Oscillation criteria for a certain class of fractional order integro-differential equations

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Abstract

In this paper, we shall give some new results about the oscillatory behavior of nonlinear fractional order integro-differential equations with forcing term $v(t)$ of form

$$D_a^\alpha x(t) = v(t) - \int_a^t K(t,s)F(s,x(s))ds, \ 0 < \alpha < 1, \ \lim_{t \to a^+} J_a^{1-\alpha} x(t) = b_1,$$

where v, K and F are continuous functions, $b_1 \in \mathbb{R}$, and D_a^α and $J_a^{1-\alpha}$ denote the Riemann-Liouville fractional order differential and integral operators respectively.

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1. Introduction

Fractional order differential equations have gained more importance in last two decades because of their various applications in different disciplines of science and engineering, such as in control theory, viscoelasticity, electromagnetic, etc. [5, 9, 14, 18]. Many mathematicians have studied the existence and uniqueness of solutions, the stability of solutions, the methods of explicit and numerical solutions of fractional order differential equations [3, 11, 13, 20, 24, 25]. For a background of the subject, we refer in particular to the books [2, 10, 19, 22, 23].

As far as the oscillation theory of fractional order differential equations are concerned, there are only a few papers. To the best of our knowledge, the paper by Grace et al. [15] has initiated the study of oscillation theory for fractional differential equations. In mentioned paper, the authors established several results criteria for solutions of equations in the form

$$D_a^\alpha x + f_1(t, x) = v(t) + f_2(t, x)$$

satisfying the initial condition

$$\lim_{t \to a^+} J_a^{1-\alpha} x(t) = b_1,$$

where D_a^α denotes the Riemann-Liouville fractional differential operator of order α with $0 < \alpha < 1$. The operator J_a^α defined by

$$J_a^\alpha x(t) := \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds$$

is called the Riemann-Liouville fractional integral operator of order α. Note that

$$f \leq g \implies J_a^\alpha(f) \leq J_a^\alpha(g). \tag{$*$}$$

The Riemann-Liouville fractional differential operator of order α with $m - 1 < \alpha < m$, $m \in \mathbb{N}$, is given by

$$D_a^\alpha x(t) := \frac{d^m}{dt^m} J_a^{m-\alpha} x(t),$$

as a special case for $0 < \alpha < 1$, Riemann-Liouville fractional differential operator is given by

$$D_a^\alpha x(t) := \frac{d}{dt} J_a^{1-\alpha} x(t).$$

Following the work developed in [15], Chen et al. in the [6] studied the fractional differential equation

$$\begin{cases} (D_a^\alpha x)(t) + f_1(t, x) = v(t) + f_2(t, x), & t > a \geq 0, \\ (D_a^{\alpha-k} x)(t) = b_k \ (k = 1, 2, ..., m-1), & \lim_{t \to a^+} (I_a^{m-\alpha} x)(t) = b_m. \end{cases}$$

and established some new results different than the ones obtained in [15]. Here, $I_a^{m-\alpha}$ is also a Riemann- Liouville fractional integral operator.

Further results on the oscillation theory of fractional order differential equations can be found in [1, 4, 7, 8, 12, 16, 21].

In the present work, we study the oscillatiory behavior of solutions of nonlinear fractional order integro-differential equations with forcing term $v(t)$ of form

$$(1.1) \qquad D_a^\alpha x(t) = v(t) - \int_a^t K(t,s) F(s, x(s)) ds, \ 0 < \alpha < 1, \ \lim_{t \to a^+} J_a^{1-\alpha} x(t) = b_1,$$

where v, K and F are continuous functions, $b_1 \in \mathbb{R}$, and D_a^α and $J_a^{1-\alpha}$ denotes the Riemann-Liouville fractional order differential and integral operators, respectively. The equation that we considered above is more general than those investigated before in the literature.

By a solution of Eq.(1.1), we mean a nontrivial function $x \in C(J, \mathbb{R})$, with $J = (a, \infty)$, satisfies Eq.(1.1) for $t \geq a \geq 0$. A solution is said to be *oscillatory* if it has arbitrarily large zeros on $(0, \infty)$; otherwise, it is called *nonoscillatory*.

