EIGENVALUES AND SCATTERING PROPERTIES OF DIFFERENCE OPERATORS WITH IMPULSIVE CONDITION

IBRAHIM ERDAL AND ŞEYHMUS YARDIMCI

ABSTRACT. In this work, we are concerned with difference operator of second order with impulsive condition. By the help of a transfer matrix M, we present scattering function of corresponding operator and examine the spectral properties of this impulsive problem.

1. INTRODUCTION

Let us shortly give an overview on the existing literature of spectral theory of Sturm–Liouville operators (SLO). Study of the spectral analysis of nonselfadjoint SLO with continuous and discrete spectrum was begun by Naimark [1]. In [1], the author proved that the spectrum of SLO consists of the continuous spectrum, the eigenvalues and the spectral singularities. The spectral singularities are poles of the kernel of the resolvent and are also embedded in the continuous spectrum, but they are not eigenvalues. Then Marchenko investigated SLO in $L_2[0,\infty)$ generated by

$$-y'' + q(x)y = \lambda^2 y, \quad 0 \le x < \infty, \tag{1}$$

with boundary condition

$$y(0) = 0, \tag{2}$$

where q is a real-valued function and λ is a spectral parameter [2]. He showed that Jost function of (1) defined by

$$e(\lambda) := 1 + \int_0^\infty K(0,t)e^{i\lambda t}dt, \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \ge 0\}$$

has a finite number of simple zeros in open half complex plain and he also defined scattering function of (1)-(2) by

$$\mathcal{S}(\lambda) := \frac{\overline{e(\lambda)}}{e(\lambda)}, \quad \lambda \in (-\infty, \infty).$$

On the other hand, difference equations have become an interesting subject in this field over the last century [3-5]. The modelling of certain problems from engineering, economics, control theory and other areas of study has led to the rapid development of the theory of the discrete equations. These developments gave rise to the study of such equations. In recent years, some problems of spectral analysis of non-selfadjoint difference operators with continuous and discrete spectrum have been investigated by some authors [6],[7]. In [8], it is proved by examples that nonselfadjoint difference operators of second order have spectral singularities. Also some problems of spectral analysis of difference and other types of operators with spectral singularities have been thoroughly studied in [9-12].

All of the studies mentioned above are of general boundary condition without discontinuities. The spectral theory of some operators with discontinuities, i.e, impulsive operators were studied in [13-15]. Also spectral properties of difference operators with impulsive condition especially scattering problem were investigated in [16]. This study is the general form of the [16], but the method used for determining the eigenvalues and spectral singularities is new and different from other methods which are found in literature.

In this work, we are concerned with difference operator on the semi axis with impulsive condition. We explore scattering theory of this problem and we give a detailed example at the end of the paper.

Consider the following difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N}, \tag{3}$$

with boundary condition

$$y_0 = 0, \tag{4}$$

where λ is a spectral parameter. Suppose that the real sequences $\{a_n\}_{n \in \mathbb{N} \cup \{0\}}$ and $\{b_n\}_{n \in \mathbb{N}}$ satisfy the condition

$$\sum_{n \in \mathbb{N}} n \left(|1 - a_n| + |b_n| \right) < \infty. \tag{5}$$

Under condition (5), equation (3) has the bounded solution satisfying the condition

$$\lim_{n \to \infty} e^{-inz} e_n(z) = 1,$$

for $\lambda = 2 \cos z$, where $z \in \overline{\mathbb{C}}_+$. $e_n(z)$ is called the Jost solution of (3). It is analytic with respect to z in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, continuous in $\overline{\mathbb{C}}_+$ and $e_n(z + 2\pi) = e_n(z)$ for all z in $\overline{\mathbb{C}}_+$. Also the function $e_n(z)$ has the representation

[17]

$$e_n(z) = \mu_n e^{inz} \left(1 + \sum_{m=1}^{\infty} A_{nm} e^{imz}\right), \quad n \in \mathbb{N}, \tag{6}$$

where μ_n and A_{nm} are expressed in terms of the sequences $\{a_n\}_{n \in \mathbb{N} \cup \{0\}}$ and $\{b_n\}_{n \in \mathbb{N}}$ as

$$\mu_n = \left\{\prod_{k=n}^{\infty} a_k\right\}^{-1},$$

$$A_{n1} = -\sum_{k=n+1}^{\infty} b_s,$$

$$A_{n2} = \sum_{k=n+1}^{\infty} \left\{\left(1 - a_k^2\right) + b_k \sum_{s=k+1}^{\infty} b_s\right\},$$

$$A_{n,m+2} = A_{n+1,m} + \sum_{k=n+1}^{\infty} \left\{\left(1 - a_k^2\right) A_{k+1,m} - b_k A_{k,m+1}\right\}, \quad m = 1, 2, ...; n \in \mathbb{N}.$$

