CONSTRUCTIVE MATHEMATICAL ANALYSIS
9 (2026), No. 1, pp. 9-18
`http://dergipark.org.tr/en/pub/cma`
ISSN 2651 - 2939

Research Article

Voronovskaya-type inequality for the MKZ-Kantorovich operator

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ABSTRACT. We prove a Voronovskaya-type inequality for the Kantorovich-type modification of Meyer-König and Zeller operator

$$\widetilde{M}_n(f,x)=\sum_{k=0}^{\infty} m_{n,k}(x)\frac{(n+k+1)(n+k+2)}{n+1}\int_{\frac{k}{n+k+1}}^{\frac{k+1}{n+k+2}} f(u)du,$$

where

$$m_{n,k}(x)=\binom{n+k}{k}x^k(1-x)^{n+1}.$$

Keywords: K-functional, Kantorovich operator, Meyer-König and Zeller operator, direct theorem, strong converse inequality.

2020 Mathematics Subject Classification: 41A36, 41A25, 41A27, 41A17.

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1. INTRODUCTION

In order to approximate unbounded functions in $[0,1)$, Meyer-König and Zeller [5] introduced a new operator M_n by the formula

$$M_n(f,x)=\sum_{k=0}^{\infty} m_{n,k}(x)f\left(\frac{k}{n+k}\right), \tag{1.1}$$

where

$$m_{n,k}(x)=\binom{n+k}{k}x^k(1-x)^{n+1}. \tag{1.2}$$

Since this operator is unbounded in L_p norm, it is needed to be modified one way or another. There are many modifications of the operator M_n:

In [7], Totik investigated approximation of functions by the next Kantorovich-type modification of (1.1)

$$M_n^*(f,x)=\sum_{k=0}^{\infty} m_{n,k}(x)\frac{(n+k)(n+k+1)}{n}\int_{\frac{k}{n+k}}^{\frac{k+1}{n+k+1}} f(u)du$$

and proved a direct and converse theorems of weak type for it. Although this definition looks as the most natural one, the operator M_n^* is not a contraction, i.e., it is not very suitable for approximating functions in L_p norm for $p<\infty$.

Received: 23.10.2025; Accepted: 01.03.2026; Published Online: 06.03.2026
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DOI: 10.33205/cma.1809592

In [6], Müller defined a Kantorovich-type modification of Meyer-König and Zeller (MKZ) operator in a slightly different way:

$$\widetilde{M}_n(f,x)=\sum_{k=0}^{\infty} m_{n,k}(x)\frac{(n+k+1)(n+k+2)}{n+1}\int_{\frac{k}{n+k+1}}^{\frac{k+1}{n+k+2}} f(u)du \tag{1.3}$$

which is a contraction. The operator $\widetilde{M}_n(f,x)$ could be written more compactly

$$\widetilde{M}_n(f,x)=\frac{n+2}{(1-x)^2}\sum_{k=0}^{\infty} m_{n+2,k}(x)\int_{\frac{k}{n+k+1}}^{\frac{k+1}{n+k+2}} f(u)du.$$

In [2], we studied the approximation of functions in L_p norm by the operator $\widetilde{M}_n(f,x)$. By defining a new K-functional we proved a direct inequality.

But first let us introduce the needed definitions and notations. By $\varphi(x)=x(1-x)^2$, we denote the weight which is naturally connected with the second derivative of MKZ operator. The first derivative operator is denoted by $D=\frac{d}{dx}$. Thus, $Dg(x)=g'(x)$ and $D^k g(x)=g^{(k)}(x)$ for every natural k. We define a differential operator $\widetilde{D}$ by the formula

$$\widetilde{D}=\frac{d}{dx}\left(\varphi(x)\frac{d}{dx}\right)=D\varphi D.$$

The space $AC_{loc}(0,1)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b]\subset(0,1)$. The spaces $\widetilde{W}_p[0,1)$ and $L_p[0,1)+\widetilde{W}_p[0,1)$ are defined as follows:

$$\widetilde{W}_p[0,1)=\{f: f, Df\in AC_{loc}(0,1), \widetilde{D}f\in L_p[0,1), \lim_{x\to 0_+}\varphi(x)Df(x)=0\},$$

$$L_p[0,1)+\widetilde{W}_p[0,1)=\left\{f: f=f_1+f_2, f_1\in L_p[0,1), f_2\in \widetilde{W}_p[0,1)\right\}.$$