2. Results for Riemann-Liouville derivative

With regard to Eq. (1.1) we assume that

(i) $v : J \to \mathbb{R}$ and $K : J \times J \to \mathbb{R}$ are continuous functions with $K(t, s) \geq 0$ for $t > s$;

(ii) there exist continuous functions $p, q : J \to [0, \infty)$ such that

$K(t, s) \leq p(t)q(s)$ for all $t \geq s$;

(iii) $F : J \times \mathbb{R} \to \mathbb{R}$, with $F(t, x) := f_1(t, x) - f_2(t, x)$, is continuous and there exist continuous functions $f_1, f_2 : J \times \mathbb{R} \to \mathbb{R}$ such that $x f_i(t, x) > 0$ $(i = 1, 2)$ for $x \neq 0$ and $t \geq a$,

(iv) there exist real constants β, γ and continuous functions $p_1, p_2 : J \to (0, \infty)$ such that

$f_1(t, x) \geq p_1(t)x^\beta$ and $f_2(t, x) \leq p_2(t)x^\gamma$, $x \neq 0$, $t \geq a$.

We will make use of the following lemma, extracted from [17].

2.1. Lemma. *If X and Y are nonnegative, then*

$$(2.1) \qquad X^\lambda + (\lambda - 1)Y^\lambda - \lambda XY^{\lambda-1} \geq 0, \ \lambda > 1,$$

and

$$(2.2) \qquad X^\lambda - (1 - \lambda)Y^\lambda - \lambda XY^{\lambda-1} \leq 0, \ \lambda < 1,$$

with equality holds if and only if $X = Y$.

Our first result is as follows.

2.2. Theorem. *Let conditions (i)-(iii) hold with $f_2 = 0$. If for every constant $k > 0$*

$$(2.3) \qquad \limsup_{t \to \infty} J_a^\alpha[v(t) - kp(t)] = +\infty$$

and

$$(2.4) \qquad \liminf_{t \to \infty} J_a^\alpha[v(t) + kp(t)] = -\infty,$$

then every solution of Eq.(1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq.(1.1) with $f_2 = 0$. We may assume that $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq a$. The case where $x(t) < 0$ for $t \geq t_1$ is similar. From Eq.(1.1), we have

$$D_a^\alpha x(t) = v(t) - \int_a^t K(t, s)F(s, x(s))ds$$

$$(2.5) \qquad = v(t) - \int_a^{t_1} K(t, s)f_1(s, x(s))ds - \int_{t_1}^t K(t, s)f_1(s, x(s))ds.$$

Letting

$$m := \min\{F(t, x(t)) : t \in [a, t_1]\} \leq 0 \text{ and } k := -m \int_a^{t_1} q(s)ds \geq 0,$$

(2.5) leads to

(2.6) $D_a^\alpha x(t) \leq v(t) + kp(t).$

Using the monotonicity property (*), we have

$$J_a^\alpha D_a^\alpha x(t) \leq J_a^\alpha [v(t) + kp(t)],$$

and hence

(2.7) $x(t) \leq \dfrac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} b_1 + J_a^\alpha [v(t) + kp(t)].$

In view of (2.4), it follows from (2.7) that

$$\liminf_{t \to \infty} x(t) = -\infty,$$

which clearly contradicts the assumption that $x(t) > 0$ eventually. This completes the proof. $\square$

Next, we have the following results.

2.3. Theorem. *Let conditions (i)-(iv) hold with $\beta > 1$ and $\gamma = 1$. In addition to conditions of Theorem 1, if*

(2.8) $\displaystyle\int_a^\infty \dfrac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \int_a^u K(u,s)p_1^{\frac{1}{1-\beta}}(s)p_2^{\frac{\beta}{\beta-1}}(s)dsdu < \infty,$

then every solution of Eq.(1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq.(1.1) with $x(t) > 0$ for $t \geq t_1$. From conditions (iii)-(iv) with $\beta > 1$ and $\gamma = 1$, we have

(2.9) $D_a^\alpha x(t) \leq v(t) + kp(t) + \displaystyle\int_{t_1}^t K(t,s)[p_2(s)x(s) - p_1(s)x^\beta(s)]ds,$

for some $k > 0$. If we take in (2.1) $\lambda = \beta$, $X = p_1^{\frac{1}{\beta}} x$ and $Y = \left(\frac{1}{\beta}p_2 p_1^{\frac{-1}{\beta}}\right)^{\frac{1}{\beta-1}}$, then we have

