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Odd Burr Lindley distribution with properties and applications

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Abstract

In this study, we introduce a new model called the Odd Burr Lindley distribution which extends the Lindley distribution and has increasing, bathtub and upside down shapes for the hazard rate function. It includes the odd Lindley distribution as a special case. Several statistical properties of the distribution are explored, such as the density, hazard rate, survival, quantile functions, and moments. Estimation using the maximum likelihood and inference of a random sample from the distribution are investigated. A simulation study is performed to compare the performance of the different parameter estimates in terms of bias and mean square error. Two real data applications are modelled with the proposed distribution to illustrate the performance of the new distribution. Based on goodness-of-fit statistics, the new distribution outperforms the generalized gamma, gamma Weibull, gamma exponentiated exponential, generalized Lindley, Kumaraswamy Lindley, and odd log-logistic Lindley distributions.

Keywords: odd Burr Family, Lindley distribution, moments, maximum likelihood.

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1. Introduction

The statistical analysis and modeling of lifetime data are essential in almost all applied sciences including, biomedical science, engineering, finance, and insurance, amongst others. A number of one-parameter continuous distributions for modeling lifetime data has been introduced in statistical literature including exponential, Lindley, gamma, lognormal, and Weibull. The exponential, Lindley and Weibull distributions are more popular than the gamma and lognormal distributions because the survival functions of the gamma and the lognormal distributions cannot be expressed in closed forms and both require numerical integration. The Lindley distribution is a very well-known distribution that has been extensively used over the past decades for modeling data in reliability, biology, insurance, finance, and lifetime analysis. It was introduced by Lindley (1958) to analyze failure time data, especially in applications of modeling stress-strength reliability. The motivation for introducing the Lindley distribution arises from its ability to model failure time data with increasing, decreasing, unimodal and bathtub shaped hazard rates. It may also be mentioned that the Lindley distribution belongs to an exponential family and it can be written as a mixture of an exponential and a gamma distributions. This distribution represents a good alternative to the exponential failure time distributions that suffer from not exhibiting unimodal and bathtub shaped failure rates (Bakouch *et al.*, 2012). The properties and inferential procedure for the Lindley distribution were studied by Ghitany *et al.* (2008, 2011). They show via a numerical example that the Lindley distribution gives better modelling than the one based on the exponential distribution when hazard rate is unimodal or bathtub shaped. Furthermore, Mazucheli and Achcar (2011) found that many of the mathematical properties are more flexible than those of the exponential distribution and proposed the Lindley distribution as a possible alternative to exponential or Weibull distributions.

The need for extended forms of the Lindley distribution arises in many applied areas. The emergence of such distributions in the statistics literature is only very recent. For some extended forms of the Lindley distribution and applications, the reader is referred to Kumaraswamy Lindley (Cakmakyapan and Ozel, 2014), beta odd log-logistic Lindley (Cordeiro *et al.*, 2015), generalized Lindley (Nadarajah *et al.*, 2011), quasi Lindley distributions (Shanker and Mishra, 2013). The probability density function (pdf) and cumulative density function (cdf) of the Lindley distribution are, respectively, given by

$$g(x;\lambda) = \frac{\lambda^2}{1+\lambda}(1+x)\,e^{-\lambda x},\ x>0,\ \lambda>0 \tag{1.1}$$

$$G(x;\lambda) = 1-\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x},\ x>0,\ \lambda>0 \tag{1.2}$$

It can be seen that this distribution is a mixture of Exponential (λ) and gamma ($2,\lambda$) distributions. Having only one parameter, the Lindley distribution does not provide enough flexibility for analyzing different types of lifetime data. To increase the flexibility for modelling purposes it will be useful to consider further alternatives to this distribution. Our purpose is to provide a generalization that may be useful to still more complex situations. Once the proposed distribution is quite flexible in terms of pdf and hazard rate function (hrf), it may provide an interesting alternative to describe income distributions and can also be applied in actuarial science, finance, bioscience, telecommunications and modelling lifetime data, for example. Therefore, the aim of this study is to introduce a new distribution using the Lindley distribution. Recenlty, Alizadeh *et al.* (2016) has introduced the OBu-G family with two extra shape parameters defined by cdf

$$(1.3)\qquad F\left(x;a,b\right)=1-\left\{1-\frac{G(x)^{a}}{G(x)^{a}+\bar{G}(x)^{a}}\right\}^{b},\ a,b>0,\ x>0$$

and the pdf

$$(1.4)\qquad f\left(x;a,b\right)=\frac{a\,b\,g(x)\,G(x)^{a-1}\,\bar{G}(x)^{a\,b-1}}{\left[G(x)^{a}+\bar{G}(x)^{a}\right]^{b+1}},\ a,b>0,\ x>0$$

where $\bar{G}\left(x\right)=1-G\left(x\right)$.

The main aim of this paper is to provide an extension of the Lindley distribution. Hence, we propose the Odd Burr Lindley (OBu-L for short) distribution by adding two extra parameters to the Lindley distribution. The article is outlined as follows: In Section 2, we introduce the OBu-L distribution and provide plots of the density and hazard rate functions. Shapes, quantile function, moments, and moment generating function are also obtained. Estimation by the method of maximum likelihood and an explicit expression for the observed information matrix are presented in Section 3. A simulation study is conducted in Section 4. Applications to real data sets are considered in Section 5. Finally, Section 6 offers some concluding remarks.

2. Main Properties

2.1. Probability Density and Cumulative Density Functions. Inserting (1.2) in (1.3), the cdf of the OBu-L with three parameters $(a,b,\lambda>0)$ is defined as

$$(2.1)\qquad F(x;a,b,\lambda)=1-\left\{1-\frac{\left[1-\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{a}}{\left[1-\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{a}+\left[\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{a}}\right\}^{b}$$

The corresponding pdf of the OBu-L is given by

$$(2.2)\qquad \begin{aligned} &f\left(x;a,b,\lambda\right)=ab\left(\tfrac{\lambda^{2}}{1+\lambda}\left(1+x\right)e^{-\lambda x}\right)\left(1-\left(1+\tfrac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right)^{a-1}\\ &\times\left(\left(1+\tfrac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right)^{ab-1}\\ &\times\left\{\left[1-\left(1+\tfrac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{a}+\left[\left(1+\tfrac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{a}\right\}^{-(b+1)}\end{aligned}$$

where λ is a scale parameter a and b are the shape parameters. Here, a and b govern the skewness of (2.2). A random variable X with the pdf (2.2) is denoted by $X\sim OBu-L\left(a,b,\lambda\right)$.

Some of the possible shapes of density function in (2.2) for the selected parameter values are illustrated in Figure 1. As seen in Figure 1, the density function can take various forms depending on the parameter values. It is evident that the OBu-L distribution is much more flexible than the Lindley distribution, i.e. the additional shape parameter *and* allow for a high degree of flexibility of the OBu-L distribution. Both unimodal and monotonically decreasing and increasing shapes appear to be possible. Because of its tractable distribution function (1.2), the OBu-L distribution can be used quite effectively even if the data are censored.