Also, we define the K-functional $\widetilde{K}(f,t)_p$ by the formula

$$\widetilde{K}(f,t)_p=\inf\left\{\|f-g\|_p+t\left\|\widetilde{D}g\right\|_p : f-g\in L_p[0,1), g\in\widetilde{W}_p[0,1)\right\}. \tag{1.4}$$

Recently we proved the following direct theorem for approximation of functions by the operator $\widetilde{M}_n(f,x)$.

Theorem 1.1 ([2])**.** *There exists an absolute constant $C>0$ such that for every natural n, $\widetilde{M}_n$ defined by (1.3), the K-functional given by (1.4) and for every $f\in L_p[0,1)+\widetilde{W}_p[0,1)$ the next inequality holds*

$$\|\widetilde{M}_n f-f\|_p\le C\widetilde{K}\left(f,\frac{1}{n}\right)_p, \quad 1\le p\le\infty.$$

In this paper we continue the study of the approximation of functions by the operator $\widetilde{M}_n(f,x)$. We prove the next Voronovskaya-type inequality.

Theorem 1.2. *For $1<p<\infty$ there exists a constant C such that for every natural $n\ge 5$ and every function $f\in C^3[0,1)$ such that the right-hand side of the inequality is finite, we have*

$$\begin{aligned}&\left\|\widetilde{M}_n f-f-\frac{1}{2n}\widetilde{D}f\right\|_p\\ &\le C\left\{\frac{\left\|\varphi^{3/2}D^3f\right\|_p}{n^{3/2}}+\frac{\left\|(1-\circ)^2Df\right\|_p}{n^2}+\frac{\left\|(1-\circ)^2D^2f\right\|_p}{n^2}+\frac{\left\|(1-\circ)^3D^3f\right\|_p}{n^3}\right\}.\end{aligned} \tag{1.5}$$

For the rest of this paper the constant C will always be absolute constant, which means it does not depend on f and n. Also, it may be different on each occurrence.

2. AUXILIARY RESULTS

In this section, we gather some properties of M_n, $\widetilde{M}_n$ and $m_{n,k}$, most of which can be found in [1], [3] and [4]. Also, we prove some technical lemmas which we will need.

Lemma 2.1. *The operators M_n and $\widetilde{M}_n$ are linear, positive operators with:*

$$\begin{aligned}
&\|M_n f\|_\infty \le \|f\|_\infty,\\
&\|\widetilde{M}_n f\|_p \le \|f\|_p \quad \textit{for} \quad 1 \le p \le \infty,\\
&M_n(1,x) = 1, \qquad M_n(t-x,x) = 0,\\
&\widetilde{M}_n(1,x) = 1.
\end{aligned}$$

In [1, Lemma 2.1] it is proved the next estimate

$$\frac{\varphi(x)}{n+1} \le M_n\left((t-x)^2,x\right) \le \frac{2\varphi(x)}{n+1}, \quad n \ge 3. \tag{2.6}$$

In [4, Lemma 2.1] it is proved

$$M_n\left((t-x)^4,x\right) \le C\frac{\varphi^2(x)}{n^2}, \quad x \ge \frac{1}{n+1}, \quad n \ge 5. \tag{2.7}$$

Lemma 2.2. *For $n \ge 3$ and $\widetilde{M}_n$ defined by* (1.3) *the next estimation is true:*

$$\widetilde{M}_n(t-x,x) = \frac{\varphi'(x)}{2n} + \alpha_n \quad \textit{where} \quad \alpha_n = O\left(\frac{(1-x)^2}{n^2}\right). \tag{2.8}$$

