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The exponentiated Kumaraswamy-power function distribution

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Abstract

In this paper, the exponentiated Kumaraswamy-power function distribution is introduced. Some structural properties of the proposed distribution including the shapes of the density, hazard and quantile functions are explored. Besides, skewness and kurtosis measures are investigated. The method of maximum likelihood is used for estimating model parameters. For different parameter settings and sample sizes, a simulation study is performed and the performance of the new distribution is compared with some well-known distributions. Then, an application is presented with a real data set to illustrate the usefulness of the proposed distribution.

Keywords: Kumaraswamy-power function distribution, maximum likelihood estimation, hazard function.

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1. Introduction

Numerous probability models are used to model lifetime distribution, but these models mathematically more complicated to manage. However, the power function (PF) distribution is very helpful in this regard. The PF distribution is a flexible lifetime distribution model that may offer a good fit to some sets of failure data [1]. Theoretically, the PF distribution is a special case of the beta distribution. It was derived from the Pareto distribution using the inverse transformation. Besides, it is a special case of Pearson type I distribution [2]. Meniconi and Barry [3] proved that the PF distribution is the best distribution to check the reliability of any electrical component. They showed from

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survival and hazard functions that PF distribution is the better than exponential, log-normal and Weibull distributions. The PF distribution can be used to fit the distribution of likelihood ratios in statistical tests.

The random variable X has the PF distribution if its cumulative distribution function (cdf) for $0 < x < \alpha$ is given by

$$G(x) = \frac{x^{\beta}}{\alpha^{\beta}}, \alpha > 0, \beta > 0, \tag{1.1}$$

where β is the shape parameter and α is the scale parameter. The probability density function (pdf) corresponding to (1.1) is given by

$$g(x) = \frac{\beta x^{\beta-1}}{\alpha^{\beta}}, \alpha > 0, \beta > 0, . \tag{1.2}$$

Characterizations of the PF distribution using order statistics and record values has been studied by Ahsanullah and Kabir [4]. Detailed discussion on parameter estimation of the PF distribution is studied by [5], [6] and [7] using various estimation procedures like method of moments, maximum likelihood, percentiles, method of least square, and Bayesian estimation with various loss functions. Bayesian analysis of the PF distribution was discussed using three single and as well as three double priors and the accuracy of these priors was assessed using simulation studies [8]. An initial test estimator for a scale parameter of the PF distribution was proposed by [9]. Abdulsathar and Renjini [10] estimated the Gini-index and Lorenz curve of the PF distribution and the shape parameter using Bayesian approach. The estimators were developed using weighted squared error and squared error loss functions. Cordeiro and dos Santos Brito [11] studied beta-PF distribution. Oguntunde et al. [12] derived the Kumaraswamy-PF distribution and Tahir et al. [13] introduced Weibull-PF distribution. Recently, Haq et al. [2] has been studied transmuted PF distribution and Shakeel [14] compared the two new robust parameter estimation methods for the PF distribution. To increase the flexibility for modeling purposes it will be useful to consider further alternatives to PF distribution. Our purpose is to provide a generalization that may be useful to still more complex situations. Once the proposed distribution is quite flexible in terms of pdf and hazard rate function, it may provide an interesting alternative to describe income distributions and can also be applied in actuarial science, finance, bioscience, and telecommunications. Therefore, the aim of this study is to introduce a new distribution using the PF distribution. For this reason, we propose exponentiated Kumaraswamy-PF distribution.

Exponentiated Kumaraswamy-PF (ExpKw-PF) distribution is a generalization to the Kumaraswamy-PF (Kw-PF) distribution through adding a new shape parameter. The construction of the exponentiated distribution is rather simple. It is based on the observation that by raising an arbitrary cdf $G(x)$ to an arbitrary power $\theta > 0$, a new cdf $F(x) = G(x)^{\theta}$ emerges with one additional parameter. The parameter θ characterizes the skewness, kurtosis and tails of the F distribution. In this construction, $G(x)$ is the baseline distribution and $F(x)$ may be referred to as the exponentiated G distribution. The relation between the corresponding density function is $f(x) = \theta G(x)^{\theta-1} g(x)$. In this study, in the same way, we generalize the Kw-PF distribution to the ExpKw-PF distribution.

The rest of paper is organized as follows. In Section 2, we define the pdf and cdf of the new distribution. Some of its properties are also presented including survival, hazard and quantile functions, skewness, kurtosis, and order statistics. The estimation of the model parameters using the method of maximum likelihood is discussed in Section 3. A simulation study is conducted in Section 4. Finally, an application on a real data set is reported in Section 5 and some conclusions are addressed in Section 6.

2. The exponentiated Kumaraswamy-power function distribution

Let $G(x)$ be the cdf of the PF distribution with parameters α and β, then $F(x) = G(x)^{\theta}$ yields the ExpKw-PF cumulative distribution (for $0 < x < \alpha$)

$$F(x) = \left[1 - \left[1 - \left(\frac{x}{\alpha}\right)^{\alpha\beta}\right]^{b}\right]^{\theta}, \tag{2.1}$$

where $a, b, \beta, and \theta > 0$ are shape parameters and $\alpha > 0$ is a scale parameter. The corresponding pdf of the ExpKw-PF distribution is obtained as

$$f(x) = \frac{\theta a b \beta x^{a\beta - 1}}{\alpha^{a\beta}} \left[1 - \left[1 - \left(\frac{x}{\alpha}\right)^{a\beta}\right]^{b}\right]^{\theta - 1} \left[1 - \left(\frac{x}{\alpha}\right)^{a\beta}\right]^{b-1}. \tag{2.2}$$

A sample contains censored observations if the only information about some of the observations is that they are below or above a specified value. Because of its tractable distribution function (2.2), the ExpKw-PF distribution can be used quite effectively even if the data are censored.

2.1. Shape of the exponentiated Kumaraswamy-power function distribution.

The shape of the ExpKw-PF distribution is revealed by means of plots. The plots of the pdf at various choices of parameters are given in Figures 1 to 5. These figures show that the ExpKw-PF distribution is much more flexible than the PF distribution and can allow for greater flexibility of tails. The ExpKw-PF distribution is mostly negatively skewed. Furthermore, as a increases and b decreases, we obtain more skewed shapes. On the other hand, as α increases, and β and θ decrease, the curve becomes more spread out. Because of its tractable distribution function (2.2), the ExpKw-PF distribution can be used quite effectively even if the data are censored.

