psi^{-1}(\delta))\right]\right.$$

$$\left.-(\frac{(\varepsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}G(\tfrac{\varepsilon+\delta}{2})+\frac{1-\omega}{B(\omega)}[G(\delta)+G(\varepsilon)])\right|$$

$$\leq\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\left(\int_{0}^{1/2}r^{\omega}dr\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{1}{2}}r^{\omega}(|G^{'}(\delta)|^{q}+|G^{'}(\varepsilon)|^{q})dr\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{1/2}^{1}(1-r)^{\omega}dr\right)^{1-\frac{1}{q}}\left(\int_{1/2}^{1}(1-r)^{\omega}(|G^{'}(\delta)|^{q}+|G^{'}(\varepsilon)|^{q})dr\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\left(\frac{1}{2^{\omega+1}(\omega+1)}\right)^{1-\frac{1}{q}}\left(\frac{1}{2^{\omega+1}(\omega+1)}(|G^{'}(\delta)|^{q}+|G^{'}(\varepsilon)|^{q})\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{2^{\omega+1}(\omega+1)}\right)^{1-\frac{1}{q}}\left(\frac{1}{2^{\omega+1}(\omega+1)}(|G^{'}(\delta)|^{q}+|G^{'}(\varepsilon)|^{q})\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\frac{1}{2^{\omega}(\omega+1)}\left(|G^{'}(\delta)|^{q}+|G^{'}(\varepsilon)|^{q}\right)^{\frac{1}{q}}.$$

This completes the proof.

Theorem 2.5 *Let* $G:[\delta,\varepsilon]\to R$ *be a differentiable function on* (δ,ε), $G^{'}\in L_1([\delta,\varepsilon])$, *and* $\delta<\varepsilon$. *If* $|G^{'}|$ *is a quasi-convex function, then the inequality*

$$|(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)})[G(\delta)+G(\varepsilon)]$$

$$-\left\lfloor {}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(G\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^{-}}(G\circ\psi)(\psi^{-1}(\delta))\right\rfloor|$$

$$\leq\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega p+1}(\omega p+1)}\right)^{\frac{1}{p}}\left(\max\{|G^{'}(\delta)|^{q},|G^{'}(\varepsilon)|^{q}\}\right)^{\frac{1}{q}}$$

holds for $\omega\in(0,1)$ and $q\geq1$.

Proof. Utilizing Lemma 1.2, the Hölder inequality and quasi-convexity of $|G^{'}|$, we have

$$\left\|\left[{}^{HAB}_{\psi}I^{\omega}_{\left(\psi^{-1}(\frac{\delta+\varepsilon}{2})\right)^{+}}(G\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\left(\psi^{-1}(\frac{\delta+\varepsilon}{2})\right)^{-}}(G\circ\psi)(\psi^{-1}(\delta))\right]\right.$$

$$\left.-(\frac{(\varepsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}G(\tfrac{\varepsilon+\delta}{2})+\frac{1-\omega}{B(\omega)}[G(\delta)+G(\varepsilon)])\right|$$

$$\leq\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\left(\int_{0}^{1/2}r^{\omega p}dr\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\max\{|G^{'}(\delta)|^{q},|G^{'}(\varepsilon)|^{q}\}dr\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{1/2}^{1}(1-r)^{\omega p}dr\right)^{\frac{1}{p}}\left(\int_{1/2}^{1}\max\{|G^{'}(\delta)|^{q},|G^{'}(\varepsilon)|^{q}\}dr\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\left(\frac{1}{2^{\omega p+1}(\omega p+1)}\right)^{\frac{1}{p}}\left(\frac{\max\{|G^{'}(\delta)|^{q},|G^{'}(\varepsilon)|^{q}\}}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{2^{\omega p+1}(\omega p+1)}\right)^{\frac{1}{p}}\left(\frac{\max\{|G^{'}(\delta)|^{q},|G^{'}(\varepsilon)|^{q}\}1}{2}\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega p+1}(\omega p+1)}\right)^{\frac{1}{p}}\left(\max\{|G^{'}(\delta)|^{q},|G^{'}(\varepsilon)|^{q}\}\right)^{\frac{1}{q}}.$$

This completes the proof.

