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Research Article

Generalized probabilistic approximation characteristic based on Birkhoff orthogonality and related conclusions in S_∞-norm

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ABSTRACT. In this article, we generalize the definition of the probabilistic Gel'fand width from the Hilbert space to the strictly convex reflexive space by giving Birkhoff left orthogonal decomposition theorem. Meanwhile, a more natural definition of Gel'fand width in the classical setting is selected to make sure probabilistic and average Gel'fand widths will not lose their meaning, so that we can give the equality relation between the probabilistic Gel'fand width and the probabilistic linear width of the Hilbert space. Meanwhile, we use this relationship to continue the study of the Gel'fand widths of the univariate Sobolev space and the multivariate Sobolev space, especially in S_∞-norm, and determine the exact order of probabilistic and average Gel'fand widths.

Keywords: Sobolev space, S_∞-norm, probabilistic and average Gel'fand widths, Birkhoff orthogonality, strictly convex reflexive space.

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1. INTRODUCTION

A central problem in approximation theory is to understand how effectively infinite-dimensional function spaces can be approximated by finite-dimensional subspaces and to quantify the inherent complexity of such approximation processes. In this context, width theory has long served as a fundamental tool for measuring the approximability of function classes. Since its emergence in the mid-20th century, width theory has not only provided theoretical lower bounds on approximation errors, but also revealed deep structural characteristics of function classes, thereby offering essential guidance for partial differential equations(PDEs) and geometric analysis, computational complexity. For example, Wenzel et al. [1] applied width theory to linear elliptic partial differential equations, which provides theoretical guarantees for the convergence of greedy algorithms in the context of kernel methods and meshfree approximation. On the other hand, Traub, Wasilkowski, and Wozniakowski [2] established relations between the Gel'fand n-widths and nth minimal radii of information in the classical setting. So, width theory is also a fundamental tool in engineering, such as in signal and image processing, algorithms and data analysis, and even in large language models(LLM).

Classical notions of width, such as Kolmogorov width, Gel'fand width, and linear width, capture the worst-case approximation errors on a given set. However, this worst-case approach does not account for the extreme distribution of functions within the class. To address this limitation, probabilistic and average settings have been introduced, where a probability measure is defined on the function space, allowing for a more nuanced analysis of approximation errors. Maiorov [3] studied the Kolmogorov width of Wiener space with L_∞ norm in average

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setting. Soon, he [4, 5] defined the Kolmogorov width and linear width in probabilistic setting, which exclude a subset of measure at most δ, and was the first to use the discretization method to estimate the asymptotic order of the width in probabilistic setting. These theoretical advances have been applied to various function spaces. For instance, Xu et al. [6] studied the probabilistic and average linear widths of Sobolev spaces on the torus in the metric of $S_q(\mathbb{T})$, establishing sharp asymptotic orders for all $1 \leq q \leq \infty$. Similarly, Wang [7] investigated the probabilistic and average linear widths of weighted Sobolev spaces on the unit ball equipped with a Gaussian measure, obtaining precise asymptotic equivalences.

The discretization method, which transforms infinite-dimensional problems into finite-dimensional ones, is discussed as a central technique in obtaining sharp asymptotic estimates. This approach has been widely used in the study of probabilistic and average widths. Fang and Ye [8] used results of Maiorov to calculate the probabilistic and average linear widths of the univariate Sobolev space W_2^r with Gaussian measure in $L_q(1 \leq q < \infty)$ norm. And then they [9] studied probabilistic and average linear widths of W_2^r in L_∞ norm. It is the first time the problem is solved when $q = \infty$. Tan and Shao et al. [10] proposed the Gel'fand width in the probabilistic setting and estimated the sharp order of the probabilistic Gel'fand width of finite-dimensional space and W_2^r in $L_q(1 \leq q < \infty)$ norm.

Moreover, as real-world problems increasingly involve high-dimensional inputs, width theory has proven highly influential in multivariate function approximation. Studies by Chen and Fang [11], Liu et al. [12], and Wu et al. [13] on the widths have laid a theoretical foundation for the width estimation of multivariate Sobolev spaces. Notably, Liu et al. [12] defined the Gel'fand width in the average setting. Wu et al. [13] calculated the probabilistic and average Gel'fand widths in $S_q(1 \leq q < \infty)$ norm. So, the result in S_∞ remained.

This paper focuses on two important tasks. On one hand, we use the Birkhoff orthogonality to generalize the definition of the probabilistic Gel'fand width from the Hilbert space to the strictly convex reflexive space. In this way, we can consider the probabilistic Gel'fand widths of L^p, l^p, $W^{k,p}$ where $1 < p < \infty, p \neq 2$ and so on. Since orthogonal decomposition has never been considered in the general normed space, this promotion has never been carried out before. On the other hand, we give the Gel'fand widths of the univariate Sobolev space and the multivariate Sobolev space in S_∞-norm. When $q = \infty$, the properties of L_q spaces and S_q spaces differ fundamentally from those when $1 \leq q < \infty$. From Ref [9], we can clearly find the differences between the proof of L_∞ and those when $1 \leq q < \infty$. In conclusion, our study in S_∞ is meaningful. We improve the proof of Ref [12] and then give the equality relation between the probabilistic Gel'fand width and the probabilistic linear width of the Hilbert space. Meanwhile, we use the relationship mentioned before to continue the study of the Gel'fand widths and determine the exact order of probabilistic and average Gel'fand widths. In this way, we can gain a deeper understanding of the essential characteristics of the space and ingeniously bypass the discretization method.

In this paper, we choose a better definition of Gel'fand width in the classical setting to make sure probabilistic and average Gel'fand widths will not lose their meaning.