(2.10) $p_2 x - p_1 x^\beta \leq (\beta-1)\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{-1}{\beta-1}} p_2^{\frac{\beta}{\beta-1}}.$

Using (2.10) in (2.9), we have

(2.11) $D_a^\alpha x(t) \leq v(t) + kp(t) + \displaystyle\int_{t_1}^t K(t,s)(\beta-1)\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{-1}{\beta-1}}(s)p_2^{\frac{\beta}{\beta-1}}(s)ds.$

Applying α order fractional operator J_a^α to (2.11), we have

$$x(t) \leq \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}b_1 + J_a^\alpha[v(t)+kp(t)]$$
$$+ \int_a^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \int_{t_1}^u K(u,s)(\beta-1)\beta^{\frac{\beta}{1-\beta}}p_1^{\frac{-1}{\beta-1}}(s)p_2^{\frac{\beta}{\beta-1}}(s)dsdu.$$

(2.12)

Taking limit inferior on both sides of (2.12) as $t \to \infty$, and using (2.4) and (2.8), we have

$$\liminf_{t\to\infty}x(t) = -\infty,$$

a contradiction with $x(t) > 0$ eventually. This completes the proof. $\square$

2.4. Theorem. *Let conditions (i)-(iv) hold with $\beta = 1$ and $\gamma < 1$. In addition to conditions of Theorem 1, if*

$$(2.13) \quad \int_a^\infty \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \int_a^u K(u,s)p_1^{\frac{\gamma}{\gamma-1}}(s)p_2^{\frac{1}{1-\gamma}}(s)dsdu < \infty,$$

then every solution of Eq.(1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq.(1.1) with $x(t) > 0$ for $t \geq t_1$. From conditions (iii)-(iv) with $\beta = 1$ and $\gamma < 1$, we have

$$(2.14) \quad D_a^\alpha x(t) \leq v(t) + kp(t) + \int_{t_1}^t K(t,s)[p_2(s)x^\gamma(s) - p_1(s)x(s)]ds,$$

for some $k > 0$. If we take $\lambda = \gamma$, $X = p_2^{\frac{1}{\gamma}}x$ and $Y = \left(\frac{1}{\gamma}p_1 p_2^{\frac{-1}{\gamma}}\right)^{\frac{1}{\gamma-1}}$ in (2.2), then we have

$$(2.15) \quad p_2 x^\gamma - p_1 x \leq (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}p_1^{\frac{\gamma}{\gamma-1}}p_2^{\frac{1}{1-\gamma}}.$$

Using (2.15) in (2.14), we have

$$(2.16) \quad D_a^\alpha x(t) \leq v(t) + kp(t) + \int_{t_1}^t K(t,s)(1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}p_1^{\frac{\gamma}{\gamma-1}}(s)p_2^{\frac{1}{1-\gamma}}(s)ds$$

The rest of the proof is similar to the proof of Theorem 2.3. and will be omitted. $\square$

Finally, we consider the case $\beta > 1$ and $\gamma < 1$.

2.5. Theorem. *Let conditions (i)-(iv) hold with $\beta > 1$ and $\gamma < 1$. In addition to conditions of Theorem 1, assume that there exists a continuous function $\xi : \mathbb{R} \to (0,\infty)$ such that*

$$(2.17) \quad \int_a^\infty \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \int_a^u K(u,s)p_1^{\frac{1}{1-\beta}}(s)\xi^{\frac{\beta}{\beta-1}}(s)dsdu < \infty,$$

and

$$(2.18) \quad \int_a^\infty \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \int_a^u K(u,s)\xi^{\frac{\gamma}{\gamma-1}}(s)p_2^{\frac{1}{1-\gamma}}(s)dsdu < \infty,$$

then every solution of Eq.(1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq.(1.1) with $x(t) > 0$ for $t \geq t_1$. Using the approach above, from conditions (iii)-(iv) with $\beta > 1$ and $\gamma < 1$, we have

$$
D_a^\alpha x(t) \leq v(t) + kp(t) + \int_{t_1}^t K(t,s)[x(s)\xi(s) - p_1(s)x^\beta(s)]ds
$$

$$
(2.19) \qquad + \int_{t_1}^t K(t,s)[p_2(s)x^\gamma(s) - x(s)\xi(s)]ds,
$$

for some $k > 0$. Taking $p_2(s) = \xi(s)$ in (2.10), and $p_1(s) = \xi(s)$ in (2.15), (2.19) yields