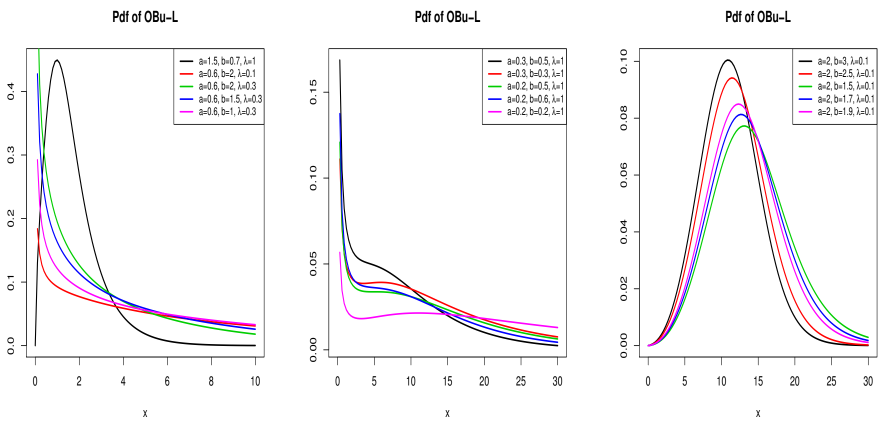

Figure 1. Plots of the pdf the OBu-L distribution for some parameter values.

2.2. Survival and Hazard Rate Functions. Central role is played in the reliability theory by the quotient of the pdf and survival function. We obtain the survival function corresponding to (2.1) as

$$S(x;a,b,\lambda)=\left\{1-\frac{\left[1-\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{a}}{\left[1-\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{a}+\left[\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{a}}\right\}^{b} \tag{2.3}$$

In reliability studies, the hrf is an important characteristic and fundamental to the design of safe systems in a wide variety of applications. Therefore, we discuss these properties of the OBu-L distribution. The hrf of X takes the form

$$\begin{aligned}&h\left(x;a,b,\lambda\right)=ab\left(\frac{\lambda^{2}}{1+\lambda}\left(1+x\right)e^{-\lambda x}\right)\left(1-\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right)^{a-1}\\&\times\left(\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right)^{ab-1}\left[\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{-a}\\&\times\left\{\left[1-\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{a}+\left[\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{a}\right\}^{-1}\end{aligned} \tag{2.4}$$

Plots for the hrf of the OBu-L distribution for several parameter values are displayed in Figure 2, respectively. Figure 2 shows that the hrf of the OBu-L distribution can have very flexible shapes, such as increasing, upside-down bathtub, and bathtub. This attractive flexibility makes the hrf of the OBu-L distribution useful and suitable for non-monotone empirical hazard behaviours which are more likely to be encountered or observed in real life situations.

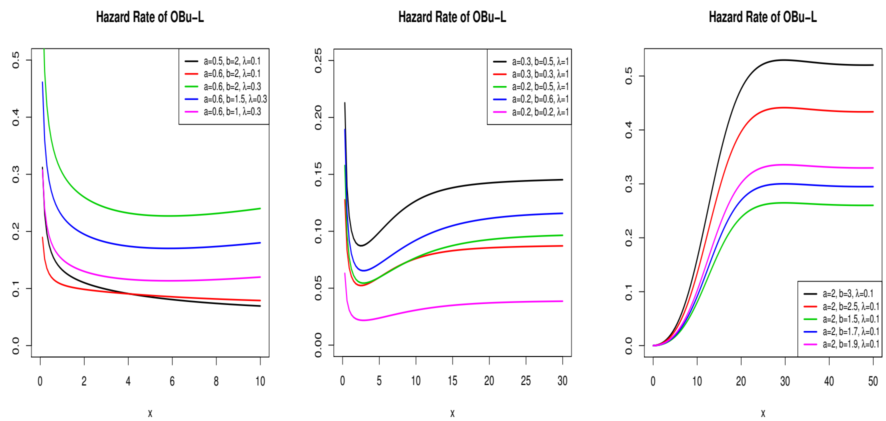

Figure 2. Plots of the hrf of the OBu-L distribution for some parameter values.

2.3. Asymptotic.

2.1. Proposition. *The asymptotics of cdf, pdf and hrf of the OBu-L distribution as* $x \to 0$ *are, respectively, given by*

$$F(x) \sim b\,(\lambda\, x)^a \qquad \text{as} \quad \mathrm{x} \to 0,$$
$$f(x) \sim a\, b\, \lambda^a\, x^{a-1} \qquad \text{as} \quad \mathrm{x} \to 0,$$
$$h(x) \sim a\, b\, \lambda^a\, x^{a-1} \qquad \text{as} \quad \mathrm{x} \to 0.$$

2.2. Proposition. *The asymptotics ofsurvival function, pdf and hrf of the OBu-L distribution as* $x \to \infty$ *are, respectively, given by*

$$1 - F(x) \sim (\frac{a\,\lambda}{1+\lambda})^b\, x^b\, \mathrm{e}^{-\mathrm{b}\,\lambda\,\mathrm{x}} \qquad \text{as} \quad \mathrm{x} \to \infty,$$
$$f(x) \sim b\,\lambda\,(\frac{a\,\lambda}{1+\lambda})^b\, x^b\, \mathrm{e}^{-\mathrm{b}\,\lambda\,\mathrm{x}} \qquad \text{as} \quad \mathrm{x} \to \infty,$$
$$h(x) \sim b\lambda \qquad \text{as} \qquad \mathrm{x} \to \infty.$$

Let us point out that these equations show the effect of parameters on tails of the OBu-L distribution.

2.4. Shapes. The density function of the OBu-L, given in (2.2), is decreasing, increasing or unimodal. In order to investigate the critical points of its density function, the first derivative of with respect to x is given by

$$\begin{aligned}\frac{d}{dx}f(x) = {} & \frac{abr\lambda^2(-(ab-1)\lambda-\lambda)(x+1)\left(ke^{\lambda x}\right)^{ab-1}(1-k)^{a-1}(k^a+(1-k)^a)^{-b-1}}{\lambda+1} + \\ & \frac{arb\lambda^2\left(ke^{\lambda x}\right)^{ab-1}(1-k)^{a-1}(k^a+(1-k)^a)^{-b-1}}{\lambda+1} \\ & + \frac{abr(ab-1)\lambda^3(x+1)\left(ke^{\lambda x}\right)^{ab-2}(1-k)^{a-1}(k^a+(1-k)^a)^{-b-1}}{\lambda+1} + \\ & \frac{(a-1)abr\lambda^2(x+1)\left(ke^{\lambda x}\right)^{ab-1}(1-k)^{a-2}(\lambda k)(k^a+(1-k)^a)^{-b-1}}{\lambda+1} \\ & + \left(\frac{a(-b-1)b\lambda^2(x+1)\left(ke^{\lambda x}\right)^{ab-1}(1-k)^{a-1}(k^a+(1-k)^a)^{-b-2}}{\lambda+1}\right) \times \\ & r\left(-a\lambda k^a + a\lambda k^a\frac{\left(ke^{\lambda x}\right)^{-1}}{(\lambda+1)} + a(1-k)^{a-1}\left(\lambda k - \frac{k-e^{-\lambda x}}{x}\right)\right)\end{aligned} \tag{2.5}$$

where $k = \frac{\lambda x+\lambda+1}{(\lambda+1)e^{\lambda x}}$ and $r = e^{-(ab-1)\lambda x-\lambda x}$. There may be more than one root to (2.5). If $x = x_0$ is a root of (2.5), then it corresponds to a local maximum if $df(x)/dx > 0$ for all $x < x_0$ and $df(x)/dx < 0$ for all $x > x_0$. It corresponds to a local minimum if $df(x)/dx < 0$ for all $x < x_0$ and $df(x)/dx > 0$ for all $x > x_0$. It corresponds to a point of inflexion if either $df(x)/dx > 0$ for all $x \neq x_0$ or $df(x)/dx < 0$ for all .