We commence by recalling the foundational concepts of width theory. We first introduce the definitions of Kolmogorov width, linear width, and Gel'fand width in the classical setting. Let X be a normed linear space with norm $\|\cdot\|_X$. $\{0\} \subset W$ is a non-empty subset of X. F_n is a linear subspace of X with dimension n, and n is a non-negative integer. We will use these letters throughout the paper. The best approximation of F_n to W in X is defined as

$$e(W, F_n, X) := \sup_{x \in W} \inf_{y \in F_n} \|x - y\|.$$

We can clearly find that $e(W, F_n, X)$ represents the deviation or error when the set W is approximated by F_n in the normed space X. Figuratively, we can understand it as the "maximum distance" from F_n to W considering the norm $\|\cdot\|_X$. It is similar to the Gromov-Hausdorff distance. Then the quantity

$$d_n(W, X) := \inf_{F_n} e(W, F_n, X) = \inf_{F_n} \sup_{x \in W} \inf_{y \in F_n} \|x - y\|$$

is called the Kolmogorov n-width of W in X. In the first infimum, we take over all n-dimensional subspaces F_n of X.

Let T_n be a bounded linear operator on X with rank at most n and let F^n be a closed linear subspace of X with co-dimension n. The definitions of the linear n-width and the Gel'fand n-width are as follows:

$$\lambda_n(W, X) := \inf_{T_n} \sup_{x \in W} \|x - T_n x\|$$
$$d^n(W, X) := \inf_{F^n} \sup_{x \in W \cap F^n} \|x\|,$$

where the infimum in $\lambda_n(W, X)$ is taken over all T_n on X and the infimum in $d^n(W, X)$ is taken over all F^n on X. In particular, since $\{0\} \subset W$ and $\{0\} \subset F^n$, $W \cap F^n$ is not empty. Further properties and foundational aspects of these widths can be found in the monograph [14].

Next, we will consider widths in the probabilistic and average setting.

Definition 1.1 ([13]). *Let X and T^n be consistent with $\lambda_n(W, X)$. Let $\mathscr{B}$ be the Borel field on W, which is the smallest σ-algebra consisting of all open subsets of W, and let μ be the probabilistic measure defined on $\mathscr{B}$. For any $\delta \in [0, 1]$, the probabilistic linear (n, δ)-width of W in X is given by*

$$\lambda_{n,\delta}(W, \mu, X) := \inf_{G_\delta} \lambda_n(W \setminus G_\delta, X) = \inf_{G_\delta} \inf_{T_n} \sup_{x \in (W \setminus G_\delta)} \|x - T_n x\|,$$

where G_δ runs over all possible subsets in $\mathscr{B}$ with $\mu(G_\delta) \le \delta$.

The p-average linear n-width of W in X is given by

$$\lambda_n^{(a)}(W, \mu, X)_p := \inf_{T_n} \left(\int_W \|x - T_n x\|^p d\mu(x) \right)^{1/p}, \ 0 < p < \infty.$$

Remark 1.1. *In particular, when $\delta = 0$ and $G_\delta = \varnothing$, the probabilistic linear (n, δ)-width is the linear width in the classical setting. When $\delta = 1$, according to the definition of the linear n-width, $G_\delta \neq W$ and $W \setminus G_\delta$ is a non-empty set of measure zero.*

Then, let us recall how the probabilistic Gel'fand width was introduced according to Ref [10]. Firstly, let H be a Hilbert space equipped with the probabilistic measure μ and $M \subset H$ be a closed subspace. Secondly, we have the Hilbert orthogonal decomposition. For any $x \in H$, x can be decomposed as $x = y + z, y \in M$, uniquely. Let P be a projection operator such that $y = Px$ is the projection of x on M. Finally, we limit the measure to M. For any Borel set G_M of M, we have $\mu_M(G_M) := \mu(\{x \in H \mid Px \in G_M\})$. Obviously, μ_M is a probabilistic measure on M. Then we have the following definition of the Gel'fand width in probabilistic setting.

Definition 1.2 ([10]). *Let X and F^n be consistent with $d^n(W, X)$. Let H be a Hilbert space and H can be continuously embedded into X. Meanwhile, μ and G_δ are defined in the same way as in Definition 1.1. Then, for any $\delta \in (0, 1]$, the probabilistic Gel'fand (n, δ)-width of H in X is given by*

$$d_\delta^n(H, \mu, X) := \inf_{G_\delta} d^n(H \setminus G_\delta, X) = \inf_{G_\delta} \inf_{F^n} \sup_{x \in (H \setminus G_\delta) \cap F^n} \|x\|.$$

Meanwhile, G_δ satisfies that for any closed subspace M of H,

$$\mu_M(G_\delta \cap M) \le \delta, \tag{1.1}$$

which guarantees $(H \setminus G_\delta) \cap F^n$ has enough elements.

Remark 1.2. *According to the definition of $d^n(W, X)$, we have at least $\{0\} \subset (H \setminus G_\delta) \cap F^n$, so $(H \setminus G_\delta) \cap F^n \neq \varnothing$, which makes sure that the probabilistic Gel'fand width makes sense. Namely, G_δ satisfies $\{0\} \not\subset G_\delta$, and when $\delta = 1$, G_δ cannot equal to H and $H \setminus G_\delta$ is a non-empty set of measure zero. Otherwise, $G_\delta = G_1 = H$ always leads to $\mu_M(G_\delta \cap M) = 1$ and $H \setminus G_\delta = \varnothing$. In this way, condition (1.1) cannot guarantee $(H \setminus G_\delta) \cap F^n$ has enough elements. Therefore, our condition $\{0\} \subset (H \setminus G_\delta) \cap F^n$ or $\{0\} \not\subset G_\delta$ is necessary. In fact, if H can be continuously embedded into X, H can be regarded as a subspace W of X. So $d_\delta^n(H, \mu, X) = d_\delta^n(W, \mu, X)$.*

Definition 1.3 ([12]). *Let W, X and F^n be the same as $d^n(W, X)$, μ be the same as Definition 1.1. The p-average Gel'fand n-width of W in X is given by*

$$d_{(a)}^n(W, \mu, X)_p := \inf_{F_n} \left(\int_{W \cap F^n} \|x\|^p d\mu(x) \right)^{1/p}, \ 0 < p < \infty.$$

Here, we provide clarifications for relevant notations. Let $c_j > 0,\ j = 0, 1, 2, \cdots$ represent positive constants, and let $a(y)$ and $b(y)$ be two arbitrary positive functions defined on the set D. $a(y) \gg b(y)$ or $a(y) \ll b(y)$ means that for all $y \in D$, $a(y) \geq c_1 b(y)$ or $a(y) \leq c_2 b(y)$, respectively. And we use $a(y) \asymp b(y)$ to express that $a(y) \gg b(y)$ and $a(y) \ll b(y),\ y \in D$.