Moreover A_{nm} satisfy

$$|A_{nm}| \leq c \sum_{k=n+\left[\frac{m}{2}\right]}^{\infty} \left(|1 - a_k| + |b_k|\right), \tag{7}$$

where $c > 0$ is a constant and $\left[\frac{m}{2}\right]$ is the integer part of $\frac{m}{2}$.

2. Scattering function of corresponding operator

Let L denote the difference operator of second order generated in $\ell^2(\mathbb{N})$ by the equation

$$a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = 2 \cos z y_n, \quad n \in \mathbb{N} \setminus \{k-1, k, k+1\} \tag{8}$$

with the boundary condition

$$y_0 = 0 \tag{9}$$

and the impulsive condition

$$\begin{pmatrix} y_{k+1} \\ \Delta y_{k+1} \end{pmatrix} = B \begin{pmatrix} y_{k-1} \\ \nabla y_{k-1} \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{10}$$

where $\alpha, \beta, \gamma, \delta$ are complex numbers, ∇ denotes the backward difference operator and Δ denotes the forward difference operator defined by $\nabla y_n := y_n - y_{n-1}$ and $\Delta y_n := y_{n+1} - y_n$, respectively. Assume that the real sequences $\{a_n\}_{n \in \mathbb{N} \cup \{0\}}$ and $\{b_n\}_{n \in \mathbb{N}}$ satisfy the condition (5). Throughout the paper, we assume that $a_n \neq 0$, for all $n \in \mathbb{N} \cup \{0\}$.

Furthermore, let us denote the solutions of equation (8) by y_n^- and y_n^+ respectively

$$\begin{cases} y_n^- := y_n(z), & n = 0, 1, 2, ..., k-1 \\ y_n^+ := y_n(z), & n = k+1, k+2, ... \ . \end{cases}$$

We shall define the semi-strip $S_0 := \left\{ z : z = \eta + i\xi, -\dfrac{\pi}{2} \leq \eta \leq \dfrac{3\pi}{2}, \xi > 0 \right\}$ and $S := S_0 \cup \left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2} \right].$

It is known that, $Q_n(z)$ and $P_n(z)$ are the fundamental solutions of (8) satisfying

$$Q_0(z) = \frac{1}{a_0}, \qquad Q_1(z) = 0$$

and

$$P_0(z) = 0, \qquad P_1(z) = 1$$

for $n = 0, 1, 2, ..., k-1$ [18]. Since the Wronskian of two solutions $y = \{y_n(z)\}$ and $u = \{u_n(z)\}$ of the difference equation (8) is defined by

$$W[y, u] := a_n[y_n(z)u_{n+1}(z) - y_{n+1}(z)u_n(z)],$$

we have

$$W[Q_n(z), P_n(z)] = 1,$$

for all $z \in \mathbb{C}$. It is clear that $Q_n(z)$ and $P_n(z)$ are entire functions of z.

On the other hand (8) admits another solution

$$e_n(-z) = \mu_n e^{-inz} \left(1 + \sum_{m=1}^{\infty} A_{nm} e^{-imz} \right), \quad n = k+1, k+2, ... \qquad (11)$$

fulfilling the asymptotic condition

$$\lim_{n \to \infty} e^{inz} e_n(-z) = 1,$$

where $z \in \overline{\mathbb{C}}_- := \{ z \in \mathbb{C} : \operatorname{Im} z \leq 0 \}$. Besides for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right] \setminus \{0, \pi\}$,

$$W[e_n(z), e_n(-z)] = -2i \sin z.$$

Now let $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right] \setminus \{0, \pi\}$. By the help of linearly independent solutions of (8), we can express the general solution of (8) by

$$\begin{cases} y_n^-(z) = A\_Q_n(z) + B_- P_n(z), & n = 0, 1, 2, ..., k-1 \\ y_n^+(z) = A_+ e_n(z) + B_+ e_n(-z), & n = k+1, k+2, ... \ , \end{cases}$$

where $A_\pm$ and $B_\pm$ are constant coefficients. By (6) and (11), we get $y_{k+1}(z)$, $\Delta y_{k+1}(z)$, $y_{k-1}(z)$ and $\nabla y_{k-1}(z)$. Next, from the impulsive condition (10), we obtain