2.5. Extreme Value. If $X_1, ..., X_n$ is a random sample from (2.1) and if $\bar{X} = (X_1 + ... + X_n)/n$ denotes the sample mean then by the usual central limit theorem $\sqrt{n}(\bar{X} - E(X))/\sqrt{\mathrm{Var}(X)}$ approaches the standard normal distribution as $n \to \infty$. One may be interested in the asymptotic of the extreme values $M_n = \max(X_1, ..., X_n)$ and $m_n = \min(X_1, ..., X_n)$. For (2.1), it can be seen that

$$\lim_{t\to 0}\frac{F(t\,x)}{F(t)} = x^a$$

and

$$\lim_{t\to\infty}\frac{1-F(t\,x)}{1-F(t)} = \mathrm{e}^{-a\,b\lambda x}.$$

Thus, it follows from Theorem 1.6.2 of Leadbetter *et al.* (1987) that there must be norming constants $a_n > 0, \quad b_n, \quad c_n > 0$ and d_n such that

$$Pr\left[a_n(M_n - b_n) \le x\right] \to \mathrm{e}^{-\mathrm{a\,b}\lambda\mathrm{x}}$$

and

$$Pr\left[a_n(m_n - b_n) \le x\right] \to 1 - \mathrm{e}^{-\mathrm{x}^{\mathrm{a}}}$$

as $n \to \infty$. Using Corollary 1.6.3 of Leadbetter *et al.* (1987), we can obtain the form of normalizing constants a_n, b_n, c_n and d_n.

2.6. Quantile Function. Let $X \sim OBu - L(a, b, \lambda)$, the quantile function, say $Q(p)$, is defined by $F(Q(p)) = p$ and the root of the following equation

$$[1+\lambda+\lambda\,Q(p)]\,\mathrm{e}^{-\lambda\,Q(p)} = \frac{-(1+\lambda)\mathrm{e}^{-1-\lambda}\,(1-\mathrm{p})^{\frac{1}{\mathrm{a\,b}}}}{(1-p)^{\frac{1}{a\,b}} + \left[1-(1-p)^{\frac{1}{b}}\right]^{\frac{1}{a}}} \tag{2.6}$$

for $0 < p < 1$. Substituting $Z(p) = -1 - \lambda - \lambda\,Q(p)$, one can write (2.6) as

$$Z(p)\,\mathrm{e}^{\mathrm{Z(p)}} = \frac{-(1+\lambda)\mathrm{e}^{-1-\lambda}\,(1-\mathrm{p})^{\frac{1}{\mathrm{a\,b}}}}{(1-\mathrm{p})^{\frac{1}{\mathrm{a\,b}}} + \left[1-(1-\mathrm{p})^{\frac{1}{\mathrm{b}}}\right]^{\frac{1}{\mathrm{a}}}} \tag{2.7}$$

Hence, the equation of $Z(p)$ is

$$Z(p) = W\left\{ \frac{-(1+\lambda)\mathrm{e}^{-1-\lambda}\,(1-\mathrm{p})^{\frac{1}{\mathrm{a}\,\mathrm{b}}}}{(1-p)^{\frac{1}{a\,b}} + \left[1-(1-p)^{\frac{1}{b}}\right]^{\frac{1}{a}}} \right\} \tag{2.8}$$

where $W(.)$ is the Lambert function (Corless *et al.*, 1996). Inserting (1.2), we obtain,

$$Q(p) = -1 - \frac{1}{\lambda} - \frac{1}{\lambda}\,W\left\{ \frac{-(1+\lambda)\mathrm{e}^{-1-\lambda}\,(1-\mathrm{p})^{\frac{1}{\mathrm{a}\,\mathrm{b}}}}{(1-p)^{\frac{1}{a\,b}} + \left[1-(1-p)^{\frac{1}{b}}\right]^{\frac{1}{a}}} \right\} \tag{2.9}$$

The particular case of (2.9) when $a = b = 1$ has been derived recently Jorda (2010). Here, we also propose different algorithms for generating random data from the OBu-L distribution as follows:

(a) The first algorithm is based on generating random data from the Lindley distribution using the exponential gamma mixture.

Algorithm 1 (Mixture form of the Lindley distribution)

- Generate $U_i \sim \mathrm{Uniform}(0,1)$, $\quad i = 1, \ldots, n$;
- Generate $V_i \sim \mathrm{Exponential}(\lambda)$, $\quad i = 1, \ldots, n$;
- Generate $W_i \sim \mathrm{Gamma}(2,\lambda)$, $\quad i = 1, \ldots, n$;
- If $\frac{\left[1-(1-U_i)^{\frac{1}{b}}\right]^{\frac{1}{a}}}{\left[1-(1-U_i)^{\frac{1}{b}}\right]^{\frac{1}{a}} + (1-U_i)^{\frac{1}{a\,b}}} \leq \frac{\lambda}{1+\lambda}$ set $X_i = V_i$, otherwise, set $X_i = W_i$, $\quad i = 1, \ldots, n$.

(b) The second algorithm is based on generating random data from the inverse cdf in (2.1) of the OBu-L distribution.

Algorithm 2 (Inverse cdf)

- Generate $U_i \sim \mathrm{Uniform}(0{,}1)$, $\quad i = 1, \ldots, n$;
- Set

$$X_i = -1 - \frac{1}{\lambda} - \frac{1}{\lambda}\,W\left\{ \frac{-(1+\lambda)\,\mathrm{e}^{-1-\lambda}\,(1-\mathrm{U_i})^{\frac{1}{\mathrm{a}\,\mathrm{b}}}}{(1-U_i)^{\frac{1}{a\,b}} + \left[1-(1-U_i)^{\frac{1}{b}}\right]^{\frac{1}{a}}} \right\}, \quad i = 1, \ldots, n.$$

2.7. Expansions. In this subsection, we provide alternative mixture representations for the pdf and cdf of X. Despite the fact that the pdf and cdf of OBu-L require mathematical functions that are widely available in modern statistical packages, frequently analytical and numerical derivations take advantage of power series for the pdf. Some useful expansions for (2.2) can be derived by using the concept of power series. We obtain the cdf of the OBu-L distribution as

$$F(x) = 1 - \sum_{i=0}^{\infty} (-1)^i \frac{G(x)^{ai}}{\left[G(x)^a + \bar{G}(x)^a\right]^i}$$

and we get

$$\frac{G(x)^{ai}}{\left[G(x)^a + \bar{G}(x)^a\right]^i} = \frac{\sum_{k=0}^{\infty} \alpha_k G(x)^k}{\sum_{k=0}^{\infty} \beta_k G(x)^k} = \sum_{k=0}^{\infty} \gamma_k G(x)^k$$

where $\alpha_k = \sum_{j=k}^{\infty} (-1)^{j+k} \left(\begin{array}{c} ai \\ j \end{array} \right) \left(\begin{array}{c} j \\ k \end{array} \right)$, $\left[G(x)^a + \bar{G}(x)^a\right]^i = \sum_{k=0}^{\infty} \beta_k G(x)^k$, and $\beta_k = h_k(\alpha, i)$ which is defined in Appendix A, $\gamma_0 = \frac{\alpha_0}{\beta_0}$ and for $k > 1$ we have $\gamma_k = \beta_0^{-1} \left[\alpha_k - \beta_0^{-1} \sum_{r=1}^{k} \beta_r \gamma_{k-r}\right]$. Then, we can write

