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Research Article

A combinatorial formula for recursive operator sequences and applications

RAÚL E. CURTO*, ABDERRAZZAK ECH-CHARYFY, KAISSAR IDRISSI, AND EL HASSAN ZEROUALI

ABSTRACT. We study sequences of bounded operators $(T_n)_{n\geq 0}$ on a complex separable Hilbert space $\mathcal{H}$ that satisfy a linear recurrence relation of the form

$$T_{n+r} = A_0T_n + A_1T_{n+1} + \cdots + A_{r-1}T_{n+r-1} \quad \text{(for all } n \geq 0),$$

where the coefficients $A_0, A_1, \ldots, A_{r-1}$ are pairwise commuting bounded operators on $\mathcal{H}$. Such relations naturally arise in the context of the operator-valued moment problem, particularly in the study of flat extensions of block Hankel operators. Our first goal is to derive an explicit combinatorial formula for T_n. As a concrete application, we provide an explicit expression for the powers of an operator-valued companion matrix. In the special case of scalar coefficients $A_k = a_kI_{\mathcal{H}}$, with $a_k \in \mathbb{R}$, we recover a Binet-type formula that allows the explicit computation of the powers and the exponential of algebraic operators in terms of Bell polynomials.

Keywords: Operator recurrence relation, operator-valued moment problem, Binet formula, combinatorial formula, companion matrix.

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1. INTRODUCTION

Let $s^{(r)} = \{s_k\}_{k=0}^r$ be a finite sequence of real numbers. Tchakaloff's Theorem [24] states that if there exists a positive Borel measure μ on $\mathbb{R}$ such that $s_k = \displaystyle\int_{\mathbb{R}} x^k \, d\mu(x)$, then there exists a positive finitely atomic measure $\nu = \displaystyle\sum_{i=1}^{m} w_i \, \delta_{x_i}$, such that

$$s_k = \int_{\mathbb{R}} x^k \, d\nu(x) = \sum_{i=1}^{m} w_i x_i^k \text{ for } k = 0, 1, \ldots, r. \tag{1.1}$$

The expression in Equation (1.1) allows us to extend our sequence in a moment sequence satisfying a recursive relation given by

$$s_{m+k} = \sum_{j=0}^{m-1} a_j s_{k+j} \text{ for all } k \geq 0 \tag{1.2}$$

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*Corresponding author: Raúl E. Curto; raul-curto@uiowa.edu
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associated with the monic polynomial

$$P(X) = \prod_{i=1}^{m}(X - x_i) = X^m - \sum_{j=0}^{m-1} a_j X^j.$$

Sequences $\{s_k\}_{k\geq 0}$ satisfying the linear recurrence relation (1.2) have been widely studied in different mathematical branches because of the large range of applications. They are known as generalized Fibonacci sequences in combinatorics and discrete mathematics, as linear difference equations in numerical analysis, and as moment recursive sequences in the context of the scalar moment problem. Hence, Tchakaloff's Theorem can be reformulated as follows: a finite sequence $s^{(r)}$ is the moment sequence of some nonnegative measure if and only if it can be extended to a full moment recursive sequence. As a consequence, the associated Hankel matrices satisfy the flat extension property; namely,

$$\operatorname{rank}(s_{i+j+k})_{0\leq i,j\leq r} = \operatorname{rank}(s_{i+j+k})_{0\leq i,j\leq r+\ell+1} \quad (\text{for all } k, \ell \geq 0).$$

This establishes the equivalence of three key properties [12, 13, 14]:

(1) the existence of a finitely atomic representing measure;
(2) the recursiveness of the extended moment sequence; and
(3) the flat extension property.

In the matrix-valued case, where the scalars in (1.2) are replaced by Hermitian matrices, this equivalence has also been investigated, and the previous observations have been recovered. In particular, the authors in [17, 18] extended Tchakaloff's Theorem and the flatness property to truncated sequences of Hermitian matrices. The connection between the scalar and matrix cases has been further explored in [11], where it is shown that the union of the supports of the scalar representing measures reflects the support of the matrix-valued measure.

In the operator-valued setting, scalar moments are replaced by self-adjoint operators acting on an infinite-dimensional Hilbert space $\mathcal{H}$. It has been shown (see [9, 17]) that Tchakaloff's Theorem does not extend to this context. More precisely, there exist finite families of self-adjoint operators that arise as moments of some nonnegative operator-valued measure, yet for which no finitely atomic operator-valued representing measure exists. Moreover, a truncated sequence of operator moments may fail to be recursively extendable in the operator moment problem. This naturally raises the following question in the operator moment problem framework:

Open question 1. *Are there suitable formulations of recursiveness and flatness under which the equivalence* $(1) \iff (2) \iff (3)$ *continues to hold, as in the finite-dimensional case?*