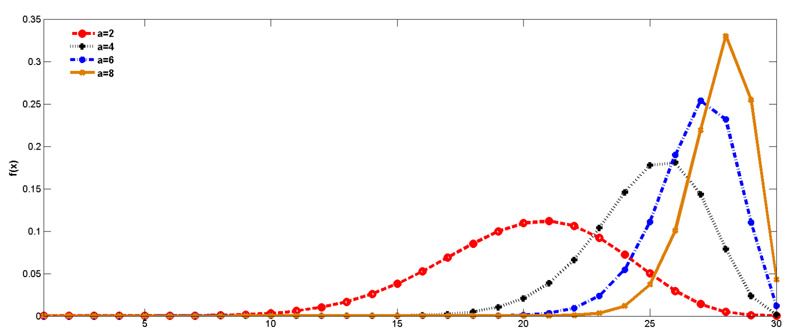

Figure 1. Plots of the pdf of ExpKw-PF for some parameter values ($b = 6, \alpha = 31, \beta = 2, \theta = 2$).

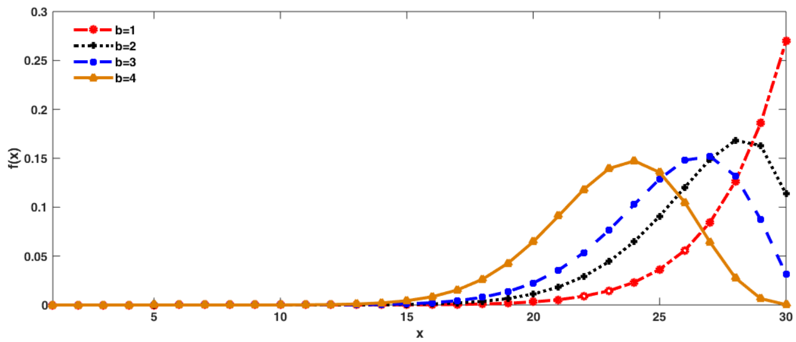

Figure 2. Plots of the pdf of ExpKw-PF for some parameter values ($a = 3, \alpha = 31, \beta = 2, \theta = 2$).

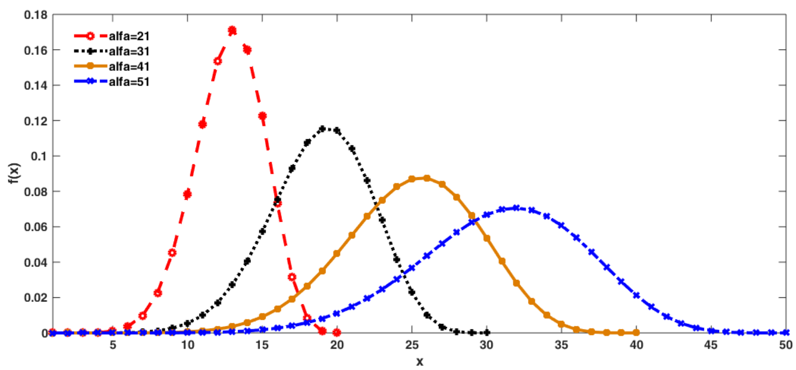

Figure 3. Plots of the pdf of ExpKw-PF for some parameter values ($a = 2, b = 8, \beta = 2, \theta = 2$).

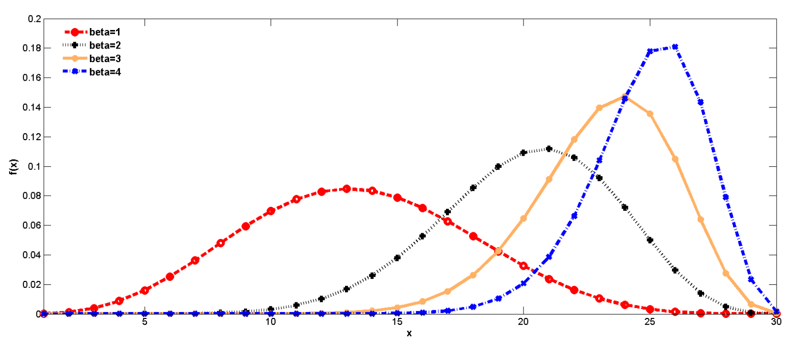

Figure 4. Plots of the pdf of ExpKw-PF for some parameter values ($a = 2, b = 6, \alpha = 31, \theta = 2$).

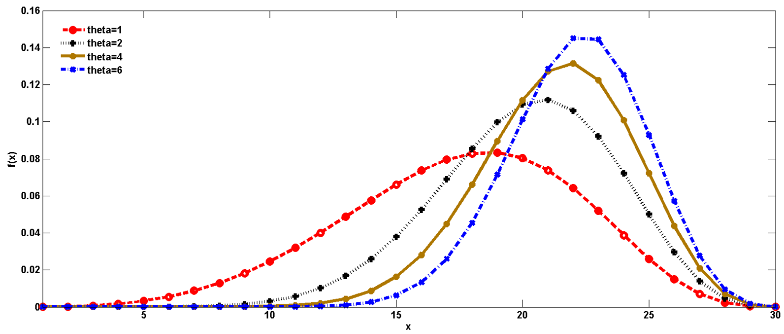

Figure 5. Plots of the pdf of ExpKw-PF for some parameter values ($a = 2, b = 6, \alpha = 31, \beta = 2$).