$$\begin{aligned} F(x) &= 1 - \sum_{i,k=0}^{\infty} (-1)^i \gamma_k(a,i) G(x)^k \\ &= 1 - \sum_{k=0}^{\infty} a_k^* G(x)^k = \sum_{k=0}^{\infty} b_k^* G(x)^k \end{aligned}$$

where $a_k^* = \sum_{i=0}^{\infty} (-1)^i \gamma_k(\alpha, i)$, $b_0^* = 1 - a_0^*$, and $b_k^* = -a_k^*$ for $k > 1$
Then, we obtain

$$F(x) = \sum_{k=0}^{\infty} b_k^* G(x)^k = \sum_{k=0}^{\infty} b_k^* H_k(x)$$

and

$$f(x) = \sum_{k=0}^{\infty} b_{k+1}^* h_{k+1}(x)$$

where $H_k(x)$ denote the cdf of the generalized Lindley with parameters λ and k.

2.8. Moments and Moment Generating Function. Some of the most important features and characteristics of a distribution can be studied through moments (e.g. tendency, dispersion, skewness and kurtosis). Now, we obtain ordinary moments and the moment generating function of the OBu-L distribution. Nadarajah *et al.* (2011) defined and computed

$$A(a,b,c,\delta) = \int_0^{\infty} x^c (1+x) \left[1 - \left(1 + \frac{bx}{b+1}\right) e^{-bx}\right]^{a-1} e^{-\delta x} dx \tag{2.10}$$

which can be used to produce ordinary moments μ_r'. Then, we have

$$A(a,b,c,\delta) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j+1} \left(\begin{array}{c} a-1 \\ i \end{array} \right) \left(\begin{array}{c} i \\ j \end{array} \right) \left(\begin{array}{c} j+1 \\ k \end{array} \right) \frac{(-1)^i b^j \Gamma(k+c+1)}{(i+b)^i (bi+\delta)^{c+k+1}} \tag{2.11}$$

From (2.10) and (2.11), we obtain

$$\mu_r' = E[X^r] = \frac{\lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+1) b_{k+1}^* A(k+1, \lambda, r, \lambda) \tag{2.12}$$

The ordinary moments of the OBu-L distribution can be calculated directly from (2.12). We now provide a formula for the conditional moments of the OBu-L distribution. Nadarajah *et al.* (2011) defined and computed the following equation for the conditional moments

$$L(a,b,c,\delta,t) = \int_t^{\infty} x^c (1+x) \left[1 - \left(1 + \frac{bx}{b+1}\right) e^{-bx}\right] e^{-\delta x} dx \tag{2.13}$$

From (2.13), we have

(2.14) $$\mathrm{L}(a,b,c,\delta,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j+1}\binom{a-1}{i}\binom{i}{j}\binom{j+1}{k}\frac{(-1)^i b^j\Gamma(k+c+1),(bi+\delta)t}{(i+b)^i(bi+\delta)^{c+k+1}}$$

where

(2.15) $$\Gamma(a,x)=\int_x^{\infty}t^{a-1}e^{-t}dt$$

denotes the incomplete gamma function. Using (2.13) and (2.14), we obtain

(2.16) $$\mu_r'(t)=E\left[X^r|X>t\right]=\frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)b_{k+1}^{*}L\left(k+1,\lambda,r,\lambda,t\right).$$

The moment generating function (mgf) of a random variable provides the basis of an alternative route to analytical results compared with working directly with its pdf and cdf. From (2.11) and (2.12), we obtain

$$M_X(t)=E\left[\mathrm{e}^{\mathrm{t\,X}}\right]=\frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)b_{k+1}^{*}A\left(k+1,\lambda,r,\lambda-t\right).$$

The central moments μ_n and cumulants of the OBu-L distribution are easily obtained as

$$\mu_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}{\mu_1'}^k\mu_{n-k}'\kappa_n=\mu_n'-\sum_{k=1}^{n-1}\binom{n-1}{k-1}\kappa_k\mu_{n-k}'$$

respectively, where $\kappa_1=\mu_1'$, $\kappa_2={\mu_1'}^2$, $\kappa_3=\mu_3'-3\mu_2'\mu_1'+2{\mu_2'}^3$ etc.

Skewness measures the degree of the long tail and kurtosis is a measure of the degree of tail heaviness. For the OBu-L distribution, The skewness can be computed by using quantile function in (2.9) as

$$S=\frac{Q\left(3/4\right)-2Q\left(1/2\right)+Q\left(1/4\right)}{Q\left(3/4\right)-Q\left(1/4\right)}$$

and the kurtosis is based on octiles as

$$K=\frac{Q\left(7/8\right)-Q\left(5/8\right)+Q\left(3/8\right)-Q\left(1/8\right)}{Q\left(6/8\right)-Q\left(2/8\right)}$$

where $Q(.)$ represents the quantile function. When the distribution is symmetric $S=0$, and when the distribution is right (or left) skewed $S>0(orS<0)$. As K increases, the tail of the distribution becomes heavier. These measures are less sensitive to outliers and they exist even for distributions without moments.

We present first four ordinary moments, skewness and kurtosis of the OBu-L distribution for various values of parameters in Table 1.

Table 1. Moments, skewness, and kurtosis of the OBu-L distribution for the some parameter values.

λ	α	β	μ_1'	μ_2'	μ_3'	μ_3'	*Skewness*	*Kurtosis*
0.5	0.5	0.5	9.608	186.867	5072	174430.543	1.585	6.540
0.5	0.5	1	4.590	49.133	746.891	14423.521	1.725	6.981
0.5	0.5	2	1.930	11.191	98.319	1112.495	2.178	8.656
0.5	0.5	5	0.458	0.900	3.133	15.072	3.501	20.257
0.5	1	0.5	5.892	58.476	809.578	14397.400	1.638	7.134
0.5	1	1	3.335	18.665	144.665	1419.200	1.469	6.089
0.5	1	2	1.898	6.026	25.882	138.923	1.436	5.878
0.5	1	5	0.893	1.373	2.860	7.364	1.395	5.524
0.5	2	0.5	4.142	23.196	170.857	1607.500	1.653	7.614
0.5	2	1	2.865	10.222	44.055	226.993	1.068	5.076
0.5	2	2	2.117	5.350	15.520	50.655	0.625	3.893
0.5	2	5	1.457	2.487	4.749	9.900	0.322	3.015

Table 1 reveals that for $\alpha < 1$, kurtosis and skewness increase when β increases. For $\alpha \geq 1$, the kurtosis and skewness decrease when β increases. Plots for skewness and kurtosis are presented in Figure 3.