Theorem 2.6 *Let* $\mathsf{G}:[\delta,\varepsilon]\rightarrow \mathsf{R}$ *be a differentiable function on* (δ,ε), $\mathsf{G}^{'}\in L_1([\delta,\varepsilon])$, *and* $\delta<\varepsilon$. *If* $|\mathsf{G}^{'}|$ *is a* P*-function, then the inequality*

$$|(\frac{(\varepsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)})[\mathsf{G}(\delta)+\mathsf{G}(\varepsilon)]$$

$$-\left[{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathsf{G}\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\psi^{-1}(\varepsilon)^{-}}(\mathsf{G}\circ\psi)(\psi^{-1}(\delta))\right]|$$

$$\leq\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega p+1}(\omega p+1)}\right)^{\frac{1}{p}}\left(\mathsf{G}^{'}(\delta)|^{q}+|\mathsf{G}^{'}(\varepsilon)|^{q}\right)^{\frac{1}{q}}$$

holds for $\omega\in(0,1)$ and $p>1$ and such that $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Utilizing Lemma 1.2, the Hölder inequality and quasi-convexity of $|\mathsf{G}^{'}|$, we have

$$\left\|\left[{}^{HAB}_{\psi}I^{\omega}_{\left(\psi^{-1}(\frac{\delta+\varepsilon}{2})\right)^{+}}(\mathsf{G}\circ\psi)(\psi^{-1}(\varepsilon))+{}^{HAB}_{\psi}I^{\omega}_{\left(\psi^{-1}(\frac{\delta+\varepsilon}{2})\right)^{-}}(\mathsf{G}\circ\psi)(\psi^{-1}(\delta))\right]\right.$$

$$\left.-(\frac{(\varepsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathsf{G}(\tfrac{\varepsilon+\delta}{2})+\frac{1-\omega}{B(\omega)}[\mathsf{G}(\delta)+\mathsf{G}(\varepsilon)])\right|$$

$$\leq\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\left(\int_0^{1/2}r^{\omega p}dr\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}(|\mathsf{G}^{'}(\delta)|^{q}+|\mathsf{G}^{'}(\varepsilon)|^{q})dr\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{1/2}^{1}(1-r)^{\omega p}\,dr\right)^{\frac{1}{p}}\left(\int_{1/2}^{1}(|\mathsf{G}^{'}(\delta)|^{q}+|\mathsf{G}^{'}(\varepsilon)|^{q})dr\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\left(\frac{1}{2^{\omega p+1}(\omega p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathsf{G}^{'}(\delta)|^{q}+|\mathsf{G}^{'}(\varepsilon)|^{q}}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{2^{\omega p+1}(\omega p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathsf{G}^{'}(\delta)|^{q}+|\mathsf{G}^{'}(\varepsilon)|^{q}}{2}\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\varepsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega p+1}(\omega p+1)}\right)^{\frac{1}{p}}\left(\mathsf{G}^{'}(\delta)|^{q}+|\mathsf{G}^{'}(\varepsilon)|^{q}\right)^{\frac{1}{q}}$$

and the proof is completed.

3. *Conclusion*

In this paper, we have established new Hermite–Hadamard type inequalities for quasi-convex and P-functions within the framework of the ψ-Hilfer–Atangana–Baleanu (ψ-HAB) fractional operators. By utilizing key identities associated with ψ-HAB operators, together with Hölder's inequality, the power-mean inequality, we derived upper and lower bounds that extend and unify several classical results in convex analysis. Our findings demonstrate that the ψ-HAB approach offers a powerful generalization capable of capturing a broader class of fractional integral inequalities.

The results presented in this study open several avenues for future research. One possible direction is the extension of these inequalities to other generalized convexity classes, such as h-convex or φ-convex functions.

References

AKDEMİR, A.O., KARAOĞLAN, A., RAGUSA, M.A., SET, E. (2021). Fractional Integral Inequalities via Atangana-Baleanu Operators for Convex and Concave Functions, *J. Funct. Spaces*, 2021(1), Article ID 1055434, 1–10.