2.2. Survival and hazard function. Central role is played in the reliability theory by the quotient of the pdf and survival function. The survival function for the ExpKw-PF distribution is given by

$$S(x) = 1 - \left[1 - \left[1 - \left(\frac{x}{\alpha}\right)^{a\beta}\right]^{b}\right]^{\theta}. \tag{2.3}$$

In reliability studies, the hazard rate function (hrf) is an important characteristic and fundamental to the design of safe systems in a wide variety of applications. Therefore, we discuss these properties of the ExpKw-PF distribution. The hrf is thus given by

$$h(x) = \frac{\frac{\theta a b \beta x^{a\beta-1}}{\alpha^{a\beta}}\left[1 - \left[1 - \left(\frac{x}{\alpha}\right)^{a\beta}\right]^{b}\right]^{\theta-1}\left[1 - \left(\frac{x}{\alpha}\right)^{a\beta}\right]^{b-1}}{1 - \left[1 - \left[1 - \left(\frac{x}{\alpha}\right)^{a\beta}\right]^{b}\right]^{\theta}}. \tag{2.4}$$

Choosing different values for the parameters $a, b, \alpha, \beta, \theta$, the plots for the hrf of the ExpKw-PF distribution are presented in Figures 6 to 10. As seen from these figures, ExpKw-PF has increasing and bath-tub hrf.

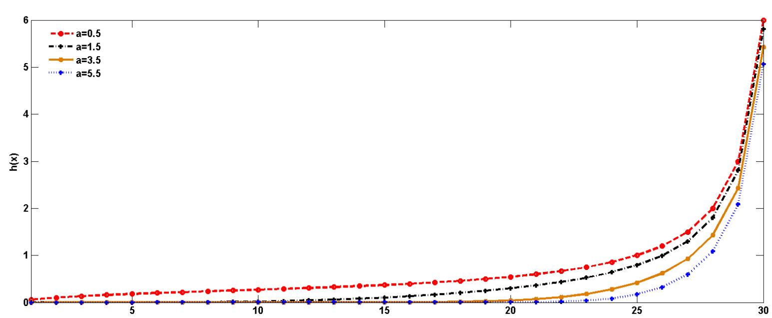

Figure 6. Plots of the hrf of ExpKw-PF for some parameter values ($b = 6, \alpha = 31, \beta = 2, \theta = 2$).

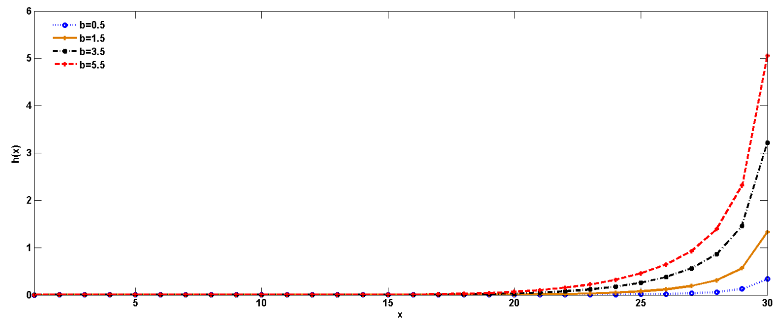

Figure 7. Plots of the hrf of ExpKw-PF for some parameter values $(a = 3, \alpha = 31, \beta = 2, \theta = 2)$.

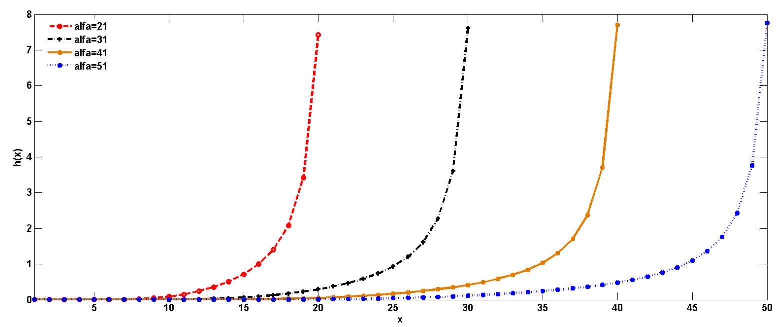

Figure 8. Plots of the hrf of ExpKw-PF for some parameter values $(a = 2, b = 8, \beta = 2, \theta = 2)$.

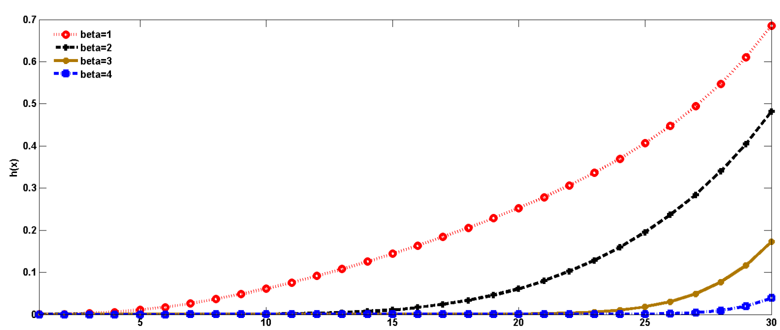

Figure 9. Plots of the hrf of ExpKw-PF for some parameter values $(a = 2, b = 6, \alpha = 31, \theta = 2)$.

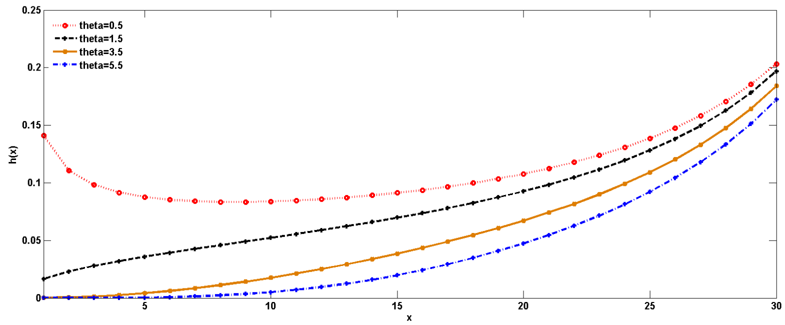

Figure 10. Plots of the hrf of ExpKw-PF for some parameter values ($a = 2, b = 6, \alpha = 31, \beta = 2$).