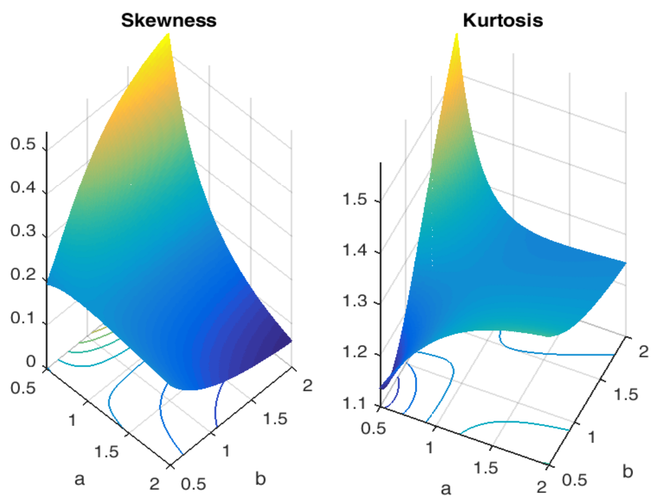

Figure 3. Plots of Galton skewness and Moor kurtosis of OBu-L distribution for several values of parameters.

3. Estimation

Several approaches for parameter estimation have been proposed in the literature but the maximum likelihood method is the most commonly employed. Here, we consider estimation of the unknown parameters of the OBu-L distribution by the method of maximum likelihood. Let $x_1, x_2, ..., x_n$ be observed values from the OBu-L distribution with parameters a, b and λ . The log-likelihood function for (a, b, λ) is given by

$$
\begin{aligned}
l_n &= n\,\log\left(\frac{a\,b\,\lambda^2}{1+\lambda}\right)+\sum_{i=1}^{n}\log(1+x_i)-\lambda\sum_{i=1}^{n}x_i+(a-1)\sum_{i=1}^{n}\log(t_i))\\
&+ (a\,b-1)\sum_{i=1}^{n}\log(1-t_i))-(b+1)\sum_{i=1}^{n}\log\left[t_i^a+(1-t_i)^a\right] \qquad (3.1)
\end{aligned}
$$

where $t_i = 1-\left(1+\frac{\lambda x_i}{1+\lambda}\right)e^{-\lambda x_i}$. The derivatives of the log-likelihood function with respect to the parameters a, b and λ are given by, respectively,

$$
\begin{aligned}
\frac{\partial l_n}{\partial a} &= \frac{n}{a}+\sum_{i=1}^{n}\log(t_i)+b\sum_{i=1}^{n}\log(1-t_i),\\
&- (b+1)\sum_{i=1}^{n}\frac{t_i^a\,\log(t_i)+(1-t_i)^a\,\log(1-t_i)}{t_i^a+\beta(1-t_i)^a}\\
\frac{\partial l_n}{\partial b} &= \frac{n}{b}+a\sum_{i=1}^{n}\log(1-t_i)-\sum_{i=1}^{n}\log\left[t_i^a+(1-t_i)^a\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial l_n}{\partial \lambda} &= \frac{2n}{\lambda}-\frac{n}{1+\lambda}-\sum_{i=1}^{n}x_i+(a-1)\sum_{i=1}^{n}\frac{t_i^{(\lambda)}}{t_i}+(1-a\,b)\sum_{i=1}^{n}\frac{t_i^{(\lambda)}}{1-t_i}\\
&-a(b+1)\sum_{i=1}^{n}t_i^{(\lambda)}\,\frac{t_i^{a-1}-(1-t_i)^{a-1}}{t_i^a+(1-t_i)^a}.
\end{aligned}
$$

where $t_i^{(\lambda)} = \frac{\lambda x_i(1+x_i)e^{-\lambda x_i}}{(1+\lambda)^2}$.

The maximum likelihood estimates (MLEs) of (a, b, λ) , say $\left(\hat{a}, \hat{b}, \hat{\lambda}\right)$, are the simultaneous solutions of the equations $\frac{\partial \ell_n}{\partial a} = 0$, $\frac{\partial \ell_n}{\partial b} = 0, \frac{\partial \ell_n}{\partial \lambda} = 0$. Note that the MLE has second derivatives with respect to the parameters, so that Fisher information matrix (FIM), $I_{ij}(\theta)$ can be expressed as

$$
I_{ij}(\theta) = E(\frac{\partial^2\ell(\theta; X_1, ..., X_n)}{\partial\theta_i\partial\theta_j}, i, j = 1, 2, 3
$$

The elements of the information matrix is given in Appendix B. The total FIM $I_n(\theta)$ can be approximated by

$$
J_n(\hat{\theta}) = \left.\frac{\partial^2\ell(\theta; X_1, ..., X_n)}{\partial\theta_i\partial\theta_j}\right|_{\theta=\hat{\theta}}, i, j = 1, 2, 3. \qquad (3.2)
$$

For real data, the matrix is obtained after the convergence of the Newton-Raphson procedure in R software. Let $\hat{\theta} = (\hat{a}, \hat{b}, \hat{\lambda})$ be the MLE of $\theta = (a, b, \lambda)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have:$\sqrt{n}(\hat{\theta}-\theta) \to N_3(0, I^{-1}(\theta))$, where $I(\theta)$ is the expected FIM. The asymptotic behaviour is still valid if $I(\theta)$ is replaced by the observed information matrix evaluated at $\hat{\theta}$, that is $J(\hat{\theta})$. The multivariate normal distribution with mean vector $0 = (0, 0, 0)^T$ and covariance matrix $I^{-1}(\theta)$ can be used to construct confidence intervals for the model parameters. That is, the approximate $100(1-\eta)$ percent two-sided confidence intervals for a, b and λ are given by

$$\hat{a} \pm Z_{\frac{\eta}{2}} \sqrt{I_{aa}^{-1}(\hat{\theta})}$$
$$\hat{b} \pm Z_{\frac{\eta}{2}} \sqrt{I_{bb}^{-1}(\hat{\theta})}$$
$$\hat{\lambda} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\lambda\lambda}^{-1}(\hat{\theta})}$$

respectively, where $I_{aa}{}^{-1}(\hat{\theta})$,$I_{bb}{}^{-1}(\hat{\theta})$, $I_{\lambda\lambda}{}^{-1}(\hat{\theta})$ are diagonal elements of $I_n{}^{-1}(\theta) = (nI_n(\hat{\theta}))^{-1}$ and $Z_{\frac{n}{2}}$ is the upper $\frac{n}{2}$ th percentile of a standard normal distribution.
Note that parameter estimation become complicated when censoring is present in the sample. Some time it is not possible to give a mathematical expression of estimated values of parameters in maximum likelihood method.

4. Simulation

In this section, we evaluate the performance of the MLEs of the parameters of OBu-L model by means of a simulation study. R software is used for simulation study and real data modelling. Inverse transform algorithm is used to generate random data from the OBu-L distribution. The used algorithm can be found in Section 2.5. We generated samples of sizes $n = 50, 100, 500$ and 1000 from the OBu-L model for different parameter combinations. We computed mean square error (MSE) of parameter estimations, estimated average length (AL) and coverage probability (CP) and this procedure is repeated 1000 times. Let $\left(\hat{a}, \hat{b}, \hat{\lambda}\right)$ be the MLEs of the parameters of the OBu-L and $\left(s_{\hat{a}_i}, s_{\hat{b}_i}, s_{\hat{\lambda}}\right)$ be the standard errors of MLEs, the estimated MSEs, AL and CP can be estimated by using following equations:

$$MSE_{\varepsilon}(n) = \frac{\sum_{i=1}^{N} (\hat{\varepsilon}_i - \varepsilon)}{N}$$
$$AL_{\varepsilon}(n) = \sum_{i=1}^{N} (s_{\hat{\varepsilon}_i}) \frac{3.919928}{N}$$
$$CP_{\varepsilon}(n) = \frac{1}{N} \sum_{i=1}^{N} I\left(\hat{\varepsilon}_i - 1.9599 s_{\hat{\varepsilon}_i}, \hat{\varepsilon}_i - 1.9599 s_{\hat{\varepsilon}_i}\right)$$

The simulation results are given in Table 2. The values in Table 2 indicate that the estimates are quite stable and, more importantly, are close to the true parameter values for these sample sizes. From Table 2, it is observed that in general MSE decreases as n increases. The simulation study also shows that the maximum likelihood method is appropriate for estimating the OBu-L parameters. In fact, the MSEs of the parameters tend to be closer to zero when n increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. The normal approximation can be oftentimes improved by using bias adjustments to these estimators. The coverage probability is near to 0.95. When the sample size increases, coverage probability approaches to nominal value and average length decreases for all cases.