2. The generalization of the probabilistic Gel'fand width

In this section, we will use the Birkhoff orthogonality to generalize the definition of the probabilistic Gel'fand width from the Hilbert space to the strictly convex reflexive space. According to the guiding process of Definition 1.2, two very important definitions are mentioned: orthogonal decomposition and projection operator. If we want to generalize the probabilistic Gel'fand width of the Hilbert space, we need to consider projection first to decompose vectors. At the same time, orthogonality should also be considered to satisfy orthogonal decomposition and geometric intuition.

Here, we give some definitions and lemmas in function approximation theory and geometry. These lemmas will greatly simplify our proof of Theorem 2.1.

Definition 2.4. *Let $(X, \|\cdot\|)$ be a normed linear space and $F \subset X$. Consider selecting elements from F in some way as approximate representations of the elements in X. Then for any $x \in X$ and $F \neq \varnothing$,*

$$\mathscr{L}_F(x) = \{u_0 \in F \mid \|x - u_0\| = \inf_{u \in F} \|x - u\|\}$$

is called the set of the best approximation elements.

(1) *If for each $x \in X$, we have $\mathscr{L}_F(x) \neq \varnothing$, then F is called an existential set in X.*
(2) *If for each $x \in X$, $\mathscr{L}_F(x)$ is an empty set or a one-point set, then F is called a unique set in X.*
(3) *If for each $x \in X$, $\mathscr{L}_F(x)$ is a one-point set, then F is called a Chebyshev set in X. Namely, a Chebyshev set is both a unique set and an existential set.*

Definition 2.5. *When F is an existential set, take any $x \in X$, and we can always find an element y in $\mathscr{L}_F(x)$ as the image of x. This kind of mapping denoted by P_F is called metric projection or the best approximation operator from X to F.*

Remark 2.3. *If F is an existential set, we have $P_F(x) = \mathscr{L}_F(x) = \{u_0 \in F \mid \|x - u_0\| = \inf_{u \in F} \|x - u\|\}$. Otherwise, P_F does not exist.*

The following discussion will be based on Banach space. Specially, a strictly convex reflexive space is a particular type of Banach space.

Definition 2.6 ([15]). *Let X be a Banach space and $x, y \in X$. Then x is said to be Birkhoff orthogonal to y, written as $x \perp_B y$, if for any scalar λ, $\|x+\lambda y\| \geq \|x\|$.*

Let $Y \neq \varnothing$ be a linear subspace in X. If for each $y \in Y$ and any scalar λ, we have $\|x+\lambda y\| \geq \|x\|$, then x is said to be Birkhoff orthogonal to Y, written as $x \perp_B Y$.

Remark 2.4. *The Birkhoff orthogonality does not satisfy symmetry.*

Remark 2.5. *In function approximation theory, we have the following variational condition: Let $Y \neq \varnothing$ be a linear subspace in X. For $x, x_0 \in X$ and $e = x - x_0$, it is easy to know that $x_0 \in \mathscr{L}_Y(x) \Leftrightarrow \|e+y\| \geq \|e\|, \forall y \in Y$. It can be described in geometric language as orthogonality in a general normed space: for each $y \in Y$, we have $\|x+y\| \geq \|x\|$, then x is said to be orthogonal to Y. Since Y is a linear subspace, for any $y \in Y$ and any scalar λ, no matter it is a real number or a complex number, specifically we can choose it to be a real number, then $\lambda y \in Y$. So, the definition of orthogonality in function approximation theory is the same as Birkhoff orthogonality. They all embody the idea that the vertical distance is the shortest.*

Lemma 2.1. *Any closed linear subspace in a reflexive Banach space is an existential set.*

Lemma 2.2. *The convex set in the strictly convex space is a unique set.*

Lemma 2.3 ([16]). *A normed linear space is strictly convex if and only if $\|x\| + \|y\| = \|x+y\|$ and $y \neq 0$ imply the existence of a number t for which $x = ty$.*

We give the following decomposition theorem based on the Birkhoff orthogonality.

Theorem 2.1. *Let X be a strictly convex reflexive space, M be a closed linear subspace in X. For any $x \in X$, there exists a unique decomposition $x = m + n$, where $m \in M$ is the best approximation element of x under the metric projection and $n \perp_B M, n \in M^{\perp_B}$. $M^{\perp_B}$ is the left orthogonal complement subspace of M in X. Meanwhile, we have $X = M \oplus M^{\perp_B}$.*

Proof. It is easy to know that a reflexive space must be a Banach space. Since M is a closed linear subspace and any linear subspace must be a convex set, M is a convex set. According to Lemma 2.1 and Lemma 2.2, we have M is a Chebyshev set. There must exist a unique m such that $P_F(x) = \{m\}$. Obviously, n is also unique. So we obtain the unique decomposition of x.

For $n = x - m$ and number field $\mathbb{K}$, assume $\exists \lambda \in \mathbb{K}$ and $m' \in M$, such that $\|n + \lambda m'\| < \|n\|$. We have

$$\begin{aligned} \|n+\lambda m'\| &= \|x - m + \lambda m'\| \\ &= \|x - (m - \lambda m')\| \\ &< \|n\| = \|x - m\| = \inf_{u \in M} \|x - u\|. \end{aligned}$$

Since M is a linear subspace and $m, m' \in M$, we have $m - \lambda m' \in M$. We obtain a contradiction. Therefore, $\forall \lambda \in \mathbb{K}$ and $m' \in M$, there must be $\|n + \lambda m'\| \geq \|n\|$. Therefore, we obtain $n \perp_B M$.