2.3. Quantile function and quantile measures. The p^{th} quantile of the ExpKw-PF distribution is given by

$$Q\left(p\right) = F^{-1}(p) = x = \alpha\left(1 - \left(1 - p^{1/\theta}\right)^{1/b}\right)^{1/a\beta}, \tag{2.5}$$

where $F^{-1}(p)$ is the inverse distribution function. Thus, the new distribution is easily simulated as $X = Q(U)$, where U has the uniform $U(0, 1)$ distribution. In particular, the median of the ExpKw-PF distribution is given by

$$median(X) = Q\left(\frac{1}{2}\right) = \alpha\left(1 - \left(1 - \frac{1}{2}^{1/\theta}\right)^{1/b}\right)^{1/a\beta}. \tag{2.6}$$

To illustrate the effect of the shape parameters a, b and θ on skewness and kurtosis of the new distribution, we consider measures based on quantiles. The shortcomings of the classical kurtosis measure are well known. There are many heavy-tailed distributions for which this measure is infinite. So, it becomes uninformative precisely when it needs to be. Indeed, our motivation to use quantile-based measures stemmed from the non-existence of classical kurtosis for many generalized distributions. The Bowley's skewness and Moor's kurtosis are based on quartiles. The Bowley's skewness and Moor's kurtosis for the ExpKw-PF distribution are given, respectively, as

$$S = \frac{Q(3/4)-2Q(1/2)+Q(1/4)}{Q(3/4)-Q(1/4)}$$

$$= \frac{\left(1-\left(1-(3/4)^{1/\theta}\right)^{1/b}\right)^{1/a\beta} - 2\left(1-\left(1-(1/2)^{1/\theta}\right)^{1/b}\right)^{1/a\beta}}{\left(1-\left(1-(3/4)^{1/\theta}\right)^{1/b}\right)^{1/a\beta} - \left(1-\left(1-(1/4)^{1/\theta}\right)^{1/b}\right)^{1/a\beta}} \tag{2.7}$$

$$+ \frac{\left(1-\left(1-(1/4)^{1/\theta}\right)^{1/b}\right)^{1/a\beta}}{\left(1-\left(1-(3/4)^{1/\theta}\right)^{1/b}\right)^{1/a\beta} - \left(1-\left(1-(1/4)^{1/\theta}\right)^{1/b}\right)^{1/a\beta}},$$

$$K = \frac{Q(7/8)-Q(5/8)-Q(3/8)+Q(1/8)}{Q(6/8)-Q(2/8)}$$

$$= \frac{\left(1-\left(1-(7/8)^{1/\theta}\right)^{1/b}\right)^{1/a\beta} - \left(1-\left(1-(5/8)^{1/\theta}\right)^{1/b}\right)^{1/a\beta}}{\left(1-\left(1-(6/8)^{1/\theta}\right)^{1/b}\right)^{1/a\beta} - \left(1-\left(1-(2/8)^{1/\theta}\right)^{1/b}\right)^{1/a\beta}} \tag{2.8}$$

$$- \frac{\left(1-\left(1-\left(\frac{3}{8}\right)^{1/\theta}\right)^{1/b}\right)^{1/a\beta} - \left(1-\left(1-(1/8)^{1/\theta}\right)^{1/b}\right)^{1/a\beta}}{\left(1-\left(1-(6/8)^{1/\theta}\right)^{1/b}\right)^{1/a\beta} - \left(1-\left(1-(2/8)^{1/\theta}\right)^{1/b}\right)^{1/a\beta}}.$$

When the distribution is symmetric, S=0 and when the distribution is right (or left) skewed. $S < 0$ (or $S > 0$). As K increases, the tail of the distribution becomes heavier. These measures are less sensitive to outliers and they exist even for distributions without moments. From (2.7) and (2.8), skewness and kurtosis of the ExpKw-PF distribution are obtained and presented in Table 1.

Table 1. Skewness and kurtosis of ExpKw-PF distribution for some values of a, b, α, β and θ.

	a	b	α=2 and β=2 Skewness	Kurtosis	α=2 and β=4 Skewness	Kurtosis	α=6 and β=3 Skewness	Kurtosis
		0.5	0.0795	0.1250	-0.1373	-0.2830	-0.0640	-0.1300
θ=0.5	0.5	2	0.3337	0.7541	0.0694	0.1692	0.1618	0.3744
		4	0.3745	0.9050	0.1039	0.2623	0.1993	0.4839
		0.5	-0.4557	-1.1796	-0.4624	-1.2155	-0.4601	-1.2035
θ=2	2	2	-0.1152	-0.3019	-0.1389	-0.3680	-0.1310	-0.3458
		4	-0.0604	-0.1598	-0.0885	-0.2367	-0.0791	-0.2109
		0.5	-0.5075	-1.3983	-0.5085	-1.4050	-0.5082	-1.4028
θ=4	3	2	-0.1270	-0.3385	-0.1342	-0.3589	-0.1318	-0.3520
		4	-0.0687	-0.1855	-0.0782	-0.2113	-0.0751	-0.2027

As seen from Table 1, the ExpKw-PF distribution should be positively and negatively skewed depending on the values of parameters. Similary, we obtain both leptokurtic and platykurtic shapes depending on the values of parameters. This result show that the ExpKw-PF distribution is quite flexible.

2.4. Order statistics. Order statistics make their appearance in many areas of statistical theory and practice. They enter in the problems of estimation and hypothesis tests in a variety of ways. Order statistics are also among the most fundamental tools in non-parametric statistics. Therefore, we now discuss some properties of the order statistic.

Let $X_{i:n}$ denote the i^{th} order statistic. Then, let $f_{i:n}(x)$ be the pdf of the i^{th} order statistic for a random sample $X_1, X_2, ..., X_n$ from the ExpKw-PF distribution. We derive the pdf of the i^{th} order statistics of the ExpKw-PF distribution:

$$\begin{aligned} f_{i:n} &= \tfrac{n!}{(i-1)!(n-i)!} f(x_i)[F(x_i)]^{i-1}[1-F(x_i)]^{n-i} \\ &= \tfrac{n!\theta ab\beta x_i^{a\beta-1}}{(i-1)!(n-i)!\alpha^{a\beta}} \left[1-\left[1-\left(\tfrac{x_i}{\alpha}\right)^{a\beta}\right]^b\right]^{\theta i-1} \left[1-\left[1-\left[1-\left(\tfrac{x_i}{\alpha}\right)^{a\beta}\right]^b\right]^\theta\right]^{n-i} \\ &\left[1-\left(\tfrac{x_i}{\alpha}\right)^{a\beta}\right]^{b-1}. \end{aligned} \tag{2.9}$$