Table 2. MSEs and means of the estimated parameters of the OBu-L distribution for several values.

				Mean			*MSE*			*AL*			*CP*		
λ	α	β	n	*lambda*	α	β	λ	α	β	λ	α	β	λ	α	β
0.5	0.5	0.5	50	0.604	0.513	0.699	0.260	0.011	0.543	1.244	0.372	1.512	0.802	0.892	0.866
			100	0.601	0.496	0.648	0.147	0.005	0.125	0.994	0.268	1.293	0.837	0.879	0.878
			250	0.530	0.501	0.594	0.035	0.002	0.0521	0.669	0.181	0.712	0.932	0.934	0.934
			500	0.514	0.500	0.514	0.015	0.001	0.014	0.459	0.128	0.447	0.926	0.938	0.940
			1000	0.498	0.500	0.512	0.006	0.0005	0.005	0.313	0.090	0.313	0.936	0.940	0.946
0.5	0.5	2	50	0.726	0.491	3.764	0.563	0.003	1.255	1.886	0.253	1.347	0.923	0.948	0.856
			100	0.576	0.494	3.821	0.113	0.001	0.953	1.138	0.172	0.759	0.922	0.955	0.882
			250	0.539	0.496	2.582	0.044	0.0007	0.125	0.728	0.108	0.544	0.938	0.956	0.908
			500	0.519	0.499	2.233	0.019	0.0003	0.025	0.506	0.074	0.219	0.942	0.956	0.924
			1000	0.506	0.499	2.041	0.009	0.0001	0.109	0.351	0.051	0.103	0.942	0.954	0.936
0.5	3	2	50	0.436	3.183	2.909	0.031	0.395	0.944	0.623	2.194	1.268	0.897	0.948	0.862
			100	0.459	3.034	2.883	0.015	0.135	0.751	0.397	1.439	1.051	0.929	0.956	0.905
			250	0.488	3.021	2.512	0.003	0.054	0.492	0.201	0.909	0.909	0.938	0.954	0.934
			500	0.491	2.998	2.305	0.001	0.028	0.337	0.134	0.631	0.885	0.948	0.946	0.945
			1000	0.498	3.012	2.086	0.0006	0.012	0.271	0.089	0.450	0.646	0.951	0.950	0.946
0.5	2.5	1.5	50	0.441	2.673	1.994	0.025	0.199	0.941	0.694	1.541	1.557	0.922	0.935	0.895
			100	0.472	2.547	1.863	0.012	0.115	0.766	0.393	1.296	1.235	0.941	0.941	0.919
			250	0.486	2.525	1.677	0.004	0.0461	0.355	0.216	0.800	0.944	0.946	0.944	0.942
			500	0.496	2.511	1.603	0.001	0.018	0.196	0.142	0.558	0.653	0.948	0.950	0.945
			1000	0.498	2.506	1.551	0.0007	0.011	0.097	0.098	0.392	0.472	0.958	0.954	0.952

Table 3. Fitted distributions and their abbreviations.

Distribution	Abbreviation	References
Odd Burr Lindley	OBu-L	Proposed
Generalized Gamma	GG	Stacy (1962)
Gamma Weibull	GW	Provost *et al.* (2011)
Gamma Exponentiated Exponential	GEE	Ristic and Balakrishnan (2012)
Generalized Lindley	GL	Nadarajah *et al.* (2011)
Kumaraswamy Lindley	KL	Cakmakyapan and Ozel (2014)
Odd Log-logistic Lindley	OLL-L	Ozel *et al.* (Accepted)

5. Applications

In this section, real data modeling performance of the OBu-L distribution is compared with several well-known distributions given in Table 3.

We used the uncensored real data set on the breaking stress of carbon fibers as reported in Nichols and Padgett (2006). The data set contains 66 observations and is given in Table 4.

Table 4. Carbon Fibers Data Set.

3.7 2.74 2.73 2.5 3.6 3.11 3.27 2.87 1.47 3.11 3.56 4.42 2.41 3.19 3.22
1.69 3.28 3.09 1.87 3.15 4.9 1.57 2.67 2.93 3.22 3.39 2.81 4.2 3.33 2.55
3.31 3.31 2.85 1.25 4.38 1.84 0.39 3.68 2.48 0.85 1.61 2.79 4.7 2.03 1.89
2.88 2.82 2.05 3.65 3.75 2.43 2.95 2.97 3.39 2.96 2.35 2.55 2.59 2.03 1.61
2.12 3.15 1.08 2.56 1.8 2.53

Table 5 gives Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and log-likelihood ($-\ell$) values for the all fitted distributions. Based on Table 5, it is clear that the OBu-L distribution provides the overall best fit and therefore could be chosen as the more adequate model from other models for explaining the data set.

Table 5. The MLEs, standard errors and the goodness-of-fit statistics for the first application.

$Model$	$Parameter Estimates$			AIC	BIC	$-\ell$
$GG(k,\lambda,\xi)$	4.073	3.346	3.092	177.83	184.404	85.917
$GW(k,\xi,\lambda)$	3.441	$1.6*10^{-7}$	3.062	178.135	184.704	86.067
$GEE(\lambda,\alpha,\delta)$	0.265	10.036	7.237	189.787	196.356	91.894
$GL(\alpha,\lambda)$	7.035	7.035	——	191.594	195.973	93.797
$KL(\alpha,\beta,\lambda)$	4.662	4.111	0.689	183.110	189.679	88.550
$OLL-L(\alpha,\lambda)$	2.965	0.488	——	179.991	184.372	87.996
$OBu-L(\mu,\sigma,\alpha,\beta)$	2.453	5.383	0.318	177.151	183.720	85.575

More information can be provided in Figure 4 by a histogram of the data with fitted lines of pdfs for all distributions. Figure 4 also suggests that the OBu-L distribution fits unimodal data very well. Figure 5 displays plots of the fitted density, cumulative and survival functions with probability-probability (P-P) plot for the OBu-L model. It is clear that the OBu-L distribution provides better fitting performance than other distributions.