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = M \begin{pmatrix} A_- \\ B_- \end{pmatrix}, \qquad (12)$$

where

$$M := \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = K^{-1} B T$$

such that

$$T := \begin{pmatrix} Q_{k-1}(z) & P_{k-1}(z) \\ \nabla Q_{k-1}(z) & \nabla P_{k-1}(z) \end{pmatrix}$$

and

$$K := \begin{pmatrix} e_{k+1}(z) & e_{k+1}(-z) \\ \Delta e_{k+1}(z) & \Delta e_{k+1}(-z) \end{pmatrix}.$$

Since $\det K = -\dfrac{2i \sin z}{a_{k+1}}$, it is easy to obtain

$$\begin{aligned} M_{22}(z) \;=\; & -\frac{a_{k+1}}{2i \sin z} \left\{ -\Delta e_{k+1}(z) \left[\alpha P_{k-1}(z) + \beta \nabla P_{k-1}(z) \right] \right. \\ & \left. + e_{k+1}(z) \left[\gamma P_{k-1}(z) + \delta \nabla P_{k-1}(z) \right] \right\}, \quad (13) \end{aligned}$$

$$\begin{aligned} M_{12}(z) \;=\; & -\frac{a_{k+1}}{2i \sin z} \left\{ \Delta e_{k+1}(-z) \left[\alpha P_{k-1}(z) + \beta \nabla P_{k-1}(z) \right] \right. \\ & \left. - e_{k+1}(-z) \left[\gamma P_{k-1}(z) + \delta \nabla P_{k-1}(z) \right] \right\}. \quad (14) \end{aligned}$$

We shall regard any two solutions of (8) denoting the coefficients $A_{\pm}$ and $B_{\pm}$ by $A_{\pm}^{\pm}$ and $B_{\pm}^{\pm}$ which are stated as

$$E_n(z) = \begin{cases} A_-^+ Q_n(z) + B_-^+ P_n(z), & n = 0, 1, 2, ..., k-1 \\ A_+^+ e_n(z) + B_+^+ e_n(-z), & n = k+1, k+2, ... \end{cases} \quad (15)$$

and

$$F_n(z) = \begin{cases} A_-^- Q_n(z) + B_-^- P_n(z), & n = 0, 1, 2, ..., k-1 \\ A_+^- e_n(z) + B_+^- e_n(-z), & n = k+1, k+2, ... \ , \end{cases} \quad (16)$$

where $A_{\pm}^{\pm}$ and $B_{\pm}^{\pm}$ are complex coefficients. Let E_n and F_n are correlated with the Jost solution of boundary value problem (8)-(10) and the boundary condition (9), respectively. Then we find

$$A_+^+ = 1, \quad B_+^+ = 0, \quad A_+^- = 0, \quad B_+^- = 1. \quad (17)$$

Furthermore considering the expression (12) and (15), we find

$$A_-^+ = \frac{M_{22}}{\det M}, \quad B_-^+ = -\frac{M_{21}}{\det M} \quad (18)$$

for the solution E_n. Similarly, for the solution F_n, considering (12) and (16), we obtain

$$A_+^- = M_{12}, \quad B_+^- = M_{22}. \quad (19)$$

Clearly, inserting the coefficients A_-^+, B_-^+, A_+^+, B_+^+ in (15) and coefficients A_-^-, B_-^-, A_+^-, B_+^- in (16), we find the solutions E_n and F_n satisfying the following asymptotics respectively

$$E_n(z) = \begin{cases} \dfrac{M_{22}}{\det M} Q_n(z) - \dfrac{M_{21}}{\det M} P_n(z), & n \to 0 \\ e_n(z), & n \to \infty \end{cases} \quad (20)$$

and

$$F_n(z) = \begin{cases} P_n(z), & n \to 0 \\ M_{12} e_n(z) + M_{22} e_n(-z), & n \to \infty \end{cases}. \quad (21)$$

Now by (20) and (21), we can give the following lemma.

Lemma 1. *The following equations hold for all* $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right] \setminus \{0, \pi\}$.

$$
\begin{aligned}
(i) \quad & W\left[E_n(z), F_n(z)\right] = \frac{M_{22}}{\det M}, & n \to 0, \\
(ii) \quad & W\left[E_n(z), F_n(z)\right] = -2i M_{22} \sin z, & n \to \infty.
\end{aligned}
$$

Moreover, by means of (5) and (7), it is understood that M_{22} has an analytic continuation from $\left[-\frac{\pi}{2}, \frac{3\pi}{2} \right]$ to S_0 and continuous up to $\left[-\frac{\pi}{2}, \frac{3\pi}{2} \right]$. By Lemma 1, we have the following.