Proof. We have

$$\begin{aligned}
\widetilde{M}_n(t,x) &= \frac{1}{2}\sum_{k=0}^{\infty} m_{n,k}(x)\left(\frac{k}{n+k+1} + \frac{k+1}{n+k+2}\right)\\
&= 1 - \frac{n+1}{2}\sum_{k=0}^{\infty} m_{n,k}(x)\left(\frac{1}{n+k+1} + \frac{1}{n+k+2}\right).
\end{aligned}$$

Since

$$\sum_{k=0}^{\infty} m_{n,k}(x) = 1, \qquad \sum_{k=0}^{\infty}\frac{m_{n-2,k}(x)}{n+k+1} < \frac{1-x}{n-2}\sum_{k=0}^{\infty} m_{n-3,k}(x) = \frac{1-x}{n-2}$$

and

$$\begin{aligned}
\frac{m_{n,k}(x)}{n+k+1} &= \frac{1-x}{n}m_{n-1,k}(x) - \frac{(1-x)^2}{n(n-1)}m_{n-2,k}(x)\\
&\quad + \frac{2(1-x)^3}{n(n-1)(n-2)}m_{n-3,k}(x) - \frac{6(1-x)^3}{n(n-1)(n-2)}\frac{m_{n-3,k}(x)}{n+k+1},
\end{aligned} \tag{2.9}$$

it follows that

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{n+k+1} &= \frac{1-x}{n} - \frac{(1-x)^2}{n(n-1)} + \frac{2(1-x)^3}{n(n-1)(n-2)} + O\left(\frac{(1-x)^4}{n^4}\right)\\
&= \frac{1-x}{n} - \frac{(1-x)^2}{n(n-1)} + O\left(\frac{(1-x)^3}{n^3}\right).
\end{aligned}$$

Analogously

$$\begin{aligned}\frac{m_{n,k}(x)}{n+k+2}&=\frac{1-x}{n}m_{n-1,k}(x)-\frac{2(1-x)^2}{n(n-1)}m_{n-2,k}(x)\\&+\frac{6(1-x)^3}{n(n-1)(n-2)}m_{n-3,k}(x)-\frac{24(1-x)^3}{n(n-1)(n-2)}\frac{m_{n-3,k}(x)}{n+k+2}\end{aligned}\tag{2.10}$$

and

$$\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{n+k+2}=\frac{1-x}{n}-\frac{2(1-x)^2}{n(n-1)}+O\left(\frac{(1-x)^3}{n^3}\right).$$

Consequently

$$\begin{aligned}&(n+1)\sum_{k=0}^{\infty}m_{n,k}(x)\left(\frac{1}{n+k+1}+\frac{1}{n+k+2}\right)\\&=\frac{2(n+1)(1-x)}{n}-\frac{3(n+1)(1-x)^2}{n(n-1)}+O\left(\frac{(1-x)^3}{n^2}\right).\end{aligned}\tag{2.11}$$

Now (2.8) easily follows. ☐

Lemma 2.3. *For every natural $n\geq 5$ and for the operator $\widetilde{M}_n$ defined by (1.3) the next estimation is true:*

$$\widetilde{M}_n\left((t-x)^2,x\right)=\frac{\varphi(x)}{n}+\beta_n,\quad \text{where}\quad \beta_n=O\left(\frac{(1-x)^2}{n^2}\right).\tag{2.12}$$

Proof. Let us denote

$$\Delta_k=\left[\frac{k}{n+k+1},\frac{k+1}{n+k+2}\right],\qquad |\Delta_k|=\frac{n+1}{(n+k+1)(n+k+2)}.$$

We have

$$\begin{aligned}\widetilde{M}_n(t^2,x)&=\frac{1}{3}\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\left[\left(\frac{k+1}{n+k+2}\right)^3-\left(\frac{k}{n+k+1}\right)^3\right]\\&=\frac{1}{3}\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\left[\left(1-\frac{n+1}{n+k+2}\right)^3-\left(1-\frac{n+1}{n+k+1}\right)^3\right]\\&=(n+1)\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\left[\frac{1}{n+k+1}-\frac{1}{n+k+2}\right]\\&-(n+1)^2\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\left[\left(\frac{1}{n+k+1}\right)^2-\left(\frac{1}{n+k+2}\right)^2\right]\\&+\frac{(n+1)^3}{3}\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\left[\left(\frac{1}{n+k+1}\right)^3-\left(\frac{1}{n+k+2}\right)^3\right].\end{aligned}$$