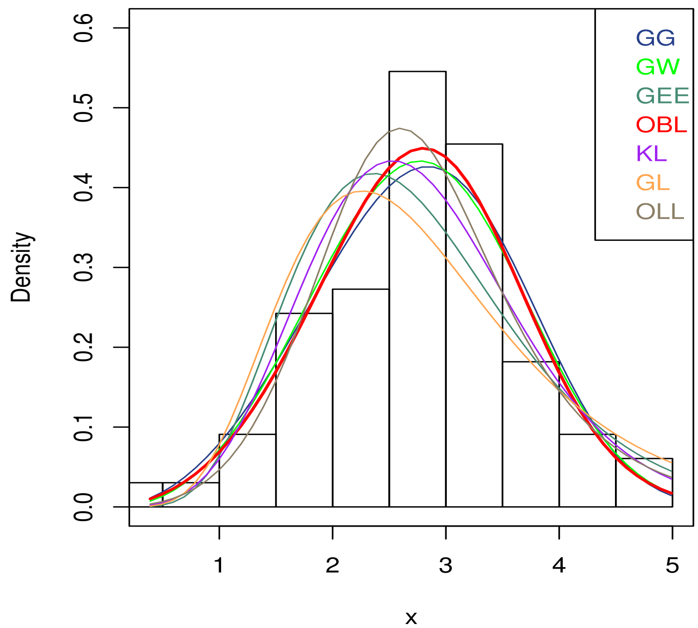

Figure 4. Fitted densities of distributions for first data set.

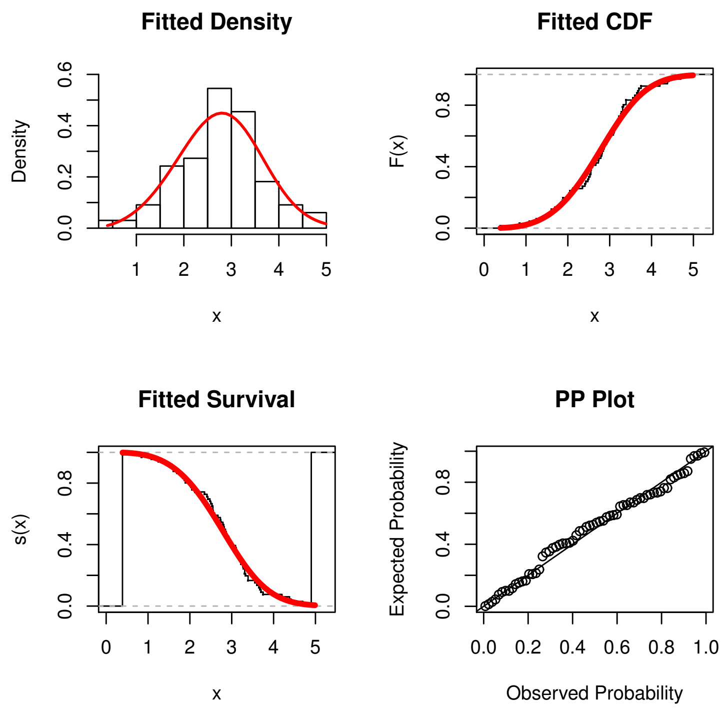

Figure 5. Plots for fitted functions of the OBu-L model for first data set.

The second data set consists of 63 observations of the strengths of 1.5 cm glass fibers, originally obtained by workers at the UK National Physical Laboratory. The data set is given in Table 6.

Table 6. Strengths of 1.5 cm Glass Fibers.

0.55 0.74 0.77 0.81 0.84 0.93 1.04 1.11 1.13 1.24
1.25 1.27 1.28 1.29 1.3 1.36 1.39 1.42 1.48 1.48
1.49 1.49 1.5 1.5 1.51 1.52 1.53 1.54 1.55 1.55
1.58 1.59 1.6 1.61 1.61 1.61 1.61 1.62 1.62 1.63
1.64 1.66 1.66 1.66 1.67 1.68 1.68 1.69 1.7 1.7
1.73 1.76 1.76 1.77 1.78 1.81 1.82 1.84 1.84 1.89
2 2.01 2.24

Table 7. The MLEs, standard errors and the goodness-of-fit statistics for the second application.

$Model$	$Parameter Estimates$	AIC	BIC	$-\ell$
$GG(k, \lambda, \xi)$	4.911 1.577 5.462	37.836	44.266	15.910
$GW(k, \xi, \lambda)$	4.428 2.236 1.458	38.836	45.265	16.419
$GEE(\lambda, \alpha, \delta)$	1.342 17.287 2.901	60.503	66.932	27.244
$GL(\alpha, \lambda)$	13.575 2.528 ——	69.725	74.011	32.829
$KL(\alpha, \beta, \lambda)$	6.269 4.069 1.340	60.887	67.317	27.407
$OLL-L(\alpha, \lambda)$	4.611 0.793 ——	44.484	48.770	20.220
$OBu-L(\mu, \sigma, \alpha, \beta)$	4.031 8 0.579	35.340	41.770	14.670

Based on Table 7, the OBu-L distribution provides better fitting performance than other distributions according to information criteria. Therefore, the OBu-L could be chosen as the more adequate model from other models for explaining the second data set and also more information can be provided in Figure 6 by a histogram of the data with fitted lines of pdfs for all distributions. Figure 7 shows the plots of the fitted density, cumulative and survival functions with P-P plot for the OBu-L model.

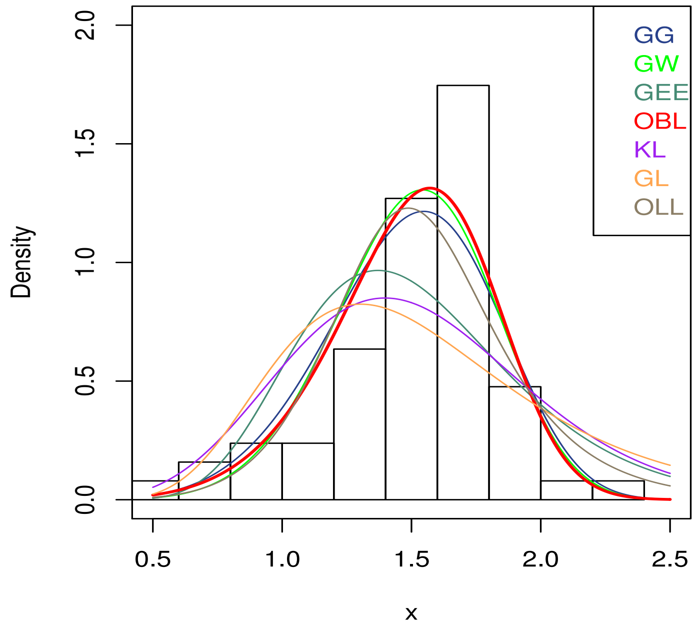

Figure 6. Fitted densities of distributions for the second data set.

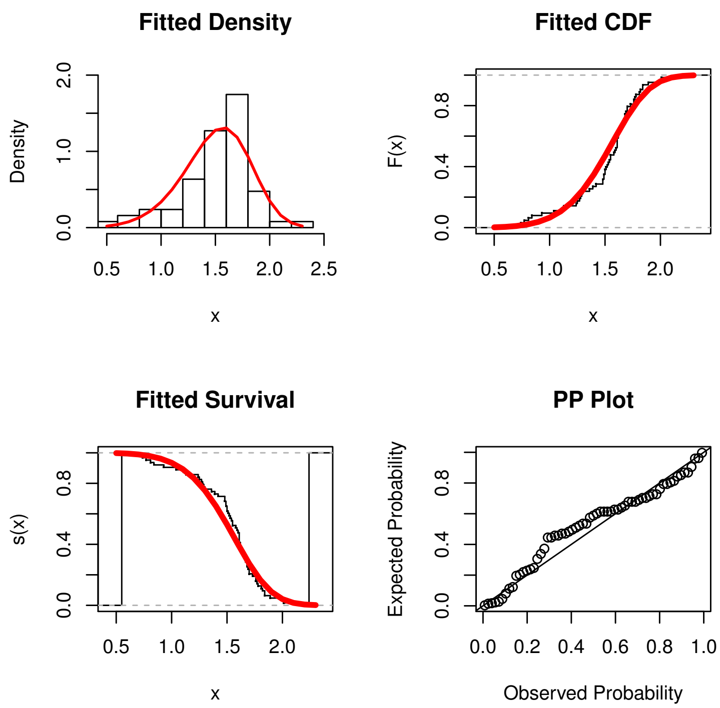

Figure 7. Plots for fitted functions of the OBu-L model for the second data set.