Let $M^{\perp_B} = \{u \in X \mid u \perp_B M\}$. It is an orthogonal complement of M based on Birkhoff orthogonality. We will prove that it is a subspace. It is obvious to maintain scalar multiplication. For addition operations, we suppose that for any scalar λ and $n_1, n_2 \in M^{\perp_B}$, we have $\|n_1 + \lambda m\| \geq \|n_1\|$ and $\|n_2 + \lambda m\| \geq \|n_2\|$. Since x is a strictly convex normed linear space, according to Lemma 2.3, we have

$$\begin{aligned} \|n_1 + n_2 + \lambda m\| &= \|n_1 + n_2\| + \|\lambda m\| \\ &= \|n_1 + n_2\| + |\lambda|\|m\| \\ &\geq \|n_1 + n_2\|. \end{aligned}$$

Therefore, $M^{\perp_B}$ is a subspace.

Because $\forall x \in X, x = m+n, m \in M, n \in M^{\perp_B}$ and this decomposition is unique, it is obvious that $X = M \oplus M^{\perp_B}$. □

Then, we can define the probabilistic Gel'fand width of the strictly convex reflexive space by imitating the way this width of Hilbert space is defined.

Definition 2.7. *Let X, F^n, μ, δ and G_δ be consistent with Definition 1.2. Let W be a strictly convex reflexive space and W can be continuously embedded into X. Then the probabilistic Gel'fand (n, δ)-width of W in X is given by*

$$d^n_\delta(W, \mu, X) := \inf_{G_\delta} d^n(W \setminus G_\delta, X) = \inf_{G_\delta} \inf_{F^n} \sup_{x \in (W \setminus G_\delta) \cap F^n} \|x\|,$$

where G_δ satisfies that for any closed subspace M of W, $\mu_M(G_\delta \cap M) \leq \delta$. Meanwhile, according to the definition of Gel'fand width in the classical setting, $\{0\} \subset (W \setminus G_\delta)$. These two conditions guarantee the set, $(W \setminus G_\delta) \cap F^n \neq \varnothing$, has enough elements.

3. The relationship between probabilistic Gel'fand width and probabilistic linear width of the Hilbert space

In Section 2, we have generalized the definition of the probabilistic Gel'fand width. Naturally, we want to study the relationship between probabilistic Gel'fand width and probabilistic linear width. Based on the better and more natural definition of Gel'fand width in the classical setting chosen before, a shortcoming in the definition of probabilistic Gel'fand width is remedied. Then, we can safely discuss the relationship we aim to study. In this section, we improve the proof of Ref [12] and then give the equality relation between the probabilistic Gel'fand width and the probabilistic linear width of the Hilbert space. Owing to the absence of linearity of the metric projection P_F in the normed space, whereas the projection P in the Hilbert space is linear, we can only get this relationship in the Hilbert space now. The following lemma is instrumental in establishing the Theorem 3.2.

Lemma 3.4 ([10]). *Suppose H is a Hilbert space, $(X, \|\cdot\|)$ is a normed linear space, H can be continuously embedded into X, $\delta \in (0, 1]$, μ is a probabilistic measure on H. Then*

$$\lambda_{n,\delta}(H, \mu, X) \leq d^n_\delta(H, \mu, X).$$

Lemma 3.5 ([14]). *Let $(X, \|\cdot\|)$ be a normed linear space and W a subset of X, then*

$$\lambda_n(W, X) \geq d^n(W, X).$$

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a normed linear space and H be a Hilbert space. H can be continuously embedded into X and $\delta \in (0, 1]$. Then*

$$\lambda_{n,\delta}(H, \mu, X) = d^n_\delta(H, \mu, X).$$

Proof. On the one hand, according to Lemma 3.4, we have $\lambda_{n,\delta}(H, \mu, X) \leq d^n_\delta(H, \mu, X)$, $\delta \in (0, 1]$.

On the other hand, for any $\delta \in (0, 1]$, according to Lemma 3.5,

$$\begin{aligned}\lambda_{n,\delta}(H, \mu, X) &= \inf_{G_\delta} \lambda_n(H \setminus G_\delta, X) \\ &\geq \inf_{G_\delta} d^n(H \setminus G_\delta, X) \\ &= d^n_\delta(H, \mu, X).\end{aligned}$$

Therefore,

$$\lambda_{n,\delta}(H, \mu, X) = d^n_\delta(H, \mu, X), \delta \in (0, 1].$$

□

Based on previous research, the theorem we have derived demonstrates the equivalence between the Gel'fand width and the linear width of the Hilbert space in the probabilistic setting. Therefore, in subsequent discussions concerning widths of Hilbert spaces, it suffices to examine either the Gel'fand width or the linear width. However, for more general normed spaces, it remains necessary to address the problem via discretization approaches.

This theorem further implies that whenever a Hilbert space H is continuously embeddable into a Banach space X, the linear width and Gel'fand width coincide in the probabilistic setting, thus exhibiting a certain independence from the particular choice of the underlying space X.

4. Gel'fand widths of Sobolev spaces in S_∞-norm

4.1. **Main results.** First, let us give some introductions to $W_2^r(\mathbb{T})$, $MW_2^r(\mathbb{T}^d)$ and S_∞.

Let d-dimensional torus $\mathbb{T}^d = [0, 2\pi]^d,\ d \in \mathbb{Z}_+$. When $d = 1$, let $\mathbb{T} := \mathbb{T}^1$. We consider the Hilbert space $L_2(\mathbb{T}^d)$ of all 2π-periodic functions $x(t)$, where $t = (t_1, \cdots, t_d) \in \mathbb{T}^d$. Let $k = (k_1, \cdots, k_d) \in \mathbb{Z}^d$ and $(k,t) = \sum_{j=1}^d k_j t_j$. The Fourier series of $x(t)$ is as follows:

$$x(t) = \sum_{k\in\mathbb{Z}^d} c_k e_k(t) = \sum_{k\in\mathbb{Z}^d} \hat{x}(k) e_k(t),$$

where $e_k(t) := \exp(i(k,t))$, Fourier coefficient $c_k = \hat{x}(k) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{\mathbb{T}^d} x(t)\exp(-i(k,t))\mathrm{d}t$.