Now

$$(n+1)\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\left[\frac{1}{n+k+1}-\frac{1}{n+k+2}\right]=\sum_{k=0}^{\infty}m_{n,k}(x)=1.$$

From (2.11) it follows that

$$\begin{aligned}
&(n+1)^2\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\left[\left(\frac{1}{n+k+1}\right)^2-\left(\frac{1}{n+k+2}\right)^2\right]\\
=&(n+1)\sum_{k=0}^{\infty}m_{n,k}(x)\left(\frac{1}{n+k+1}+\frac{1}{n+k+2}\right)\\
=&\frac{2(n+1)(1-x)}{n}-\frac{3(n+1)(1-x)^2}{n(n-1)}+O\left(\frac{(1-x)^3}{n^2}\right)\\
=&\frac{2(n+1)(1-x)}{n}-\frac{3(1-x)^2}{n}+O\left(\frac{(1-x)^2}{n^2}\right).
\end{aligned}$$

Also,

$$\begin{aligned}
&(n+1)^3\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\left[\left(\frac{1}{n+k+1}\right)^3-\left(\frac{1}{n+k+2}\right)^3\right]\\
=&(n+1)^2\sum_{k=0}^{\infty}m_{n,k}(x)\left[\left(\frac{1}{n+k+1}\right)^2+\frac{1}{(n+k+1)(n+k+2)}+\left(\frac{1}{n+k+2}\right)^2\right].
\end{aligned}$$

Since

$$\begin{aligned}
\frac{m_{n,k}(x)}{(n+k+1)^2}=&\frac{(1-x)^2m_{n-2,k}(x)}{n(n-1)}-\frac{3(1-x)^3m_{n-3,k}(x)}{n(n-1)(n-2)}\\
&+\frac{9(1-x)^3m_{n-3,k}(x)}{n(n-1)(n-2)(n+k+1)}+\frac{2(1-x)^2m_{n-2,k}(x)}{n(n-1)(n+k+1)^2},
\end{aligned}$$

it follows that

$$\begin{aligned}
(n+1)^2\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{(n+k+1)^2}=&\frac{(n+1)^2(1-x)^2}{n(n-1)}-\frac{3(n+1)^2(1-x)^3}{n(n-1)(n-2)}\\
&+\frac{9(n+1)^2(1-x)^3}{n(n-1)(n-2)}\sum_{k=0}^{\infty}\frac{m_{n-3,k}(x)}{n+k+1}-\frac{2(n+1)^2(1-x)^2}{n(n-1)}\sum_{k=0}^{\infty}\frac{m_{n-2,k}(x)}{(n+k+1)^2}\\
=&\frac{(n+1)^2(1-x)^2}{n(n-1)}-\frac{3(n+1)^2(1-x)^3}{n(n-1)(n-2)}+O\left(\frac{(1-x)^4}{n^2}\right).
\end{aligned}$$

From (2.9) and (2.10), we have

$$\begin{aligned}
(n+1)^2\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{(n+k+1)(n+k+2)}=&(n+1)^2\sum_{k=0}^{\infty}\left(\frac{m_{n,k}(x)}{n+k+1}-\frac{m_{n,k}(x)}{n+k+2}\right)\\
=&\frac{(n+1)^2(1-x)^2}{n(n-1)}-\frac{4(n+1)^2(1-x)^3}{n(n-1)(n-2)}+O\left(\frac{(1-x)^4}{n^2}\right),
\end{aligned}$$