$$
D_a^\alpha x(t) \leq v(t) + kp(t) + \int_{t_1}^t K(t,s)(\beta - 1)\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{-1}{\beta-1}}(s)\xi^{\frac{\beta}{\beta-1}}(s)ds
$$

$$
(2.20) \qquad + \int_{t_1}^t K(t,s)(1 - \gamma)\gamma^{\frac{\gamma}{1-\gamma}}\xi^{\frac{\gamma}{\gamma-1}}(s)p_2^{\frac{1}{1-\gamma}}(s)ds.
$$

The rest of the proof is similar to that of Theorem 2.3. $\qquad\square$

2.6. Example. Consider the integro-differential equation with Riemann-Liouville fractional derivative

$$
(2.21) \quad D_0^{1/3}x(t) = \frac{t^{2/3}}{\Gamma(5/3)} + \frac{t^4}{3} - t\int_0^t sx(s)ds, \quad \lim_{t\to0^+} J_0^{2/3}x(t) = 0.
$$

Comparing with Eq.(1.1) with $f_2 = 0$, we have

$$
\alpha = \frac{1}{3}, \ a = b_1 = 0, \ f_1(t,x) = x, \ v(t) = \frac{t^{2/3}}{\Gamma(5/3)} + \frac{t^4}{3}, \ K(t,s) = ts.
$$

Conditions $(i) - (iii)$ are satisfied. But, condition (2.4) fails, since

$$
\liminf_{t\to\infty}\left[t + \frac{8t^{13/3}}{\Gamma(16/3)}\right] = \infty.
$$

One can easily verify that $x(t) = t$ is a *nonoscillatory* solution of Eq.(2.21). Here,

$$
\lim_{t\to0^+} J_0^{2/3}x(t) = \lim_{t\to0^+} \frac{9t^{5/3}}{10\Gamma(2/3)} = 0,
$$

Note that here $m = k = 0$.

2.7. Example. Consider the integro-differential equation with Riemann-Liouville fractional derivative

$$
(2.22) \quad D_0^{1/2}x(t) = \frac{8t^{3/2}}{3\sqrt{\pi}} + \frac{t^5}{4} - \frac{t^4}{3} - t\int_0^t s\left[x(s) - \frac{x(s)}{s}\right]ds, \quad \lim_{t\to0^+} J_0^{1/2}x(t) = 0.
$$

Now we have

$$
\alpha = \frac{1}{2}, \ a = b_1 = 0, \ f_1(t,x) = x, \ f_2(t,x) = \frac{x}{t}, \ v(t) = \frac{8t^{3/2}}{3\sqrt{\pi}} + \frac{t^5}{4} - \frac{t^4}{3}, \ K(t,s) = ts.
$$

Conditions $(i) - (iv)$ are satisfied with $\gamma = 1$, $\beta = 2$ and $p_1(t) = t^{-3}$, $p_2(t) = t$. But, condition (2.8) is not, because

$$\lim_{t\to\infty} \frac{1}{7\sqrt{\pi}} \int\limits_0^t (t-u)^{-1/2} u^8 du = \infty.$$

One can easily verify that $x(t) = t^2$ is a *nonoscillatory* solution of Eq.(2.22), as

$$\lim_{t\to 0^+} J_0^{1/2} x(t) = \lim_{t\to 0^+} \frac{16 t^{5/2}}{15\sqrt{\pi}} = 0.$$

3. Results for Caputo Fractional Derivative

If we replace the Caputo fractional derivative by the Riemann-Liouville fractional derivative defined by

$$^C D_a^q f(t) = J_a^{m-q} f^{(m)}(t), \ m-1 < q < m, \ m \in \mathbb{N},$$

Eq.(1.1) turns into

(3.1)
$$^C D_a^\alpha x(t) = v(t) - \int\limits_a^t K(t,s) F(s, x(s)) ds, \quad m-1 < \alpha < m,$$
$$D^i x(a) = b_i \in \mathbb{R}, \ i = 0, 1, ..., m-1$$

Below we provide corresponding results for (3.1). Since the arguments are similar to the Riemann-Liouville case, we only give the proof of the first theorem.