The inner product defined on $L_2(\mathbb{T}^d)$ is

$$\langle x, y\rangle = \frac{1}{(2\pi)^d}\int_{T^d} x(t)\overline{y(t)}\mathrm{d}t, \quad x, y \in L_2(\mathbb{T}^d).$$

For any $r = (r_1, \cdots, r_d) \in \mathbb{R}^d$, Weyl r-fractional derivative is given by

$$x^{(r)}(t) := (D^r x)(t) = \sum_{k\in\mathbb{Z}^d} (ik)^r c_k e_k(t),$$

where $(ik)^r = \displaystyle\prod_{j=1}^d |k_j|^{r_j}\exp(\frac{\pi i}{2}\mathrm{sgn} r_j)$. The univariate Sobolev space is denoted by

$$W_2^r(\mathbb{T}) = \{x(t) \in L_2(\mathbb{T}) \mid x^{(r)}(t) \in L_2(\mathbb{T}), \hat{x}(0) = 0\}, r \in \mathbb{R}_+.$$

For any $|\alpha| \le r$, it is obvious that $x^{(\alpha)}(t) \in L_2(\mathbb{T})$. The norm defined on $W_2^r(\mathbb{T})$ is $\|x\|_{W_2^r} := \langle x^{(r)}, x^{(r)}\rangle^{\frac{1}{2}}$. Therefore, $W_2^r(\mathbb{T})$ is a Hilbert space with inner product $\langle x, y\rangle_1 := \langle x^{(r)}, y^{(r)}\rangle$.

Equip $W_2^r(\mathbb{T})$ with a Gaussian measure μ. The mean of μ is 0, and the correlation operator C_μ has eigenfunctions $e_k = \exp(ik(\cdot))$ and eigenvalues $\lambda_k = |k|^{-\rho}$, $k \in \mathbb{Z}_0$, $\mathbb{Z}_0 := \mathbb{Z}\setminus\{0\}$, $\rho > 1$. Then, $C_\mu e_k = \lambda_k e_k$.

Let $\mathscr{B}$ be any Borel subset in $\mathbb{R}^n$. For any orthonormal system $y_1, y_2, \cdots, y_n$ in $L_2(\mathbb{T})$, the cylindrical subsets in $W_2^r(\mathbb{T})$ is denoted by

$$S = \{x \in W_2^r(\mathbb{T}) \mid (\langle x, y_1^{(-r)}\rangle_1, \cdots, \langle x, y_n^{(-r)}\rangle_1) \in \mathscr{B}\}.$$

Let $\sigma_j = \langle C_\mu y_j, y_j\rangle,\ j = 1, \cdots, n$, the Gaussian measure μ on S is defined by

$$\mu(S) := \prod_{j=1}^n (2\pi\sigma_j)^{-\frac{1}{2}}\int_{\mathscr{B}}\exp\left(-\sum_{j=1}^n\frac{|u_j|^2}{2\sigma_j}\right)\mathrm{d}u_1\cdots\mathrm{d}u_n.$$

The multivariate Sobolev space with mixed derivative is denoted by

$$MW_2^r(\mathbb{T}^d) = \{x \in L_2(\mathbb{T}^d) \mid x^{(r)} \in L_2(\mathbb{T}^d), \int_0^{2\pi} x(t)\mathrm{d}t_j = 0, j = 1, \cdots, d\},$$

where $r = (r_1, \cdots, r_d) \in \mathbb{R}_+^d$. The integral condition in the set means if $k_1 k_2 \cdots k_d = 0$, $c_k = 0$. The norm defined on $MW_2^r(\mathbb{T}^d)$ is $\|x\|_{MW_2^r}^2 := \langle x^{(r)}, x^{(r)}\rangle$. Therefore, $MW_2^r(\mathbb{T}^d)$ is a Hilbert space with inner product $\langle x, y\rangle_d := \langle x^{(r)}, y^{(r)}\rangle$.

Let $\mathbb{Z}_0^d = \{k = (k_1, \cdots, k_d) \in \mathbb{Z}^d \mid k_i \neq 0, i = 1, \cdots, d\}$ and $\rho > 1$. Equip $MW_2^r(\mathbb{T}^d)$ with a Gaussian measure μ. For the convenience of writing, we continue to use μ. However, μ here is not the same as in the univariate Sobolev space. The mean of μ is 0, and the correlation operator C_μ has eigenfunctions $e_k = \exp(i(k, \cdot))$ and eigenvalues $\lambda_k = |k|^{-\rho}$, $k \in \mathbb{Z}_0^d$. Then, $C_\mu e_k = \lambda_k e_k$.

For any orthonormal system $s_1, s_2, \cdots, s_n$ in $L_2(\mathbb{T}^d)$, the cylindrical subsets in $MW_2^r(\mathbb{T}^d)$ is denoted by

$$MS = \{x \in MW_2^r(\mathbb{T}^d) \mid (\langle x, s_1^{(-r)}\rangle_d, \cdots, \langle x, s_n^{(-r)}\rangle_d) \in \mathscr{B}\}.$$

Let $\sigma_j = \langle C_\mu s_j, s_j\rangle,\ s = 1, \cdots, n$, the Gaussian measure μ on MS is defined by

$$\mu(MS) := \prod_{j=1}^{n} (2\pi\sigma_j)^{-\frac{1}{2}} \int_{\mathscr{B}} \exp\left(-\sum_{j=1}^{n} \frac{|u_j|^2}{2\sigma_j}\right) \mathrm{d}u_1 \cdots \mathrm{d}u_n.$$

