COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASS OF MEROMORPHIC AND BI-UNIVALENT FUNCTIONS

SAIDEH HAJIPARVANEH AND AHMAD ZIREH

ABSTRACT. In this paper, we introduce and investigate an interesting subclass of meromorphic and bi-univalent functions on $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. Furthermore, for functions belonging to this class, estimates on the initial coefficients are obtained. The results presented in this paper would generalize and improve some recent works of several earlier authors.

1. INTRODUCTION

Let Σ be the family of meromorphic functions f of the form

$$f(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}, \tag{1.1}$$

that are univalent in the domain $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$.

Since $f \in \Sigma$ is univalent, it has an inverse f^{-1} that satisfies

$$f^{-1}(f(z)) = z \ (z \in \Delta),$$

and

$$f(f^{-1})(w) = w \ (M < |w| < \infty, M > 0).$$

Furthermore, the inverse function f^{-1} has a series expansion of the form

$$f^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n},$$

where $M < |w| < \infty$.

A simple calculation shows that

$$g(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2}{w^3} + \dots . \tag{1.2}$$

A function $f \in \Sigma$ is said to be meromorphic bi-univalent if both f and f^{-1} are meromorphic univalent in Δ. We denote by Σ_M the class of all meromorphic bi-univalent functions in Δ given by (1.1).

The coefficient problem was investigated for various interesting subclasses of the meromorphic univalent functions. Estimates on the coefficients of meromorphic univalent functions were widely investigated in the literature; for example, Schiffer [5] obtained the estimate $|b_2| \leqslant \frac{2}{3}$ for meromorphic univalent functions $f \in \Sigma$ with $|b_0| = 0$ and Duren [1] proved that $|b_n| \leqslant \frac{2}{(n+1)}$ for $f \in \Sigma$ with $b_k = 0, 1 \leqslant k \leqslant \frac{n}{2}$.

For the coefficients of inverses of meromorphic univalent functions, Springer [7] proved that

$$|B_3| \leq 1 \ and \ |B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2}$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!} \ (n = 1, 2, ...).$$

In 1977, Kubota [3] proved that the Springer conjecture is true for $n = 3, 4, 5$ and subsequently Schober [6] obtained a sharp bounds for the coefficients B_{2n-1}, $1 \leq n \leq 7$.
Recently Orhan and Magash [4] introduced the following two subclasses of the meromorphic bi-univalent function class Σ and found estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in the each of these subclasses.

Definition 1.1. (see [4]) Let $0 < \alpha \leq 1$, $\lambda \geq 1$, $\mu \geq 0$, $\lambda > \mu$, a function $f(z)$ given by (1.1) is said to be in the class $\tilde{\Sigma}_M^\star(\alpha, \mu, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma_M \ and \ \left| arg \left[(1-\lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right] \right| < \frac{\alpha\pi}{2} \ (z \in \Delta),$$

and

$$\left| arg \left[(1-\lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} \right] \right| < \frac{\alpha\pi}{2} \ (w \in \Delta),$$

where g is given by (1.2).

Theorem 1.2. (see [4]) Let $f(z)$ given by (1.1) be in the class $\tilde{\Sigma}_M^\star(\alpha, \mu, \lambda)$. Then

$$|b_0| \leq \frac{2\alpha}{\lambda - \mu},$$

and

$$|b_1| \leq 2\alpha^2 \sqrt{\frac{1}{(2\lambda - \mu)^2} + \frac{(1-\mu)^2}{(\lambda - \mu)^4}}.$$

Definition 1.3. (see [4]) Let $0 \leq \beta < 1$, $\lambda \geq 1$, $\mu \geq 0$, $\lambda > \mu$, a function $f(z)$ given by (1.1) is said to be in the class $\Sigma_M^\star(\beta, \mu, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma_M \text{ and } \mathfrak{Re}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right] > \beta \ (z \in \Delta),$$

and

$$\mathfrak{Re}\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^\mu + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right] > \beta \ (w \in \Delta),$$

where the function g is given by (1.2).

Theorem 1.4. (see [4]) Let $f(z)$ given by (1.1) be in the class $\Sigma_M^\star(\beta, \mu, \lambda)$. Then

$$|b_0| \leq \frac{2(1-\beta)}{\lambda - \mu},$$

and

$$|b_1| \leq 2(1-\beta)\sqrt{\frac{1}{(2\lambda-\mu)^2} + \frac{(1-\mu)^2(1-\beta)^2}{(\lambda-\mu)^4}}.$$

The purpose of this paper is to investigate the meromorphic bi-univalent function class $\Sigma_M^{h,p}(\mu, \lambda)$ introduced in Definition 2.1 and estimates for the coefficients $|b_0|$ and $|b_1|$ of functions in the newly introduced subclass are obtained. Our results for the meromorphic bi-univalent function class $f \in \Sigma_M^{h,p}(\mu, \lambda)$ would generalize and improve some recent results obtained in [2, 4].