ATANGANA, A., BALEANU, D. (2016). New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Thermal Science*, 20(2), 763–769.

AVCI-ARDIÇ, M., AKDEMİR, A.O., KAVURMACI, H. (2023). Integral inequalities for differentiable s-convex functions in the second sense via Atangana-Baleanu fractional integral operators, *Filomat*, 37(18), 6229–6244.

AZZOUZ, N., BENAISSA, B., BUDAK, H., DEMİR, İ. (2025). Hermite-Hadamard-Mercer type inequalities for fractional integrals: A study with h-convexity and ψ-Hilfer operators, *Bound Value Probl.*, 2025, Article number 14.

BUDAK, H., ALI, M.A., KASHURI, A. (2022). New inequalities of Hermite-Hadamard type for twice differentiable functions via generalized fractional integrals, *Turkish J. Ineq.*, 6(2), 27–39.

BUTT, S.I., KHAN, D., JAIN, S., OROS, G.I., AGARWAL, P., MOMANI, S. (2025). Fractional integral inequalities for Superquadratic functions via Atangana Baleanu's operator with applications, *Fractals*, 33(8), 1–24.

COŞKUN, M., YILDIZ, Ç., COTIRLA, L.-I. (2024). Novel Estimations of Hadamard-Type Integral Inequalities for Raina's Fractional Operators, *Fractal and Fractional*, 8(5), 302.

ÇELİK, B., ÖZDEMİR, M.E., AKDEMİR, A.O., SET, E. (2021). Integral Inequalities for some convexity classes via Atangana-Baleanu Integral Operators, *Turkish J. Ineq.*, 5(2), 82–92.

HEZENCİ, F., KARA, H., BUDAK, H. (2023). Conformable fractional versions of Hermite-Hadamard-type inequalities for twice-differentiable functions, *Bound. Value Probl.*, 2023(1), Article ID 48.

ION, D.A. (2007). Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Annals of University of Craiova, *Math. Comp. Sci. Ser.*, 34, 82–87.

KERMAUSUOR, S., DENU, D., Olanipekun, P.O. (2026). ψ-Hilfer-Atangana-Baleanu fractional integral operators and some fractional integral inequalities, Kragujevac J. Math., 50(10), 1585–1604.

KILBAS, A.A., SRIVASTAVA, H.M., TRUJILLO, J. Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.

MUNIR, A., KASHURI, A., HEZENCI, F., KARA, H. (2025). Novelty Hermite-Hadamard-type inequalities for various convexity functions and their applications, *Sahand Commun. Math. Anal.*, 2(4), 213–239.

PACHPATTE, B.G. (2005). Mathematical Inequalities, Elsevier, Amsterdam.

PEARCE, C.E.M., RUBINOV, A.M. (1999). *P*-functions, quasi-convex functions and Hadamard-type inequalities, *J. Math. Anal. Appl.*, 240(1), 92–104.

PECARIC, J., PROSCHAN, F., TONG, Y.L. (1992). Convex Functions, Partial Orderings and Statistical Applications, Academic Press, Boston.

SARIKAYA, M.Z., SET, E., YALDIZ, H., BAŞAK, N. (2013). Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, 57, 2403–2407.

SET, E., ÇELİK, B. (2018). Generalized fractional Hermite-Hadamard type inequalities for convex and convex functions, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 67(1), 333–344.

SET E., CHOI, J., ÇELİK, B. (2018). New Hermite-Hadamard type inequalities for product of different convex functions involving certain fractional integral operators, *J. Math. Comp. Sci.*, 18(1), 29–36.

XI, B.-Y., BAI, R.F., QI, F. (2012). Hermite-Hadamard type inequalities for the m- and (α, m)-geometrically convex functions, *Aequationes Math.*, 84(3), 261–269.

YALÇIN, A., GÜL, E., AKDEMİR, A.O. (2025). Hermite-Hadamard type inequalities for Co-ordinated convex functions with variable-order fractional integrals, *Appl. Comput. Math.*, 24(2), 326–343.