Corollary 2. *A necessary and sufficient condition to investigate the eigenvalues and spectral singularities of the difference operator L with impulsive condition is to investigate the zeros of the function M_{22}.*

Thus, from the definition of spectral singularities and eigenvalues [18], we can introduce the sets of spectral singularities and eigenvalues of operator L,

$$
\sigma_{ss}(L) = \left\{ \lambda \in \mathbb{C} : \lambda = 2\cos z, \ z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right] \setminus \{0, \pi\}, \ M_{22}(z) = 0 \right\}
$$

and

$$
\sigma_d(L) = \left\{ \lambda \in \mathbb{C} : \lambda = 2\cos z, \ z \in S_0, \ M_{22}(z) = 0 \right\},
$$

respectively.

Theorem 3. *Under the condition (5), the function M_{22} satisfy the following asymptotics for $\xi \to \infty$, where $z = \eta + i\xi$,*

(i) *If* $\alpha + \beta + \gamma + \delta \neq 0$,

$$
M_{22} = e^{4iz} \left(\prod_{n=1}^{k-2} a_n \right)^{-1} \left[(\alpha + \beta + \gamma + \delta) \mu_{k+1} + o(1) \right]. \tag{22}
$$

(ii) *If* $\alpha + \beta + \gamma + \delta = 0$,

$$
M_{22} = e^{5iz} \left(\prod_{n=1}^{k-3} a_n \right)^{-1} \left[-a_{k-2}^{-1} (\alpha + \beta) \mu_{k+2} - (\beta + \delta) \mu_{k+1} + o(1) \right]. \tag{23}
$$

Proof. From (13), we get

$$
\begin{aligned}
M_{22} \quad = \quad & -\frac{a_{k+1}}{2i \sin z} \left\{ \beta e_{k+2}(z) P_{k-2}(z) - (\alpha + \beta) e_{k+2}(z) P_{k-1}(z) \right. \\
& \left. + (\alpha + \beta + \gamma + \delta) e_{k+1}(z) P_{k-1}(z) - (\beta + \delta) e_{k+1}(z) P_{k-2}(z) \right\}. \tag{24}
\end{aligned}
$$

Also it is known that

$$
e_n(z) = \mu_n e^{inz} \left[1 + o(1) \right], \quad n \in \mathbb{N}, \quad z = \eta + i\xi, \quad \xi \to \infty. \tag{25}
$$

By (24) and (25), we obtain asymptotic equations of M_{22}. This completes the proof.   □

Theorem 4. *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. *For all* $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right] \setminus \{0, \pi\}$, $M_{22}(z) \neq 0$.

Proof. Since E_n and F_n are the solutions of $(8) - (10)$ impulsive boundary value problem, it follows from (13), (14), (18) and (19) that

$$\overline{B_+^-} = \overline{M_{22}(z)} = M_{12}(z) = A_+^-, \tag{26}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$. Assume that, there exists a $z_0 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$ such that $M_{22}(z_0) = 0$. According to (26), we find $A_+^-(z_0) = B_+^-(z_0) = 0$. Then the solution $F_n(z_0)$ is equal to zero identically. So, F_n is a trivial solution of (8)-(10) which gives a contradiction with our assumption, i.e., $M_{22}(z) \neq 0$ for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$. $\qquad \square$

Corollary 5. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. The operator L has no spectral singularities.*

Definition 6. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Then the scattering function of the operator L is defined by*

$$\mathcal{S}(z) := \frac{E(0, -z)}{E(0, z)}.$$

Since $\{a_n\}_{n \in \mathbb{N} \cup \{0\}}$ and $\{b_n\}_{n \in \mathbb{N}}$ are real sequences, it can be easily seen from (15) that

$$\overline{E_n(z)} = E_n(-z)$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$. Then the scattering function transforms

$$\mathcal{S}(z) = \frac{\overline{E(0, z)}}{E(0, z)} = \frac{\overline{M_{22}(z)}}{M_{22}(z)} = \frac{M_{12}(z)}{M_{22}(z)}. \tag{27}$$