2. THE SUBCLASS $\Sigma_M^{h,p}(\mu, \lambda)$

In this section, we introduce and investigate the general subclass $\Sigma_M^{h,p}(\mu, \lambda)$.

Definition 2.1. Let the functions $h, p : \Delta \to \mathbb{C}$ be analytic functions and

$$h(z) = 1 + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots, \quad p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^2} + \cdots,$$

such that

$$\min\{Re\,(h(z))\,, Re\,(p(z))\} > 0 \ , \ z \in \Delta.$$

A function f given by (1.1) is said to be in the class $\Sigma_M^{h,p}(\mu, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma_M \text{ and } (1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \in h(\Delta), \qquad (2.1)$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^\mu + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \in p(\Delta), \qquad (2.2)$$

where $\lambda \geqslant 1, \lambda > \mu, \mu \geq 0$ and the function g is defined by (1.2).

Remark 2.2. There are many choices of h and p which would provide interesting subclasses of class $\Sigma_M^{h,p}(\mu, \lambda)$.

(1) If we take

$$h(z) = p(z) = \left(\frac{1 + \frac{1}{z}}{1 - \frac{1}{z}} \right)^\alpha = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \ldots \ \ (0 < \alpha \le 1, z \in \Delta),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f \in \Sigma_M^{h,p}(\mu, \lambda)$, then

$$f \in \Sigma_M \ and \ \left| arg \left[(1 - \lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right] \right| < \frac{\alpha\pi}{2} \ (z \in \Delta),$$

and

$$\left| arg \left[(1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} \right] \right| < \frac{\alpha\pi}{2} \ (w \in \Delta).$$

Therefore in this case, the class $\Sigma_M^{h,p}(\mu, \lambda)$ reduce to class $\tilde{\Sigma}_M^\star(\alpha, \mu, \lambda)$ in Definition 1.1.

(2) If we take

$$h(z) = p(z) = \frac{1 + \frac{1-2\beta}{z}}{1 - \frac{1}{z}} = 1 + \frac{2(1-\beta)}{z} + \frac{2(1-\beta)}{z^2} + \ldots \ \ (0 \le \beta < 1, z \in \Delta),$$

then the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f \in \Sigma_M^{h,p}(\mu, \lambda)$, then

$$f \in \Sigma_M \ and \ \mathfrak{Re} \left[(1 - \lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right] > \beta \ (z \in \Delta),$$

and

$$\mathfrak{Re} \left[(1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} \right] > \beta \ (w \in \Delta).$$

Therefore in this case, the class $\Sigma_M^{h,p}(\mu, \lambda)$ reduce to class $\Sigma_M^\star(\beta, \mu, \lambda)$ in Definition 1.3.

2.1. **Coefficient Estimates.** Now, we obtain the estimates on the coefficients $|b_0|$ and $|b_1|$ for subclass $\Sigma_M^{h,p}(\mu, \lambda)$ which generalize and improve several well-known conventional results that recently published.

Theorem 2.3. *Let $f(z)$ given by (1.1) be in the class $\Sigma_M^{h,p}(\mu, \lambda)$. Then*

$$|b_0| \le \min \left\{ \sqrt{\frac{|h_1|^2 + |p_1|^2}{2(\lambda - \mu)^2}}, \sqrt{\frac{|h_2| + |p_2|}{(2\lambda - \mu)|1 - \mu|}} \right\}, \tag{2.3}$$

and

$$|b_1| \le \min\left\{\frac{|h_2| + |p_2|}{2(2\lambda - \mu)}, \sqrt{\frac{|h_2|^2 + |p_2|^2}{2(2\lambda - \mu)^2} + \frac{(1 - \mu)^2}{16(\lambda - \mu)^4}(|h_1|^2 + |p_1|^2)^2}\right\}. \quad (2.4)$$

Proof. From definition (2.1) and (2.2) we have

$$(1 - \lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = h(z) \ (z \in \Delta), \quad (2.5)$$

and

$$(1 - \lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = p(w) \ (w \in \Delta), \quad (2.6)$$

where functions h and p satisfy the conditions of Definition 2.1. Also, the functions h and p have the following forms:

$$h(z) = 1 + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots, \quad (2.7)$$

and

$$p(w) = 1 + \frac{p_1}{w} + \frac{p_2}{w^2} + \frac{p_3}{w^3} + \cdots. \quad (2.8)$$