Let $l_q, 1 \leq q < \infty$ be the classical sequence space and l_∞ be the normed linear space consisting of all bounded sequences endowed with the norm:

$$\|x\|_{l_\infty} = \sup_{j \in \mathbb{Z}} |x_j|.$$

The definition of $S_q(\mathbb{T}^d), 1 \leq q \leq \infty$ is given by

$$S_q(\mathbb{T}^d) := \{x \in L_1(\mathbb{T}^d) \mid \{\hat{x}(k)\}_{k \in \mathbb{Z}^d} \in l_q, \|x\|_{S_q} := \|\{\hat{x}(k)\}\|_{l_q}\}.$$

When $d = 1$, we get $S_q(\mathbb{T}), 1 \leq q \leq \infty$. According to Parseval equalities, if $2 < q \leq \infty$, $S_q(\mathbb{T}^d) \supset L_q(\mathbb{T}^d)$. Therefore, according to the Sobolev embedding theorem, when $2 < q \leq \infty$ and $r > \max\{0, \frac{1}{2} - \frac{1}{q}\}$, $MW_2^r(\mathbb{T}^d)$ can be continuously embedded into the space $S_q(\mathbb{T}^d)$. Here, $r > \max\{0, \frac{1}{2} - \frac{1}{q}\}$ means $r = (r_1, \cdots, r_d)$ satisfies $r_i > \max\{0, \frac{1}{2} - \frac{1}{q}\}, i = 1, \cdots, d$. In the same way, when $r > \max\{0, \frac{1}{2} - \frac{1}{q}\}$, $W_2^r(\mathbb{T})$ can be continuously embedded into the space $S_q(\mathbb{T})$. In this paper, we only need to consider $q = \infty$ and we always assume $r > \frac{1}{2}$.

Xu [6] has studied the probabilistic linear width of $W_2^r(\mathbb{T})$ in $S_q(\mathbb{T})$ and Wang [17] has studied the probabilistic linear width of $MW_2^r(\mathbb{T}^d)$ in $S_q(\mathbb{T}^d)$. From the relation connecting probabilistic Gel'fand width width and probabilistic linear width, the following lemmas will be instrumental in determining the probabilistic Gel'fand widths.

Lemma 4.6 ([6]). *Let $1 \leq q \leq \infty$, $\rho > 1$, $r > \frac{1}{2}$, and $\delta \in \left(0, \frac{1}{2}\right]$. Then, the probabilistic linear (n, δ)-widths of $W_2^r(\mathbb{T})$ with Gaussian measure μ in the metric of $S_q(\mathbb{T})$ satisfy the following asymptotic relation:*

$$\lambda_{n,\delta}(W_2^r(\mathbb{T}), \mu, S_q(\mathbb{T})) \asymp \begin{cases} n^{-(r+\frac{\rho}{2}-\frac{1}{q})}\sqrt{1 + \frac{1}{n}\ln\frac{1}{\delta}}, & 1 \leq q \leq 2, \\ n^{-(r+\frac{\rho}{2}-\frac{1}{q})}\left(1 + n^{-\frac{1}{q}}\sqrt{\ln\frac{1}{\delta}}\right), & 2 < q < \infty, \\ n^{-(r+\frac{\rho}{2})}\sqrt{\ln\frac{n}{\delta}}, & q = \infty. \end{cases}$$

Lemma 4.7 ([17]). *Let $1 \le q \le \infty$, $\rho > 1$, $r = (r_1, \cdots, r_d) \in \mathbb{R}^d$, $\frac{1}{2} < r_1 = \cdots = r_v \le r_{v+1} \le \cdots \le r_d$, and $\delta \in \left(0, \frac{1}{2}\right]$. Then, the probabilistic linear (n, δ)-widths of $MW_2^r(\mathbb{T}^d)$ with Gaussian measure μ in the metric of $S_q(\mathbb{T}^d)$ satisfy the following asymptotic relation:*

(1) $1 \le q \le 2$,

$$\lambda_{n,\delta}(MW_2^r(\mathbb{T}^d), \mu, S_q(\mathbb{T}^d)) \asymp (n^{-1} \ln^{v-1} n)^{r_1 + \frac{\rho}{2} - \frac{1}{q}} (\ln^{\frac{v-1}{q}} n) \sqrt{1 + \frac{1}{n} \ln \frac{1}{\delta}}.$$

(2) $2 < q < \infty$,

$$\lambda_{n,\delta}(MW_2^r(\mathbb{T}^d), \mu, S_q(\mathbb{T}^d)) \asymp (n^{-1} \ln^{v-1} n)^{r_1 + \frac{\rho}{2} - \frac{1}{q}} (\ln^{\frac{v-1}{q}} n) \left(1 + n^{-\frac{1}{q}} \sqrt{\ln \frac{1}{\delta}}\right).$$

(3) $q = \infty$,

$$\lambda_{n,\delta}(MW_2^r(\mathbb{T}^d), \mu, S_q(\mathbb{T}^d)) \asymp (n^{-1} \ln^{v-1} n)^{r_1 + \frac{\rho}{2}} \sqrt{\ln \frac{n}{\delta}}.$$

By using the previously established relationship and the aforementioned lemmas, the probabilistic Gel'fand width can be straightforwardly derived. The central outcomes of this work are formulated in the following manner:

Theorem 4.3. *Let $\rho > 1$, $r > \frac{1}{2}$, and $\delta \in \left(0, \frac{1}{2}\right]$, $0 < p < \infty$. Then, as $n \to \infty$, the Gel'fand widths of the univariate Sobolev space $W_2^r(\mathbb{T})$ equipped with Gaussian measure μ in $S_\infty(\mathbb{T})$ satisfy the following asymptotic equalities:*

(1) Probabilistic Gel'fand width

$$d_\delta^n(W_2^r(\mathbb{T}), \mu, S_\infty(\mathbb{T})) \asymp n^{-(r + \frac{\rho}{2})} \sqrt{\ln \frac{n}{\delta}}.$$

(2) Average Gel'fand width

$$d_{(a)}^n(W_2^r(\mathbb{T}), \mu, S_\infty(\mathbb{T}))_p \asymp n^{-(r + \frac{\rho}{2})} \sqrt{\ln n}.$$