Theorem 7. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. For all $z \in \left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] \setminus \{0, \pi\}$, the scattering function satisfies*

$$\mathcal{S}(-z) = \mathcal{S}^{-1}(z) = \overline{\mathcal{S}(z)}.$$

Proof. By (27), we obtain

$$\mathcal{S}(-z) = \frac{M_{12}(-z)}{M_{22}(-z)}.$$

Since $\overline{M_{22}(-z)} = M_{22}(z)$ and $\overline{M_{12}(-z)} = M_{12}(z)$ for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, we get

$$\mathcal{S}(-z) = \mathcal{S}^{-1}(z) = \overline{\mathcal{S}(z)}.$$

It completes the proof. $\qquad \square$

3. An example

Let us consider the difference operator L_0 in $\ell^2(\mathbb{N})$ created by the following difference equation

$$y_{n-1} + y_{n+1} = 2 \cos z y_n, \quad \mathbb{N} \setminus \{1, 2, 3\} \tag{28}$$

and boundary condition

$$y_0 = 0, \tag{29}$$

with impulsive condition

$$\begin{pmatrix} y_3 \\ \Delta y_3 \end{pmatrix} = B \begin{pmatrix} y_1 \\ \nabla y_1 \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{30}$$

where $\alpha, \beta, \gamma, \delta$ are complex numbers. Here, since $a_n = 1$ and $b_n = 0$ for all $n \in \mathbb{N}$. We obtain directly from (13) and (14) that

$$M_{22}(z) = -\frac{e^{3iz}}{2i\sin z} \left[-(\alpha + \beta) e^{iz} + \alpha + \beta + \gamma + \delta \right] \tag{31}$$

and

$$M_{12}(z) = -\frac{e^{-3iz}}{2i\sin z} \left[(\alpha + \beta) e^{-iz} + \alpha + \beta - \gamma - \delta \right] \tag{32}$$

for $k = 2$. In order to examine eigenvalues and spectral singularities of L_0, we investigate zeros of M_{22}. For this purpose, we see

$$e^{iz} = 1 + \frac{\gamma + \delta}{\alpha + \beta}$$

by (31). Using the last equation, we find

$$z_m = -i \ln |1 + H| + Arg\,(1 + H) + 2m\pi, \quad m \in \mathbb{Z}, \tag{33}$$

where $H = \dfrac{\gamma + \delta}{\alpha + \beta}$. Now we need to investigate some special cases.

Case 1: Let $H = e^{i\theta} - 1$, where $\theta \in \mathbb{R}$. Since $Arg\,(1 + H) = \theta$, we get

$$z_m = \theta + 2m\pi, \quad m \in \mathbb{Z}.$$

In this case $z_m \in \mathbb{R}$. Thus, the numbers are spectral singularities of L_0 in
$\{z_m : m \in \mathbb{Z}\} \cap \left\{ \left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2} \right] \setminus \{0, \pi\} \right\}$.

Case 2: Let $D := \{z \in \mathbb{C} : |z + 1| = 1\}$. In this case, since $|1 + H| = 1$, we get

$$z_m = \theta + 2m\pi, \quad m \in \mathbb{Z}.$$

From (33) and this implies that spectral singularities of L_0 are in $\{z_m : m \in \mathbb{Z}\} \cap D$. Furthermore, let $D_* := \{z \in \mathbb{C} : |z + 1| < 1\}$. Since $|1 + H| < 1$, then it is easy to see that the eigenvalues of L_0 are in $S_0 \cap D_*$.

Case 3: Let $H \in \mathbb{R}$. If $-2 < H < 0$, then the operator has eigenvalues. Otherwise, the operator has no eigenvalues. Note that, the operator has spectral singularities for $H = -2$.

Moreover, let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Then using (27), (31) and (32), we get the scattering function of impulsive boundary value problem (28)-(30) by

$$\mathcal{S}(z) = e^{-6iz} \left[\frac{(\alpha + \beta) e^{-iz} + (\alpha + \beta - \gamma - \delta)}{-(\alpha + \beta) e^{iz} + (\alpha + \beta + \gamma + \delta)} \right]$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right] \setminus \{0, \pi\}$.

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Current address: Ibrahim Erdal: Ankara University, Faculty of Science, Department of Mathematics, 06100, Ankara, Turkey.

E-mail address: `ierdal@ankara.edu.tr`

ORCID Address: `http://orcid.org/0000-0002-4445-2389`

Current address: Şeyhmus Yardımcı: Ankara University, Faculty of Science, Department of Mathematics, 06100, Ankara, Turkey.

E-mail address: `smyardimci@ankara.edu.tr`

ORCID Address: `http://orcid.org/0000-0002-1062-9000`