Now, upon substituting from (2.7) and (2.8) into (2.5) and (2.6), respectively, and equating the coefficients, we get

$$(\mu - \lambda)b_0 = h_1, \quad (2.9)$$

$$(\mu - 2\lambda)(b_1 + (\mu - 1)\frac{b_0^2}{2}) = h_2, \quad (2.10)$$

$$-(\mu - \lambda)b_0 = p_1, \quad (2.11)$$

and

$$(2\lambda - \mu)(b_1 - (\mu - 1)\frac{b_0^2}{2}) = p_2. \quad (2.12)$$

From (2.9) and (2.11), we get

$$h_1 = -p_1, \quad (2.13)$$

and

$$2(\lambda - \mu)^2 b_0^2 = h_1^2 + p_1^2. \quad (2.14)$$

Adding (2.10) and (2.12), we get

$$(\mu - 1)(\mu - 2\lambda)b_0^2 = h_2 + p_2. \quad (2.15)$$

Therefore, from (2.14) and (2.15), we have

$$b_0^2 = \frac{h_1^2 + p_1^2}{2(\lambda - \mu)^2}, \quad (2.16)$$

and

$$b_0^2 = \frac{h_2 + p_2}{(\mu - 1)(\mu - 2\lambda)}, \tag{2.17}$$

respectively. Therefore, we find from the equations (2.16) and (2.17), that

$$|b_0|^2 \leq \frac{|h_1|^2 + |p_1|^2}{2(\lambda - \mu)^2},$$

and

$$|b_0|^2 \leq \frac{|h_2| + |p_2|}{|1 - \mu|(2\lambda - \mu)},$$

respectively. So we get the desired estimate on the coefficient $|b_0|$ as asserted in (2.3).

Next, in order to find the bound on the coefficient b_1, by subtracting (2.12) from (2.10), we get

$$2(\mu - 2\lambda)b_1 = h_2 - p_2. \tag{2.18}$$

By squaring and adding (2.10) and (2.12), using (2.14) in the computation leads to

$$b_1^2 = \frac{h_2^2 + p_2^2}{2(2\lambda - \mu)^2} - \frac{(1 - \mu)^2}{16(\lambda - \mu)^4}(h_1^2 + p_1^2)^2. \tag{2.19}$$

Therefore, we find from the equations (2.18) and (2.19) that

$$|b_1| \leq \frac{|h_2| + |p_2|}{2(2\lambda - \mu)}, \tag{2.20}$$

and

$$|b_1| \leqslant \sqrt{\frac{|h_2|^2 + |p_2|^2}{2(2\lambda - \mu)^2} + \frac{(1 - \mu)^2}{16(\lambda - \mu)^4}(|h_1|^2 + |p_1|^2)^2}. \tag{2.21}$$

So we obtain from (2.20) and (2.21) the desired estimate on the coefficient $|b_1|$ as asserted in (2.4). This completes the proof. $\square$

3. Conclusions

If we take

$$h(z) = p(z) = \left(\frac{1 + \frac{1}{z}}{1 - \frac{1}{z}} \right)^\alpha = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \dots \quad (0 < \alpha \leq 1, \ z \in \Delta),$$

in Theorem 2.3, we conclude the following result.

Corollary 3.1. *Let the function $f(z)$ given by (1.1) be in the class $\tilde{\Sigma}_M^\star(\alpha, \mu, \lambda)$. Then*

$$|b_0| \leq \min \left\{ \frac{2\alpha}{\lambda - \mu}, \frac{2\alpha}{\sqrt{|1 - \mu|(2\lambda - \mu)}} \right\},$$

and

$$|b_1| \leq \min \left\{ \frac{2\alpha^2}{2\lambda - \mu}, 2\alpha^2 \sqrt{\frac{1}{(2\lambda - \mu)^2} + \frac{(1-\mu)^2}{(\lambda - \mu)^4}} \right\}.$$

Remark 3.2. Corollary 3.1 is an improvement of estimates obtained by Orhan in Theorem 1.2. Because it is easy to see, for the coefficient $|b_0|$, if $0 \leq \mu < 1, 1 \leq \lambda \leq 1 + \sqrt{1 - \mu}$, then

$$\frac{2\alpha}{\sqrt{(2\lambda - \mu)(1 - \mu)}} \leqslant \frac{2\alpha}{\lambda - \mu},$$

and if $\mu > 1, \mu < \lambda \leqslant (2\mu - 1) + \sqrt{2\mu^2 - 3\mu + 1}$, then

$$\frac{2\alpha}{\sqrt{(2\lambda - \mu)(\mu - 1)}} \leqslant \frac{2\alpha}{\lambda - \mu}.$$

Also for the coefficient $|b_1|$, we have

$$\frac{2\alpha^2}{(2\lambda - \mu)} \leqslant 2\alpha^2 \sqrt{\frac{1}{(2\lambda - \mu)^2} + \frac{(1-\mu)^2}{(\lambda - \mu)^4}}.$$

If we take $\mu = 0, \lambda = 1$ in Corollary 3.1, we obtain the following result which is an improvement of estimates obtained by Halim et. al. [2, Theorem 2].