From (2.9), the pdf of the minimum order statistics of the ExpKw-PF distribution is given by

$$\begin{aligned} f_{1:n} &= nf(x_1)[1-F(x_1)]^{n-1} \\ &= \tfrac{n\theta ab\beta x_1^{a\beta-1}}{\alpha^{a\beta}} \left[1-\left[1-\left[1-\left(\tfrac{x_1}{\alpha}\right)^{a\beta}\right]^b\right]^\theta\right]^{n-1} \left[1-\left[1-\left(\tfrac{x_1}{\alpha}\right)^{a\beta}\right]^b\right]^{\theta-1} \\ &\left[1-\left(\tfrac{x_1}{\alpha}\right)^{a\beta}\right]^{b-1}, \end{aligned} \tag{2.10}$$

and the pdf of the maximum order statistics of the ExpKw-PF distribution is given by

$$\begin{aligned} f_{n:n} &= nf(x_n)[F(x_n)]^{n-1} \\ &= \tfrac{n\theta ab\beta x_n^{a\beta-1}}{\alpha^{a\beta}} \left[1-\left[1-\left(\tfrac{x_n}{\alpha}\right)^{a\beta}\right]^b\right]^{\theta n-1} \left[1-\left(\tfrac{x_n}{\alpha}\right)^{a\beta}\right]^{b-1}. \end{aligned} \tag{2.11}$$

3. Maximum Likelihood Estimation

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters $(a, b, \alpha, \beta, \theta)$ of the ExpKw-PF distribution. Suppose $X_1, X_2, ..., X_n$ is a random sample of size n from the ExpKw-PF distribution. Then, the likelihood function is given by

$$L = \theta^n a^n b^n \beta^n \alpha^{-na\beta} \prod_{i=1}^{n} {x_i}^{a\beta-1} \prod_{i=1}^{n} \left[1 - \left[1 - \left(\tfrac{x_i}{\alpha}\right)^{a\beta}\right]^b\right]^{\theta-1} \prod_{i=1}^{n} \left[1 - \left(\tfrac{x_i}{\alpha}\right)^{a\beta}\right]^{b-1}, \tag{3.1}$$

and the log-likelihood function is given by

$$\begin{aligned} \ln(L) = l &= n\ln(\theta) + n\ln(a) + n\ln(b) + n\ln(\beta) - na\beta\ln(\alpha) \\ &+ (a\beta - 1)\sum_{i=1}^{n}\ln(x_i) + (\theta-1)\sum_{i=1}^{n}\ln\left(1 - \left[1-\left(\tfrac{x_i}{\alpha}\right)^{a\beta}\right]^b\right) \\ &+ (b-1)\sum_{i=1}^{n}\ln\left(1-\left(\tfrac{x_i}{\alpha}\right)^{a\beta}\right). \end{aligned} \tag{3.2}$$

The estimates of the parameters maximize the likelihood function. Taking the partial derivatives of the log-likelihood function with respect to $(a, b, \alpha, \beta, \theta)$ and equalizing the obtained expressions to zero yields to likelihood equations can be written as

$$\begin{aligned} \frac{\partial l}{\partial a} &= \frac{n}{a} - n\beta\ln(\alpha) + \beta\sum_{i=1}^{n}\ln(x_i) + (\theta-1)\sum_{i=1}^{n}\frac{b\beta x_i^{a\beta}\left[1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right]^{b-1}\ln\left(\frac{x_i}{\alpha}\right)}{\alpha^{a\beta}\left(1-\left[1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right]^b\right)} \\ &- (b-1)\sum_{i=1}^{n}\frac{\beta x_i^{a\beta}\ln\left(\frac{x_i}{\alpha}\right)}{\alpha^{a\beta}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \frac{\partial l}{\partial b} &= \frac{n}{b} - (\theta-1)\sum_{i=1}^{n}\frac{\ln\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)}{1-\left[1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right]^b} \\ &+ \sum_{i=1}^{n}\ln\left(1-\left[1-\left(\tfrac{x_i}{\alpha}\right)^{a\beta}\right]^b\right), \end{aligned} \tag{3.4}$$

$$\begin{aligned} \frac{\partial l}{\partial \alpha} &= \frac{-na\beta}{\alpha} - (\theta-1)\sum_{i=1}^{n}\frac{ab\beta x_i^{a\beta}\left[1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right]^{b-1}}{\alpha^{a\beta+1}\left(1-\left[1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right]^b\right)} \\ &+ (b-1)\sum_{i=1}^{n}\frac{a\beta x_i^{a\beta}}{\alpha^{a\beta+1}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \frac{\partial l}{\partial \beta} &= \frac{n}{\beta} - na\ln(\alpha) + a\sum_{i=1}^{n}\ln(x_i) + (\theta-1)\sum_{i=1}^{n}\frac{ab\beta x_i^{a\beta}\left[1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right]^{b-1}\ln\left(\frac{x_i}{\alpha}\right)}{\alpha^{a\beta}\left(1-\left[1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right]^b\right)} \\ &- (b-1)\sum_{i=1}^{n}\frac{a x_i^{a\beta}\ln\left(\frac{x_i}{\alpha}\right)}{\alpha^{a\beta}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)}, \end{aligned} \tag{3.6}$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln\left(1-\left[1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right]^b\right). \tag{3.7}$$

The maximum likelihood estimates $\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta}, \hat{\theta}$ of the parameters $(a, b, \alpha, \beta, \theta)$ can be obtained numerically by solving simultaneously the non-linear equations $\frac{\partial l}{\partial a} = 0$, $\frac{\partial l}{\partial b} = 0$, $\frac{\partial l}{\partial \alpha} = 0$, $\frac{\partial l}{\partial \beta} = 0$, $\frac{\partial l}{\partial \theta} = 0$.