Theorem 4.4. *Let $\rho > 1$, $r = (r_1, \cdots, r_d) \in \mathbb{R}^d$, $\frac{1}{2} < r_1 = \cdots = r_v \le r_{v+1} \le \cdots \le r_d$, and $\delta \in \left(0, \frac{1}{2}\right]$, $0 < p < \infty$. Then, as $n \to \infty$, the Gel'fand widths of the multivariate Sobolev space $MW_2^r(\mathbb{T}^d)$ equipped with Gaussian measure μ in $S_\infty(\mathbb{T}^d)$ satisfy the following asymptotic equalities:*

(1) Probabilistic Gel'fand width

$$d_\delta^n(MW_2^r(\mathbb{T}^d), \mu, S_\infty(\mathbb{T}^d)) \asymp (n^{-1} \ln^{v-1} n)^{r_1 + \frac{\rho}{2}} \sqrt{\ln \frac{n}{\delta}}.$$

(2) Average Gel'fand width

$$d_{(a)}^n(MW_2^r(\mathbb{T}^d), \mu, S_\infty(\mathbb{T}^d))_p \asymp (n^{-1} \ln^{v-1} n)^{r_1 + \frac{\rho}{2}} \sqrt{\ln n}.$$

Theorem 4.3 and Theorem 4.4 will be proved in the next subsection.

4.2. **Proof of Theorems 4.3 and 4.4.** In this part, since the proof method of the Gel'fand widths of the univariate Sobolev space $W_2^r(\mathbb{T})$ is almost exactly the same as that of the multivariate Sobolev space $MW_2^r(\mathbb{T}^d)$, we mainly discuss the Gel'fand widths of $MW_2^r(\mathbb{T}^d)$ in $S_\infty(\mathbb{T}^d)$. For the convenience of writing, we will use d_δ^n instead of $d_\delta^n(MW_2^r(\mathbb{T}^d), \mu, S_\infty(\mathbb{T}^d))$ and $d_{(a)}^n$ instead of $d_{(a)}^n(MW_2^r(\mathbb{T}^d), \mu, S_\infty(\mathbb{T}^d))_p$ in this part.

Proof. According to the relationship given in Theorem 3.2, and Lemma 4.7, we obtain the probabilistic Gel'fand width of $MW_2^r(\mathbb{T}^d)$:

$$d_\delta^n = \lambda_{n,\delta}(MW_2^r(\mathbb{T}^d), \mu, S_\infty(\mathbb{T}^d)) \asymp (n^{-1}\ln^{v-1} n)^{r_1+\frac{\varrho}{2}}\sqrt{\ln\frac{n}{\delta}}. \tag{4.2.1}$$

We will calculate the asymptotic order from two aspects. First, we give the upper bound of the average Gel'fand width.

According to the Definition 2.7 of probabilistic Gel'fand width, there exists a linear subspace F^n of $S_\infty(\mathbb{T}^d)$ with co-dimension at most n, such that for any $\delta \in \left(0, \frac{1}{2}\right]$ and some subset $G_\delta \subset MW_2^r(\mathbb{T}^d)$, $\mu(G_\delta) \le \delta$, we have

$$\sup_{x\in(MW_2^r(\mathbb{T}^d)\setminus G_\delta)\cap F^n} \|x\|_{S_\infty} \le d_\delta^n(MW_2^r(\mathbb{T}^d), \mu, S_\infty(\mathbb{T}^d)) + \epsilon,\ \text{for any } \epsilon > 0.$$

Due to the arbitrariness of ϵ, we have

$$\sup_{x\in(MW_2^r(\mathbb{T}^d)\setminus G_\delta)\cap F^n} \|x\|_{S_\infty} \ll d_\delta^n(MW_2^r(\mathbb{T}^d), \mu, S_\infty(\mathbb{T}^d)). \tag{4.2.2}$$

Let the set sequence $\{G_{2^{-k}}\}_{k=0}^\infty$ satisfy the condition that for any k, $\mu(G_{2^{-k}}) \le 2^{-k}$ and $G_1 = MW_2^r(\mathbb{T}^d)$. Therefore, $MW_2^r(\mathbb{T}^d) = \bigcup_{k=0}^\infty (G_2^{-k} \setminus G_2^{-k-1})$. We have

$$\begin{aligned}
\left(d_{(a)}^N\right)^p &\le \int_{MW_2^r(\mathbb{T}^d)\cap F^n} \|x\|_{S_\infty}^p \mathrm{d}\mu \\
&= \sum_{k=0}^\infty \int_{(G_{2^{-k}}\setminus G_{2^{-k-1}})\cap F^n} \|x\|_{S_\infty}^p \mathrm{d}\mu \\
&\le \sum_{k=0}^\infty \int_{(G_{2^{-k}}\setminus G_{2^{-k-1}})\cap F^n} \sup_{x\in(G_{2^{-k}}\setminus G_{2^{-k-1}})\cap F^n} \|x\|_{S_\infty}^p \mathrm{d}\mu \\
&\le \sum_{k=0}^\infty \int_{(G_{2^{-k}}\setminus G_{2^{-k-1}})\cap F^n} \sup_{x\in(W_2^r(\mathbb{T})\setminus G_{2^{-k-1}})\cap F^n} \|x\|_{S_\infty}^p \mathrm{d}\mu.
\end{aligned}$$

Then, according to the inequality (4.2.2), we have

$$\begin{aligned}
\left(d_{(a)}^N\right)^p &\ll \sum_{k=0}^\infty \int_{(G_{2^{-k}}\setminus G_{2^{-k-1}})\cap F^n} d_{2^{-k-1}}^n \left(MW_2^r(\mathbb{T}^d), \mu, S_\infty(\mathbb{T}^d)\right)^p \mathrm{d}\mu. \\
&\ll \sum_{k=0}^\infty d_{2^{-k-1}}^n \left(MW_2^r(\mathbb{T}^d), \mu, S_\infty(\mathbb{T}^d)\right)^p \mu(G_{2^{-k}}).
\end{aligned}$$