$$\begin{aligned}
\frac{m_{n,k}(x)}{(n+k+2)^2}=&\frac{(1-x)^2m_{n-2,k}(x)}{n(n-1)}-\frac{5(1-x)^3m_{n-3,k}(x)}{n(n-1)(n-2)}\\
&+\frac{20(1-x)^3m_{n-3,k}(x)}{n(n-1)(n-2)(n+k+2)}+\frac{6(1-x)^2m_{n-2,k}(x)}{n(n-1)(n+k+2)^2}
\end{aligned}$$

and

$$\begin{aligned}&(n+1)^2\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{(n+k+2)^2}\\ =&\frac{(n+1)^2(1-x)^2}{n(n-1)}-\frac{5(n+1)^2(1-x)^3}{n(n-1)(n-2)}+O\left(\frac{(1-x)^4}{n^2}\right).\end{aligned}$$

Then

$$\begin{aligned}&\frac{(n+1)^3}{3}\sum_{k=0}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\left[\left(\frac{1}{n+k+1}\right)^3-\left(\frac{1}{n+k+2}\right)^3\right]\\ &=\frac{(n+1)^2(1-x)^2}{n(n-1)}-\frac{4(n+1)^2(1-x)^3}{n(n-1)(n-2)}+O\left(\frac{(1-x)^4}{n^2}\right)\\ &=\frac{(n+2)(1-x)^2}{n-1}-\frac{4(1-x)^3}{n}+O\left(\frac{(1-x)^2}{n^2}\right).\end{aligned}$$

Consequently

$$\begin{aligned}\widetilde{M}_n(t^2,x)=1&-\frac{2(n+1)(1-x)}{n}+\frac{3(1-x)^2}{n}\\ &+\frac{(n+2)(1-x)^2}{n-1}-\frac{4(1-x)^3}{n}+O\left(\frac{(1-x)^2}{n^2}\right)\end{aligned}$$

and from Lemma 2.2, it follows that

$$\begin{aligned}\widetilde{M}_n((t-x)^2,x)&=\widetilde{M}_n(t^2,x)-2x\widetilde{M}_n(t-x,x)-x^2\\ &=1-\frac{2(n+1)(1-x)}{n}+\frac{3(1-x)^2}{n}+\frac{(n+2)(1-x)^2}{n-1}\\ &-\frac{4(1-x)^3}{n}+O\left(\frac{(1-x)^2}{n^2}\right)-2x\left(\frac{(1-x)(1-3x)}{2n}+\alpha_n\right)-x^2\\ &=\frac{\varphi(x)}{n}+O\left(\frac{(1-x)^2}{n^2}\right).\end{aligned}$$

The lemma is proved.

□

3. PROOF OF THE THEOREM 1.2

By Taylor's formula

$$f(u)=f(x)+(u-x)Df(x)+\frac{1}{2}(u-x)^2D^2f(x)+\frac{1}{2}\int_x^u(u-v)^2D^3f(v)dv.$$

Applying $\widetilde{M}_n$ and using Lemma 2.2 and Lemma 2.3, we obtain:

$$\begin{aligned}\widetilde{M}_n(f,x)=f(x)&+\frac{\varphi'(x)}{2n}Df(x)+Df(x)\alpha_n+\frac{\varphi(x)}{2n}D^2f(x)\\ &+D^2f(x)\beta_n+\frac{1}{2}\widetilde{M}_n\left(\int_x^u(u-v)^2D^3f(v)dv\right)\\ &=f(x)+\frac{1}{2n}\widetilde{D}f(x)+Df(x)\alpha_n+D^2f(x)\beta_n+I_n,\end{aligned}$$

where

$$I_n = \frac{1}{2}\sum_{k=0}^{\infty} m_{n,k}(x)|\Delta_k|^{-1}\int_{\Delta_k}\left(\int_x^u (u-v)^2 D^3 f(v)dv\right)du.$$

Now we estimate I_n. We consider two cases:

Case 1. $x \geq \frac{1}{n+1}$. Let $x \in \Delta_N$. Then

$$\begin{aligned}2I_n &= \sum_{k=0}^{N-1}\frac{m_{n,k}(x)}{|\Delta_k|}\int_{\Delta_k}\left(\int_x^u (u-v)^2 D^3 f(v)dv\right)du\\ &+\frac{m_{n,N}(x)}{|\Delta_N|}\int_{\Delta_N}\left(\int_x^u (u-v)^2 D^3 f(v)dv\right)du\\ &+\sum_{k=N+1}^{\infty}|\frac{m_{n,k}(x)}{|\Delta_k|}\int_{\Delta_k}\left(\int_x^u (u-v)^2 D^3 f(v)dv\right)du = I_n^1 + I_n^2 + I_n^3.\end{aligned}$$

For I_n^1 we have $u < x$ and in this case for $u \leq v \leq x$:

$$\frac{(u-v)^2}{\varphi^{3/2}(v)} \leq \left(\frac{v-u}{v}\right)^{3/2}\frac{(v-u)^{1/2}}{(1-x)^3} \leq \left(1-\frac{u}{x}\right)^{3/2}\frac{(x-u)^{1/2}}{(1-x)^3} = \frac{(x-u)^2}{\varphi^{3/2}(x)}.$$

Consequently

$$\begin{aligned}\left|I_n^1\right| &\leq \sum_{k=0}^{N-1}\frac{m_{n,k}(x)}{|\Delta_k|}\int_{\Delta_k}\frac{(u-x)^2}{\varphi^{3/2}(x)}\left(\int_u^x \varphi^{3/2}(v)\left|D^3 f(v)\right|dv\right)du\\ &= \sum_{k=0}^{N-1}\frac{m_{n,k}(x)}{|\Delta_k|}\int_{\Delta_k}\frac{|u-x|^3}{\varphi^{3/2}(x)}\left(\frac{1}{x-u}\int_u^x \varphi^{3/2}(v)\left|D^3 f(v)\right|dv\right)du\\ &\leq \varphi^{-3/2}(x)M\left(\varphi^{3/2}\left|D^3 f\right|, x\right)\sum_{k=0}^{N-1}\frac{m_{n,k}(x)}{|\Delta_k|}\int_{\Delta_k}|u-x|^3 du\\ &\leq C\varphi^{-3/2}(x)M\left(\varphi^{3/2}\left|D^3 f\right|, x\right)\sum_{k=0}^{N-1} m_{n,k}(x)\left|x-\frac{k}{n+k+1}\right|^3\\ &\leq C\varphi^{-3/2}(x)M\left(\varphi^{3/2}\left|D^3 f\right|, x\right)\sum_{k=0}^{\infty} m_{n,k}(x)\left|x-\frac{k}{n+k+1}\right|^3\end{aligned}$$

where

$$M(h,x) = \sup_{x\in\Delta}\frac{1}{|\Delta|}\int_{\Delta}|h(t)|dt$$

is the Hardy's maximal function.

Now by using Cauchy's inequality, (2.6) and (2.7) we obtain

$$\begin{aligned}&\sum_{k=0}^{\infty} m_{n,k}(x)\left|x-\frac{k}{n+k+1}\right|^3\\ &\leq\left\{\sum_{k=0}^{\infty} m_{n,k}(x)\left(x-\frac{k}{n+k+1}\right)^2\right\}^{1/2}\left\{\sum_{k=0}^{\infty} m_{n,k}(x)\left(x-\frac{k}{n+k+1}\right)^4\right\}^{1/2}\\ &\leq Cn^{-3/2}\varphi^{3/2}(x),\end{aligned}$$

i.e.,

$$I_n^1 \le C n^{-3/2} M\left(\varphi^{3/2}\left|D^3 f\right|, x\right).$$

For I_n^2 we have $\varphi(v) \sim \varphi(x) \sim \frac{Nn^2}{(n+N)^3}$, $|\Delta_N| \sim \frac{n}{(n+N)^2}$, and consequently