Corollary 3.3. *Let the function $f(z)$ given by (1.1) be in the class $\tilde{\Sigma}_M^*(\alpha)$. Then*

$$|b_0| \leqslant \sqrt{2}\alpha,$$

and

$$|b_1| \leqslant \alpha^2.$$

By setting

$$h(z) = p(z) = \frac{1 + \frac{1-2\beta}{z}}{1 - \frac{1}{z}} = 1 + \frac{2(1-\beta)}{z} + \frac{2(1-\beta)}{z^2} + \dots \quad (0 \leq \beta < 1, \ z \in \Delta),$$

in Theorem 2.3, we deduce the following result.

Corollary 3.4. *Let the function $f(z)$ given by (1.1) be in the class $\Sigma_M^{\star}(\beta, \mu, \lambda)$. Then*

$$|b_0| \leq \min \left\{ \frac{2(1-\beta)}{(\lambda - \mu)}, 2\sqrt{\frac{(1-\beta)}{|1-\mu|(2\lambda - \mu)}} \right\},$$

and

$$|b_1| \leq \min \left\{ \frac{2(1-\beta)}{2\lambda - \mu}, 2(1-\beta)\sqrt{\frac{1}{(2\lambda - \mu)^2} + \frac{(1-\mu)^2(1-\beta)^2}{(\lambda - \mu)^4}} \right\}.$$

Remark 3.5. It is easy to see that

$$|b_0| \leq \begin{cases} 2\sqrt{\dfrac{1-\beta}{|1-\mu|(2\lambda-\mu)}}, & 0 \leqslant \beta < 1 - \dfrac{(\lambda-\mu)^2}{|1-\mu|(2\lambda-\mu)} \\[3mm] \dfrac{2(1-\beta)}{(\lambda-\mu)}, & 1 - \dfrac{(\lambda-\mu)^2}{|1-\mu|(2\lambda-\mu)} \leq \beta < 1 \end{cases}$$

and

$$\frac{2(1-\beta)}{2\lambda-\mu} \leq 2(1-\beta)\sqrt{\frac{1}{(2\lambda-\mu)^2} + \frac{(1-\mu)^2(1-\beta)^2}{(\lambda-\mu)^4}},$$

which, in conjunction with Corollary 3.4, would obviously yield an improvement of Theorem 1.4.

If we take $\mu = 0, \lambda = 1$ in Corollary 3.4, we obtain the following result which is an improvement of estimates obtained by Halim et. al. [2, Theorem 1]

Corollary 3.6. *Let the function $f(z)$ given by (1.1) be in the class $\Sigma_M^*(\beta)$. Then*

$$|b_0| \leq \begin{cases} \sqrt{2(1-\beta)}, & 0 \leqslant \beta < \frac{1}{2} \\[3mm] 2(1-\beta), & \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|b_1| \leqslant 1 - \beta.$$

References

[1] Duren, P. L., Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[2] Halim, S. A., Hamidi S. G. and Ravichandran, V., Coefficient estimates for mero- morphic bi-univalent functions, *arXiv:1108.4089v1* (2011), 1-9.

[3] Kubota, Y., Coefficients of meromorphic univalent functions, *Kodai Math. Sem. Rep.* **28**(2-3) (1976/77), 253-261.

[4] Orhan, H., Magesh N. and Balaji, V. K., Initial coefficient bounds for certain classes of meromorphic bi-univalent functions, *Asian European J. Math.* 7(1) (2014), 1-9.

[5] Schiffer, M., Surun probleme dextremum de la representation conforme, *Bull. Soc. Math. France* **66** (1938), 48-55.

[6] Schober, G., Coefficients of inverses of meromorphic univalent functions, *Proc. Amer. Math. Soc.* **67**(1) (1977), 111-116.

[7] Springer, G., The coefficient problem for Schlicht mappings of the exterior of the unit circle, *Trans. Amer. Math. Soc.* **70** (1951), 421-450.

Current address: Saideh Hajiparvaneh: Department of Mathematics, Shahrood University of Technology, P.O.Box 316-36155, Shahrood, Iran

E-mail address: `sa.parvaneh64@gmail.com`

ORCID Address: `http://orcid.org//0000-0002-3405-853X`

Current address: Ahmad Zireh (Corresponding author): Department of Mathematics, Shahrood University of Technology, P.O.Box 316-36155, Shahrood, Iran

E-mail address: `azireh@shahroodut.ac.ir`

ORCID Address: `http://orcid.org/0000-0003-0602-9122`