3.1. Theorem. *Let conditions (i)-(iii) hold with $f_2 = 0$. If for every constant $k > 0$*

(3.2)
$$\limsup_{t\to\infty} \ t^{1-m} J_a^\alpha [v(t) - kp(t)] = +\infty$$

and

(3.3)
$$\liminf_{t\to\infty} \ t^{1-m} J_a^\alpha [v(t) + kp(t)] = -\infty,$$

then every solution of Eq.(3.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq.(3.1) with $f_2 = 0$. We may assume that $x(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 1, we have

(3.4)
$$^C D_a^\alpha x(t) \leq v(t) + kp(t).$$

Applying the α order fractional operator J_a^α to (3.4), we have

(3.5)
$$t^{1-m} x(t) \leq \sum_{k=0}^{m-1} \frac{D^k x(a)}{k!} (t-a)^{k-m+1} + t^{1-m} J_a^\alpha [v(t) + kp(t)]$$

Using (3.3) we see that

$$\liminf_{t\to\infty} \frac{x(t)}{t^{m-1}} = -\infty.$$

This however contradicts the fact that $x(t) > 0$ eventually. This completes the proof. $\square$

3.2. Theorem. *Let conditions (i)-(iv) hold with $\beta > 1$ and $\gamma = 1$. In addition to conditions of Theorem 5, if*

(3.6)
$$\lim_{t\to\infty} \ t^{1-m} \int\limits_a^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \int\limits_a^u K(u,s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s) ds du < \infty,$$

then every solution of Eq.(3.1) is oscillatory.

3.3. Theorem. *Let conditions (i)-(iv) hold with $\beta = 1$ and $\gamma < 1$. In addition to conditions of Theorem 5, if*

$$(3.7) \qquad \lim_{t\to\infty} \ t^{1-m} \int\limits_a^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \int\limits_a^u K(u,s)p_1^{\frac{\gamma}{\gamma-1}}(s)p_2^{\frac{1}{1-\gamma}}(s)dsdu < \infty,$$

then every solution of Eq.(3.1) is oscillatory.

3.4. Theorem. *Let conditions (i)-(iv) hold with $\beta > 1$ and $\gamma < 1$. In addition to conditions of Theorem 5, assume that there exists a continuous function $\xi : \mathbb{R} \to (0,\infty)$ such that*

$$(3.8) \qquad \lim_{t\to\infty} \ t^{1-m} \int\limits_a^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \int\limits_a^u K(u,s)p_1^{\frac{1}{1-\beta}}(s)\xi^{\frac{\beta}{\beta-1}}(s)dsdu < \infty,$$

and

$$(3.9) \qquad \lim_{t\to\infty} \ t^{1-m} \int\limits_a^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \int\limits_a^u K(u,s)\xi^{\frac{\gamma}{\gamma-1}}(s)p_2^{\frac{1}{1-\gamma}}(s)dsdu < \infty,$$

then every solution of Eq.(3.1) is oscillatory.,

3.5. Example. Consider the integro-differential equation with Caputo fractional derivative

$$(3.10) \qquad {}^C D_0^{3/2}x(t) = \frac{2t^{1/2}}{\sqrt{\pi}} + \frac{t^5}{4} - t\int\limits_0^t sx(s)ds, \ x(0) = 0, \ x'(0) = 0.$$

Comparing with Eq.(3.1) with $f_2 = 0$, we see

$$\alpha = \frac{3}{2}, \ a = b_1 = b_2 = 0, \ f_1(x,t) = x, \ v(t) = \frac{2t^{1/2}}{\sqrt{\pi}} + \frac{t^5}{4}, \ K(t,s) = ts.$$

Conditions $(i) - (iii)$ are satisfied. But, condition (3.3) does not satisfy, because

$$\liminf_{t\to\infty} \left[\frac{t}{2} + \frac{256t^{11/2}}{9009\sqrt{\pi}} \right] = \infty.$$

One can effortlessly verify that $x(t) = t^2$ is a *nonoscillatory* solution of Eq.(3.1).

We end this work with following remark.

3.6. Remark. We noting that the function

$$K(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$$

satisfies assumptions (i) and (ii). Therefore, the results given above are valid for equations of the form

$$D_a^\alpha x(t) = v(t) - J_a^\alpha F(t,x(t)), \ 0 < \alpha < 1, \ \lim_{t\to a^+} J_a^{1-\alpha}x(t) = b_3.$$

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