Note that the MLE has second derivatives with respect to the parameters, so that Fisher information matrix (FIM), $I_{ij}(\xi)$ can be expressed as

$$I_{ij}(\xi) = E\left(\frac{\partial^2 l(\xi; X_1, ..., X_n)}{\partial \xi_i \partial \xi_j}\right), i, j = 1, ..., 5 \tag{3.8}$$

The elements of the information matrix is given in Appendix. The total FIM $I_n(\xi)$ can be approximated by

$$J_n\left(\hat{\xi}\right)=\left(\frac{\partial^2 l\left(\xi;X_1,...,X_n\right)}{\partial\xi_i\partial\xi_j}\Big|_{\xi=\hat{\xi}}\right),i,j=1,...,5 \tag{3.9}$$

For real data, the matrix given in (3.9) is obtained after the convergence of the Newton-Raphson procedure in R software. Let $\hat{\xi}=\left(\hat{a},\hat{b},\hat{\alpha},\hat{\beta},\hat{\theta}\right)$ be the maximum likelihood estimate of $\xi=(a,b,\alpha,\beta,\theta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}\left(\hat{\xi}-\xi\right)\xrightarrow{d}N_5\left(0,I^{-1}(\xi)\right)$, where $I(\xi)$ is the expected FIM. The asymtotic behavior is still valid if $I(\xi)$ is replaced by the observed information matrix evaluated at $\hat{\theta}$, that is $J\left(\hat{\xi}\right)$.

The multivariate normal distribution with mean vector $0=(0,0,0,0,0)^T$ and covariance matrix $I^{-1}(\xi)$ can be used to construct confidence intervals for the model parameters. That is, the approximate $100(1-\eta)$ percent two-sided confidence intervals for $\xi=(a,b,\alpha,\beta,\theta)$ are given by

$$\hat{a}\pm Z_{\frac{\eta}{2}}\sqrt{I_{aa}^{-1}\left(\hat{\xi}\right)},\hat{b}\pm Z_{\frac{\eta}{2}}\sqrt{I_{bb}^{-1}\left(\hat{\xi}\right)},\hat{\alpha}\pm Z_{\frac{\eta}{2}}\sqrt{I_{\alpha\alpha}^{-1}\left(\hat{\xi}\right)},\hat{\beta}\pm Z_{\frac{\eta}{2}}\sqrt{I_{\beta\beta}^{-1}\left(\hat{\xi}\right)}\hat{\theta}\pm Z_{\frac{\eta}{2}}\sqrt{I_{\theta\theta}^{-1}\left(\hat{\xi}\right)} \tag{3.10}$$

respectively, where

$$I_{aa}^{-1}\left(\hat{\xi}\right),\ I_{bb}^{-1}\left(\hat{\xi}\right),\ I_{\alpha\alpha}^{-1}\left(\hat{\xi}\right),\ I_{\beta\beta}^{-1}\left(\hat{\xi}\right),\ I_{\theta\theta}^{-1}\left(\hat{\xi}\right), \tag{3.11}$$

are diagonal elements of $I_n^{-1}(\xi)=\left(nI_n\left(\hat{\xi}\right)\right)^{-1}$ and $Z_{\frac{\eta}{2}}$ is the upper $\frac{n^{th}}{2}$ percentile of a standard normal distribution. "numDeriv" package of R language can be used to compute the Hessian matrix and its inverse, standard errors and asymptotic confidence intervals.

Note that parameter estimation become complicated when censoring is present in the sample. Some time it is not possible to give a mathematical expression of estimated values of parameters in maximum likelihood method.

4. Simulation Study

In this section, we evaluate the performance of the MLEs of the parameters of the ExpKw-PF model by means of a simulation study. We generated samples of sizes n =10, 25, 50, 100, 200 from the ExpKw-PF model for different parameter combinations. We computed mean and mean square error (MSE) of parameter estimations, and this procedure is repeated 1000 times. Let $\hat{\xi}=(\hat{a},\hat{b},\hat{\alpha},\hat{\beta},\hat{\theta})$ be the MLEs of the parameters of the ExpKw-PF. The estimated MSEs can be estimated by using following equation:

$$MSE_{\varepsilon}(n)=\frac{\sum\limits_{i=1}^{N}\left(\hat{\varepsilon}_i-\varepsilon\right)}{N} \tag{4.1}$$

R software is used for simulation study and real data modeling. The simulation results are presented in Table 2.

The values in Table 2 indicate that the estimates are quite stable and, more importantly, are close to the true parameter values for these sample sizes. From Table 2, it is observed that in general MSE decreases as n increases. The simulation study also shows that the maximum likelihood method is appropriate for estimating the ExpKw-PF parameters. In fact, the MSEs of the parameters tend to be closer to zero when n increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs.

Table 2. Maximum likelihood estimation of the ExpKw-PF parameters for $\theta = 2$ and $\alpha = 1$.

				Mean					MSE				
a	b	β	n	a	b	α	β	θ	a	b	α	β	θ
0.5	0.5	1	10	0.514	0.669	1.843	0.669	2.571	0.013	0.543	0.269	1.050	0.176
			25	0.498	0.647	1.647	0.647	2.354	0.006	0.125	0.146	1.507	0.299
			50	0.501	0.553	1.328	0.553	2.146	0.003	0.0521	0.034	0.084	0.105
			100	0.500	0.512	1.195	0.512	2.009	0.001	0.014	0.018	0.032	0.053
			200	0.500	0.511	1.003	0.511	2.024	0.0005	0.005	0.009	0.01	0.016
0.5	2	1.5	10	0.493	2.764	1.640	2.764	3.011	0.008	1.255	0.564	2.417	0.432
			25	0.498	2.421	1.528	2.421	2.445	0.005	0.953	0.118	2.180	0.218
			50	0.497	2.282	1.303	2.282	2.042	0.0009	0.125	0.046	1.073	0.065
			100	0.499	2.133	1.141	2.133	2.013	0.0002	0.025	0.022	0.568	0.032
			200	0.499	2.054	1.002	2.054	2.003	0.0001	0.109	0.012	0.246	0.019
3	2	2	10	3.154	2.309	1.326	2.309	2.321	0.495	0.944	0.037	1.372	0.385
			25	3.022	2.298	1.232	2.298	2.298	0.136	0.751	0.019	0.909	0.249
			50	3.018	2.223	1.113	2.223	2.109	0.059	0.492	0.004	0.763	0.152
			100	2.999	2.105	1.000	2.105	2.001	0.024	0.337	0.001	0.239	0.056
			200	3.002	2.054	1.000	2.054	2.000	0.011	0.271	0.0005	0.002	0.002

5. Application

In this section, we study on a real data set to illustrate the usefulness of the ExpKw-PF distribution. We make comparison of the results with PF, Kw-PF and ExpKw-PF distributions. We used data set consists of annual maximum daily precipitation (unit:mm) at Busan, Korea for the 1904-2011 period. This data set has recently used Mansoor et al. [15].