Meanwhile, according to equation (4.2.1), we have

$$d_\delta^n \ll (n^{-1}\ln^{v-1} n)^{r_1+\frac{\varrho}{2}}\sqrt{\ln\frac{n}{\delta}}.$$

Therefore,

$$\begin{aligned}\left(d_{(a)}^{N}\right)^{p} &\ll \sum_{k=0}^{\infty}\left((n^{-1}\ln^{v-1}n)^{r_1+\frac{\varrho}{2}}\sqrt{\ln\frac{n}{2^{-k-1}}}\right)^{p}\mu(G_{2^{-k}})\\ &\ll (n^{-1}\ln^{v-1}n)^{(r_1+\frac{\varrho}{2})p}\sum_{k=0}^{\infty}\left(\sqrt{\ln n+(k+1)\ln 2}\right)^{p}2^{-k}\\ &\ll (n^{-1}\ln^{v-1}n)^{(r_1+\frac{\varrho}{2})p}\sum_{k=0}^{\infty}\left(\left(\sqrt{\ln n}\right)^{p}+\left(\sqrt{(k+1)\ln 2}\right)^{p}\right)2^{-k}.\end{aligned}$$

Since $\sum_{k=0}^{\infty}2^{-k}(\sqrt{\ln n})^{p}$ and $\sum_{k=0}^{\infty}2^{-k}(\sqrt{(k+1)\ln 2})^{p}$ are convergent, we have

$$\left(d_{(a)}^{n}(MW_2^{r}(\mathbb{T}^d),\mu,S_\infty(\mathbb{T}^d)\right)_p)^{p}\ll (n^{-1}\ln^{v-1}n)^{(r_1+\frac{\varrho}{2})p}(\sqrt{\ln n})^{p}.$$

Namely,

$$d_{(a)}^{n}(MW_2^{r}(\mathbb{T}^d),\mu,S_\infty(\mathbb{T}^d))_p\ll (n^{-1}\ln^{v-1}n)^{r_1+\frac{\varrho}{2}}\sqrt{\ln n}.$$

Next, we give the lower bound of the average Gel'fand width. According to equation (4.2.1), there exists a constant $c>0$, then

$$d_{\frac{1}{2}}^{n}(MW_2^{r}(\mathbb{T}^d),\mu,S_\infty(\mathbb{T}^d))>c(n^{-1}\ln^{v-1}n)^{r_1+\frac{\varrho}{2}}\sqrt{\ln 2n}. \tag{4.2.3}$$

Consider the set

$$G=\left\{x\in MW_2^{r}(\mathbb{T}^d)\cap F^n \mid \|x\|_{S_\infty}>c(n^{-1}\ln^{v-1}n)^{r_1+\frac{\varrho}{2}}\sqrt{\ln 2n}\right\},$$

We claim $\mu(G)>\frac{1}{2}$. According to the definition of the Gel'fand width in the probabilistic setting, we obtain

$$\begin{aligned}d_{\frac{1}{2}}^{n}(MW_2^{r}(\mathbb{T}^d),\mu,S_\infty(\mathbb{T}^d)) &\le \sup_{x\in(MW_2^{r}(\mathbb{T}^d)\setminus G)\cap F^n}\|x\|_{S_\infty}\\ &\le c(n^{-1}\ln^{v-1}n)^{r_1+\frac{\varrho}{2}}\sqrt{\ln 2n}.\end{aligned}$$

It contradicts the inequality (4.2.3). Then,

$$\begin{aligned}\int_{MW_2^{r}(\mathbb{T}^d)\cap F^n}\|x\|_{S_\infty}^{p}\,\mathrm{d}\mu &\gg \int_G \|x\|_{S_\infty}^{p}\,\mathrm{d}\mu\\ &\gg \left((n^{-1}\ln^{v-1}n)^{r_1+\frac{\varrho}{2}}\sqrt{\ln 2n}\right)^{p}\mu(G)\\ &\gg \left((n^{-1}\ln^{v-1}n)^{r_1+\frac{\varrho}{2}}\sqrt{\ln 2n}\right)^{p}.\end{aligned}$$

Namely,

$$d_{(a)}^{n}(MW_2^{r}(\mathbb{T}^d),\mu,S_\infty(\mathbb{T}^d))_p\gg (n^{-1}\ln^{v-1}n)^{r_1+\frac{\varrho}{2}}\sqrt{\ln n}.$$

So, we obtain

$$d_{(a)}^{n}(MW_2^{r}(\mathbb{T}^d),\mu,S_\infty(\mathbb{T}^d))_p\asymp (n^{-1}\ln^{v-1}n)^{r_1+\frac{\varrho}{2}}\sqrt{\ln n}.$$

Similarly, we have

$$d_{\delta}^{n}(W_2^{r}(\mathbb{T}),\mu,S_\infty(\mathbb{T}))\asymp n^{-(r+\frac{\varrho}{2})}\sqrt{\ln\frac{n}{\delta}}.$$

and

$$d_{(a)}^{n}(W_2^{r}(\mathbb{T}),\mu,S_\infty(\mathbb{T}))_p\asymp n^{-(r+\frac{\varrho}{2})}\sqrt{\ln n}.$$

□

5. Summary

In this article, we mainly studied the generalization of the definition of the probabilistic Gel'fand width. By using the Birkhoff orthogonality, we generalize it from the Hilbert space to the strictly convex reflexive space. Then, we improve many details of previous works and then give the equality relation between the probabilistic Gel'fand width and the probabilistic linear width of the Hilbert space. In the end, we use the relation to obtain the Gel'fand widths of the univariate Sobolev space and the multivariate Sobolev space in the S_∞-norm in the probabilistic settings and calculate their exact order of average Gel'fand widths. As we all know, with the development of computing science and internal intersections within mathematics, the complexity of algorithms and properties of functional spaces are more and more important. Our research in the width theory is an effective way to clearly articulate the essence of the problem.

However, our research still has some shortcomings. For example, since the metric projection operator does not possess the linear property, we still don't know whether the relationship between probabilistic Gel'fand width and probabilistic linear width can be extended to the strictly convex reflexive space. This is a question that we still need to study in the future.

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