6. Conclusion

In this study, a new three-parameter distribution is introduced. A characteristic of the OBu-L distribution is that its hrf can be increasing, bathtub-shaped, and unimodal depending on its parameter values. Several properties of the new distribution such as pdf, hrf, and moments are obtained. The MLE procedure is presented. Real data applications and a simulation study indicate the flexibility and capacity of the proposed distribution in data modeling. The new model provides consistently a better fit than the other models, namely: Generalized Gamma, Gamma Weibull, Gamma Exponentiated Exponential, Generalized Lindley, Kumaraswamy Lindley, and Odd Log-logistic Lindley distributions. In view of the density function and hrf shapes, it seems that the proposed model can be considered as a suitable candidate model in reliability analysis, biological systems, data modeling, and related fields.

In many practical applications, the lifetimes are affected by explanatory variables such as the cholesterol level, blood pressure, weight and many others. Parametric models to estimate univariate survival functions for censored data regression problems are widely used. A parametric model that provides a good fit to lifetime data tends to yield more precise estimates of the quantities of interest. Therefore, we are planing to extend this study for censored data sets in future.

Appendix A: Three useful power series

First, expanding z^{λ} in Taylor series, we can write

$$z^{\lambda} = \sum_{k=0}^{\infty} (\lambda)_k \, (z-1)^k / k! = \sum_{i=0}^{\infty} f_i \, z^i \tag{6}$$

where

$$f_i = f_i(\lambda) = \sum_{k=i}^{\infty} \frac{(-1)^{k-i}}{k!} \binom{k}{i} (\lambda)_k \tag{7}$$

and $(\lambda)_k = \lambda(\lambda-1)\ldots(\lambda-k+1)$ denotes the descending factorial.

Second, we obtain an expansion for $[G(x)^a + \bar{G}(x)^a]^c$. We can write from equation (6) and (7)

$$[G(x)^a + \bar{G}(x)^a] = \sum_{j=0}^{\infty} t_j \, G(x)^j, \tag{8}$$

where $t_j = t_j(a) = a_j(a) + (-1)^j \binom{a}{j}$ and $a_j(a)$ is defined by (7). Then, using (8), we have

$$[G(x)^a + \bar{G}(x)^a]^c = \sum_{i=0}^{\infty} f_i \left(\sum_{j=0}^{\infty} t_j \, G(x)^j \right)^i,$$

where $f_i = f_i(c)$.

Finally, using again equations (7) and (8), we have

$$[G(x)^a + \bar{G}(x)^a]^c = \sum_{j=0}^{\infty} h_j(a,c) \, G(x)^j,$$

where $h_j(a,c) = \sum_{i=0}^{\infty} f_i \, m_{i,j}$ and (for $i \geq 0$) $m_{i,j} = (j \, t_0)^{-1} \sum_{m=1}^{j} [m(j+1)-j] \, t_m \, m_{i,j-m}$ (for $j \geq 1$) and $m_{i,0} = t_0^i$.

Appendix B

Here, we provide formulas for the second-order partial derivatives of the log-likelihood function. After some algebraic manipulations, we obtain

$$\frac{\partial^2 {}_n}{\partial a^2} = \frac{-n}{a^2} - (b+1) \sum_{i=1}^{n} \frac{t_i^a \left[\log(t_i)\right]^2 + (1-t_i)^a \left[\log(1-t_i)\right]^2}{t_i^a + (1-t_i)^a}$$

$$+(b+1) \sum_{i=1}^{n} \left[\frac{t_i^a \log(t_i) + (1-t_i)^a \log(1-t_i)}{t_i^a + (1-t_i)^a} \right]^2$$

$$\frac{\partial^2 {}_n}{\partial a \partial b} = \sum_{i=1}^{n} \log(t_i) - \sum_{i=1}^{n} \frac{t_i^a \log(t_i) + (1-t_i)^a \log(1-t_i)}{t_i^a + (1-t_i)^a}$$

$$\frac{\partial^2 {}_n}{\partial a \partial \lambda} = \sum_{i=1}^{n} \frac{t_i^{(\lambda)}}{t_i} - b \sum_{i=1}^{n} \frac{t_i^{(\lambda)}}{1-t_i}$$

$$-(b+1) \sum_{i=1}^{n} t_i^{(\lambda)} \frac{t_i^a \; [1 + a\, \log(t_i)] + (1-t_i)^a \; [1 + a\, \log(1-t_i)]}{t_i^a + (1-t_i)^a}$$

$$\frac{\partial^2 {}_n}{\partial b^2} = \frac{-n}{b^2}$$

$$\frac{\partial^2 {}_n}{\partial b \partial \lambda} = -a \sum_{i=1}^{n} \frac{t_i^{(\lambda)}}{1-t_i} - a \sum_{i=1}^{n} t_i^{(\lambda)} \frac{t_i^{a-1} - (1-t_i)^{a-1}}{t_i^a + (1-t_i)^a} to$$

$$\frac{\partial^2 {}_n}{\partial \lambda^2} = \frac{-2n}{\lambda^2} + \frac{n}{(1+\lambda)^2} + (a-1) \sum_{i=1}^{n} \frac{t_i^{(\lambda\,\lambda)}\, t_i - \left[t_i^{(\lambda)}\right]^2}{t_i^2}$$

$$+(1 - a\, b) \sum_{i=1}^{n} \frac{t_i^{(\lambda\,\lambda)}\, (1-t_i) + \left[t_i^{(\lambda)}\right]^2}{(1-t_i)^2}$$

$$-a(b+1) \sum_{i=1}^{n} t_i^{(\lambda\,\lambda)} \frac{t_i^{a-1} - (1-t_i)^{a-1}}{t_i^a + (1-t_i)^a}$$

$$-a(a-1)(b+1) \sum_{i=1}^{n} t_i^{(\lambda\,\lambda)} \frac{t_i^{a-2} + (1-t_i)^{a-2}}{t_i^a + (1-t_i)^a}$$

$$+a^2\,(b+1) \sum_{i=1}^{n} \left[t_i^{(\lambda)} \frac{t_i^{a-1} - (1-t_i)^{a-1}}{t_i^a + (1-t_i)^a} \right]^2$$

(6.-15)

where

$$t_i^{(\lambda)} = \frac{\lambda x_i \,(1 + x_i)\, e^{-\lambda x_i}}{(1+\lambda)^2}$$

$$t_i^{(\lambda\lambda)} = \frac{\lambda\, x_i\, (1 + x_i)\, [1 - \lambda - \lambda(1+\lambda) x_i]\; \mathrm{e}^{-\lambda\, \mathrm{x_i}}}{(1+\lambda)^3}$$

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