The data are: 24.8, 140.9, 54.1, 153.5, 47.9, 165.5, 68.5, 153.1, 254.7, 175.3, 87.6, 150.6, 147.9, 354.7, 128.5, 150.4, 119.2, 69.7, 185.1, 153.4, 121.7, 99.3, 126.9, 150.1, 149.1, 143, 125.2, 97.2, 79.3, 125.8, 101, 89.8, 54.6, 283.9, 94.3, 165.4, 48.3, 69.2, 147.1, 114.2, 159.4, 114.9, 58.5, 76.6, 20.7, 107.1, 244.5, 126, 122.2, 219.9, 153.2, 145.3, 101.9, 135.3, 103.1, 74.7, 174, 126, 144.9, 226.3, 96.2, 149.3, 122.3, 164.8, 188.6, 273.2, 61.2, 84.3, 130.5, 96.2, 155.8, 194.6, 92, 131, 137, 106.8, 131.6, 268.2, 124.5, 147.8, 294.6, 101.6, 103.1, 274.51, 40.2, 153.3, 91.8, 79.4, 149.2, 168.6, 127.7, 332.8, 261.6, 122.9, 273.4, 178, 177, 108.5, 115. 241, 76, 127.5, 190, 259.5, 301.5.

Table 3. Descriptive statistics for the precipitation data.

n	Minimum	Median	Mean	Maximum	Variance	Skewness	Kurtosis
104	20.7	130.75	142.973	354.7	4495.71	0.97	0.936

In application, we maximized the log-likelihood function using "nlm" function in R statistical package. For each maximization, the "nlm" function is executed for a wide range of initial values.

The maximum likelihood estimates, the corresponding values of log-likelihood and the Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) values for fitted distributions reported in Table 4.

Table 4. Maximum likelihood estimates, AIC, BIC statistics under considered distributions based on precipitation data

Model	Maximum Likelihood Estimates					LogL	AIC	BIC	K-S statistic	p-value
	a	b	α	β	θ					
PF	-	-	0.954	369.995	-	614.806	1233.730	1238.901	0.317	0.000
Kw-PF	6.985	2.759	360.047	0.087	-	581.208	1172.416	1185.638	0.127	0.070
ExpKw-PF	0.328	2.804	416.801	0.543	78.897	577.620	1165.241	1178.463	0.097	0.284

The results in Table 4 show that the ExpKw-PF distribution provides a significantly better fit than the other models.

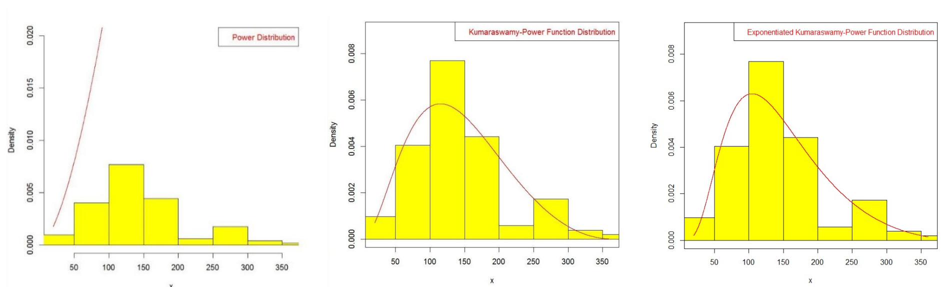

Figure 11. Fitting performance of distributions.

6. Conclusion

In this study, a five-parameter distribution, called exponentiated Kumaraswamy-power function distribution (denoted by ExpKw-PF distribution), is introduced. A detailed study on the statistical properties of the new distribution is presented. We obtain the survival, hazard and quantile functions, order statistics, skewness and kurtosis. The results show that the distribution is mostly negatively-skewed. Then, parameters of the ExpKw-PF distribution are estimated by maximum likelihood and a simulation study is conducted. An application of the ExpKw-PF to real data shows that the new distribution can be used quite effectively to provide better fits than PF, Kw-PF distributions. We hope that the proposed model may attract wider applications in statistics and other areas.

Appendix

Elements of the information matrix:

$$I_{\alpha\alpha} = \frac{na\beta}{\alpha^2} - ab\beta\,(\theta-1)\sum_{i=1}^{n} \frac{a(b-1){x_i}^{2a\beta}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-2}\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]}{\alpha^{a\beta+1}\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^2}$$

$$-\frac{{x_i}^{a\beta}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-1}\left\{(a\beta+1)\alpha^{a\beta}\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]+b\alpha^{a\beta+1}\left(1-\frac{a\beta}{\alpha}\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)\right\}}{\alpha^{a\beta+1}\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^2}$$

$$I_{\alpha\beta} = -\frac{na}{\alpha} - \frac{ab}{\alpha}\,(\theta-1)\sum_{i=1}^{n} \frac{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]\left\{\left(\frac{x_i}{\alpha}\right)^{a\beta}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-2}\left[\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)-a\beta(b-1)\left(\frac{x_i}{\alpha}\right)^{a\beta}\ln\left(\frac{x_i}{\alpha}\right)\right]\right\}}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^2}$$

$$-\frac{ab\beta\left(\frac{x_i}{\alpha}\right)^{2ab}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{2b-2}\ln\left(\frac{x_i}{\alpha}\right)}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^2} + (b-1)\,\frac{a}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{a\beta}\left(a\beta\ln\left(\frac{x_i}{\alpha}\right)+1\right)\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)+a\beta\left(\frac{x_i}{\alpha}\right)^{2a\beta}\ln\left(\frac{x_i}{\alpha}\right)}{\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^2}$$

$$I_{\alpha\theta} = -\frac{ab\beta}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{a\beta}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-1}}{1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}}$$