$$\begin{aligned}
|I_n^2| &= \frac{m_{n,N}(x)}{|\Delta_N|}\int_{\Delta_N}\left|\int_x^u (u-v)^2 D^3 f(v)dv\right| du\\
&\le \frac{m_{n,N}(x)}{|\Delta_N|}\int_{\Delta_N}\frac{|u-x|^3}{\varphi^{3/2}(x)}\left|\frac{1}{x-u}\int_u^x \varphi^{3/2}(v)\left|D^3 f(v)\right| dv\right| du\\
&\le m_{n,N}(x)|\Delta_N|^3 \varphi^{-3/2}(x) M\left(\varphi^{3/2}\left|D^3 f\right|, x\right)\\
&\le C n^{-3/2} M\left(\varphi^{3/2}\left|D^3 f\right|, x\right).
\end{aligned}$$

For I_n^3 we have $x < u$ and in this case for $x \le v \le u$:

$$\frac{(u-v)^2}{\varphi^{3/2}(v)} \le \frac{1}{x^{3/2}}\left(\frac{u-v}{1-v}\right)^2\frac{1}{1-u} \le \frac{(u-x)^2}{x^{3/2}(1-x)^2}\frac{1}{1-u} = \frac{(u-x)^2}{\varphi^{3/2}(x)}\frac{1-x}{1-u}.$$

Then

$$\begin{aligned}
\left|I_n^3\right| &= \sum_{k=N+1}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\int_{\Delta_k}\left(\int_x^u \frac{(u-v)^2}{\varphi^{3/2}(v)}\varphi^{3/2}(v)\left|D^3 f(v)\right| dv\right) du\\
&\le (1-x)\sum_{k=N+1}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\int_{\Delta_k}\frac{(u-x)^2}{\varphi^{3/2}(x)}\left(\int_x^u \varphi^{3/2}(v)\left|D^3 f(v)\right| dv\right)\frac{du}{1-u}\\
&= (1-x)\sum_{k=N+1}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\int_{\Delta_k}\frac{|u-x|^3}{\varphi^{3/2}(x)}\left(\frac{1}{u-x}\int_x^u \varphi^{3/2}(v)\left|D^3 f(v)\right| dv\right)\frac{du}{1-u}\\
&\le (1-x)\varphi^{-3/2}(x) M\left(\varphi^{3/2}\left|D^3 f\right|, x\right)\sum_{k=N+1}^{\infty}\frac{m_{n,k}(x)}{|\Delta_k|}\int_{\Delta_k}|u-x|^3\frac{du}{1-u}\\
&\le \frac{1-x}{\varphi^{3/2}(x)} M\left(\varphi^{3/2}\left|D^3 f\right|, x\right)\sum_{k=N+1}^{\infty} m_{n,k}(x)\frac{n+k+2}{n+1}\left(\frac{k+1}{n+k+2}-x\right)^3\\
&\le \frac{1-x}{\varphi^{3/2}(x)} M\left(\varphi^{3/2}\left|D^3 f\right|, x\right)\sum_{k=0}^{\infty} m_{n,k}(x)\frac{n+k+2}{n+1}\left(\frac{k+1}{n+k+2}-x\right)^3\\
&\le C\varphi^{-3/2}(x) M\left(\varphi^{3/2}\left|D^3 f\right|, x\right)\sum_{k=0}^{\infty} m_{n+1,k}(x)\left(\frac{k+1}{n+k+2}-x\right)^3.
\end{aligned}$$

Analogously to the above estimation of I_n^1 we have

$$\left|I_n^3\right| \le C n^{-3/2} M\left(\varphi^{3/2}\left|D^3 f\right|, x\right).$$

Case 2. $x < \frac{1}{n+1}$. In this case we have $1 - x \sim 1$.
We will estimate the terms in the sum of I_n separately for $k = 0, 1, 2$ and for $k \geq 3$. For $k = 0$:

$$\begin{aligned}
&\frac{m_{n,0}(x)}{|\Delta_0|} \int_{\Delta_0} \left(\int_x^u (u-v)^2 \left| D^3 f(v) \right| dv \right) du \\
\leq &(n+2)(1-x)^{n+1} \int_0^{\frac{1}{n+2}} |u-x|^3 \left(\frac{1}{u-x} \int_x^u \left| D^3 f(v) \right| dv \right) du \\
\leq &(n+2)(1-x)^{n+1} M\left(\left| D^3 f \right|, x \right) \int_0^{\frac{1}{n+2}} |u-x|^3 du. \\
\leq &\frac{(1-x)^{n+1}}{(n+1)^3} M\left(\left| D^3 f \right|, x \right) \leq \frac{C}{n^3} M\left((1-\circ)^3 \left| D^3 f \right|, x \right).
\end{aligned}$$