$$I_{\alpha a} = -\frac{n\beta}{\alpha} - \frac{b\beta}{\alpha}\,(\theta-1)\sum_{i=1}^{n}\frac{\beta\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]\left(\frac{x_i}{\alpha}\right)^{a\beta}}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^2}$$

$$+\frac{\ln\left(\frac{x_i}{\alpha}\right)\left\{a\left[1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right]^{b-1}-a\beta(b-1)\left(\frac{x_i}{\alpha}\right)^{a\beta}\ln\left(\frac{x_i}{\alpha}\right)\right\}}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^2}$$

$$+\frac{\beta}{\alpha}\,(b-1)\sum_{i=1}^{n}\frac{\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)\left(\frac{x_i}{\alpha}\right)^{a\beta}\left(a\beta\ln\left(\frac{x_i}{\alpha}\right)+1\right)+a\beta\left(\frac{x_i}{\alpha}\right)^{2a\beta}\ln\left(\frac{x_i}{\alpha}\right)}{\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^2}$$

$$I_{\alpha b} = -\frac{a\beta}{\alpha}\,(\theta-1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{a\beta}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-1}\left\{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]\left[b\ln\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)+1\right]\right\}}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^2}$$

$$-\frac{b\left(\frac{x_i}{\alpha}\right)^{a\beta}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-1}\left[\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\ln\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)\right]}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^2} + \frac{a\beta}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{a\beta}}{1-\left(\frac{x_i}{\alpha}\right)^{a\beta}}$$

$$I_{\beta\beta} = -\frac{n}{\beta^2} + 2a^2b\,(\theta-1)\sum_{i=1}^{n}\ln\left(\frac{x_i}{\alpha}\right)\frac{\left(\frac{x_i}{\alpha}\right)^{a\beta}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-2}\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)\right]^{b}\left[\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-1}-(b-1)\left(\frac{x_i}{\alpha}\right)^{a\beta}\right]}{\left\{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)\right]^{b}\right\}^2}$$

$$+\frac{b\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{2b-2}\ln\left(\frac{x_i}{\alpha}\right)\left(\frac{x_i}{\alpha}\right)^{2ab}}{\left\{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)\right]^{b}\right\}^2} - 2a\,(b-1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{a\beta}\ln\left(\frac{x_i}{\alpha}\right)}{\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^2}$$

$$I_{\beta\theta} = \sum_{i=1}^{n} \frac{ab\left(\frac{x_i}{\alpha}\right)^{a\beta}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-1}\ln\left(\frac{x_i}{\alpha}\right)}{1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}}$$

$$I_{\beta a} = -n\ln(\alpha) + \sum_{i=1}^{n}\ln(x_i) + b(\theta-1)\sum_{i=1}^{n} \frac{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-1}\left(\frac{x_i}{\alpha}\right)^{a\beta}\ln\left(\frac{x_i}{\alpha}\right)}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]}$$

$$\cdot\frac{\left[a\ln\left(\frac{x_i}{\alpha}\right)+1-a\beta(b-1)\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{-1}\left(\frac{x_i}{\alpha}\right)^{a\beta}\ln\left(\frac{x_i}{\alpha}\right)\right]}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]}$$

$$-(b-1)\sum_{i=1}^{n} \frac{\ln\left(\frac{x_i}{\alpha}\right)\left(\frac{x_i}{\alpha}\right)^{a\beta}\left(a\beta\ln\left(\frac{x_i}{\alpha}\right)+1\right)\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)+2a\beta\left(\frac{x_i}{\alpha}\right)^{2a\beta}\ln\left(\frac{x_i}{\alpha}\right)}{\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{2}}$$

$$I_{\beta b} = a(\theta-1)\sum_{i=1}^{n} \frac{\ln\left(\frac{x_i}{\alpha}\right)\left(\frac{x_i}{\alpha}\right)^{a\beta}\left\{\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-1}\left[b\ln\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-1}+1\right]\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]\right\}}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^{2}}$$

$$+\frac{b\ln\left(\frac{x_i}{\alpha}\right)\left(\frac{x_i}{\alpha}\right)\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{2b-1}\ln\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^{2}} - a\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{a\beta}\ln\left(\frac{x_i}{\alpha}\right)}{1-\left(\frac{x_i}{\alpha}\right)^{a\beta}}$$

$$I_{a\theta} = b\beta\sum_{i=1}^{n} \frac{\left(\frac{x_i}{\alpha}\right)^{a\beta}\ln\left(\frac{x_i}{\alpha}\right)\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-1}}{1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-1}}$$

$$I_{aa} = -\frac{n}{a^2} + 2b\beta^2(\theta-1)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{a\beta}\ln\left(\frac{x_i}{\alpha}\right) \frac{\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-2}\left\{1-(b-1)\left(\frac{x_i}{\alpha}\right)^{a\beta}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]\right\}}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^{2}}$$

$$-2b\beta^2(\theta-1)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{a\beta}\ln\left(\frac{x_i}{\alpha}\right)\frac{\left[b\left(\frac{x_i}{\alpha}\right)^{a\beta}\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^{2}} - 2\beta^2(b-1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{a\beta}\ln\left(\frac{x_i}{\alpha}\right)}{\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{2}}$$

$$I_{ab} = \beta(\theta-1)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)\ln\left(\frac{x_i}{\alpha}\right)\frac{\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b-1}\left\{\left[b\ln\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)+1\right]\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]\right\}}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^{2}}$$

$$+\beta(\theta-1)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)\ln\left(\frac{x_i}{\alpha}\right)\frac{\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)\ln\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right]^{2}} - \sum_{i=1}^{n}\frac{\beta\left(\frac{x_i}{\alpha}\right)\ln\left(\frac{x_i}{\alpha}\right)}{1-\left(\frac{x_i}{\alpha}\right)}$$

$$I_{b\theta} = -\sum_{i=1}^{n}\frac{\ln\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)}{1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}}$$

$$I_{bb} = -\frac{n}{b^2} - (\theta-1)\sum_{i=1}^{n}\frac{2\ln\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b+1}}{\left[1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)\right]^{2}} + \sum_{i=1}^{n}\ln\left(1-\left(1-\left(\frac{x_i}{\alpha}\right)^{a\beta}\right)^{b}\right)$$

$I_{\theta\theta} = -\frac{n}{\theta^2}$

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