The estimations in the cases $k = 1, 2$ are similar. For $k \geq 3$, we have

$$\begin{aligned}
&\sum_{k=3}^{\infty} \frac{m_{n,k}(x)}{|\Delta_k|} \int_{\Delta_k} \left(\int_x^u (u-v)^2 \left| D^3 f(v) \right| dv \right) du \\
\leq &\sum_{k=3}^{\infty} \frac{m_{n,k}(x)}{|\Delta_k|} \int_{\Delta_k} |u-x|^3 \left(\frac{1}{u-x} \int_x^u \left| D^3 f(v) \right| dv \right) du \\
\leq &M\left(\left| D^3 f \right|, x \right) \sum_{k=3}^{\infty} \frac{m_{n,k}(x)}{|\Delta_k|} \int_{\Delta_k} |u-x|^3 du.
\end{aligned}$$

Now for $x < \frac{1}{n+1}$ and $k \geq 3$, we have

$$\int_{\Delta_k} |u-x|^3 du \leq |\Delta_k| \left(\frac{k+1}{n+k+2} \right)^3 \leq C |\Delta_k| \left(\frac{k}{n+k+1} \right)^3.$$

Then

$$\begin{aligned}
&\sum_{k=3}^{\infty} \frac{m_{n,k}(x)}{|\Delta_k|} \int_{\Delta_k} \left(\int_x^u (u-v)^2 \left| D^3 f(v) \right| dv \right) du \\
\leq &CM\left(\left| D^3 f \right|, x \right) \sum_{k=3}^{\infty} m_{n,k}(x) \left(\frac{k}{n+k+1} \right)^3 \\
\leq &CM\left(\left| D^3 f \right|, x \right) \sum_{k=3}^{\infty} \binom{n+k}{k} \left(\frac{k}{n+k} \right)^3 x^k (1-x)^{n+1} \\
\leq &CM\left(\left| D^3 f \right|, x \right) x^3 \sum_{k=0}^{\infty} m_{n-3,k}(x) \\
= &CM\left(\left| D^3 f \right|, x \right) x^3 \leq \frac{C}{n^3} M\left((1-x)^3 \left| D^3 f \right|, x \right).
\end{aligned}$$

By applying the Hardy's inequality about maximal function (for $p > 1$) we complete the proof of the theorem.

Acknowledgements. This study is financed by the European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project No BG-RRP-2.004-0008.

REFERENCES

[1] M. Becker, R. Nessel: *A global approximation theorem for Meyer-König and Zeller operators*, Math. Z., **160** (1978), 195–206.
[2] I. Gadjev: *A Direct Theorem for MKZ-Kantorovich Operator*, Anal. Math., **45** (2019), 25–38.
[3] I. Gadjev: *Strong converse result for uniform approximation by Meyer-König and Zeller operator*, J. Math. Anal. Appl., **428** (2015), 32–42.
[4] S. Guo, Q. Qi and C. Li: *Strong converse inequalities for Meyer-König and Zeller operators*. J. Math. Anal. Appl., **337** (2008), 994–1001.
[5] W. Meyer-König, K. Zeller: *Bernsteinsche Potenzreihen*, Studia Math., **19** (1960), 89–94.
[6] M. Müller: *L_p-Approximation by the method of integral Meyer-König and Zeller operators*, Studia Mathematica, **T.LXIII** (1978), 81–88.
[7] V. Totik: *Approximation by Meyer-König and Zeller Type Operators*, Math. Z., **182** (1983), 425–446.

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