In recent work [10], the authors have extended the classical notion of flatness from the finite-dimensional to the infinite-dimensional operator setting. Specifically, consider a block operator

$$X = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

acting on the Hilbert space $\mathcal{H} \oplus \mathcal{K}$. We say that X is a flat extension of A if and only if

$$\mathcal{H} \oplus \mathcal{K} = (\mathcal{H}, 0_{\mathcal{K}}) + \ker X.$$

This definition preserves the essential geometric structure of flatness and naturally generalizes the finite-dimensional concept (see [16, Proposition 2.1]). It provides a robust framework for extending moment-theoretic and algebraic ideas to operator-valued contexts, where positivity and recursive properties coexist in infinite dimensions. As an application, we consider a sequence $(T_n)_{n\geq 0}$ of bounded self-adjoint operators on a Hilbert space $\mathcal{H}$. The block Hankel operator $H_n = (T_{i+j})_{0\leq i,j\leq n}$ acting on the Hilbert space $\mathcal{H}^{(n)} = \underbrace{\mathcal{H}\oplus\cdots\oplus\mathcal{H}}_{n+1 \text{ times}}$ admits a flat extension $H_{n+1} = (T_{i+j})_{0\leq i,j\leq n+1}$ if and only if $\mathcal{H}^{(n+1)} = (\mathcal{H}^{(n)}, 0_{\mathcal{H}}) + \ker H_{n+1}$. Under this formulation, the flatness of the Hankel operator encodes the recursive structure of the sequence $(T_n)_{n\geq 0}$. Indeed, one can show that a truncated sequence admitting a flat extension determines a sequence of bounded self-adjoint operators satisfying a linear operator recurrence relation of order r:

$$T_{n+r} = A_0T_n + A_1T_{n+1} + \cdots + A_{r-1}T_{n+r-1}, \quad \forall n \geq 0, \tag{1.3}$$

where the coefficients $\mathbf{A} = (A_0, A_1, \ldots, A_{r-1})$ are bounded self-adjoint operators on $\mathcal{H}$ with $A_{r-1} \neq 0_{\mathcal{H}}$.

Our objective in this work is to study and classify the types of sequences that satisfy a linear recurrence relation of the form (1.3) in the case where the r-tuple $\mathbf{A} = (A_0, A_1, \ldots, A_{r-1})$ consists of pairwise commuting self-adjoint operators.

1.1. **Our approach.** Our methodology can be outlined as follows:

- We extend the recurrence relation (1.3) to a more general vectorial framework.
- We rely on the spectral theory of commuting r-tuples of self-adjoint operators. By means of the joint spectral theorem, the vectorial framework is reduced to a family of scalar recurrence problems on a measurable space.
- For the scalar setting, we recover classical results, which are then lifted back to the operator framework by means of continuous functional calculus. This spectral approach enables us to construct explicit formulas for operator sequences satisfying (1.3).

1.2. **Main contributions.** The main contributions of this paper can be summarized as follows:

- We establish an operator-valued analogue of the combinatorial formula for linear recurrences (1.3), valid in the context of commuting self-adjoint operator coefficients.
- As a concrete application, we provide an explicit formula for the powers of the operator companion matrix, obtained by reformulating the recurrence as a first-order dynamical system.
- In the case where the coefficients are scalar, we obtain a Binet-type formula, which allows us to explicitly characterize both the powers and the exponential of an algebraic operator in terms of Bell polynomials.

2. Preliminaries

2.1. **Notation and terminology.** For an integer $r \geq 2$, a multi-index of length r is a vector

$$\mathbf{k} := (k_0, k_1, \ldots, k_{r-1}) \in \mathbb{Z}_+^r.$$

Its length is defined by

$$|\mathbf{k}| := k_0 + k_1 + \cdots + k_{r-1},$$

and its weighted degree, i.e., the scalar product with the vector $\mathbf{d} := (1, 2, \ldots, r)$, is given by

$$\langle \mathbf{k}, \mathbf{d}\rangle := \sum_{j=0}^{r-1}(j+1)k_j = k_0 + 2k_1 + \cdots + rk_{r-1}.$$

The multinomial coefficient associated with $\mathbf{k} \in \mathbb{Z}_+^r$ is

$$\binom{|\mathbf{k}|}{\mathbf{k}} := \frac{(|\mathbf{k}|)!}{k_0! \cdots k_{r-1}!},$$

while the multivariate monomial of index $\mathbf{k}$ associated with $\mathbf{a} = (a_0, a_1, \ldots, a_{r-1}) \in \mathbb{C}^r$ is defined as

$$\mathbf{a}^{\mathbf{k}} := a_0^{k_0} a_1^{k_1} \cdots a_{r-1}^{k_{r-1}}.$$

Throughout, $\mathbf{B}(\mathcal{H})$ denotes the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$; $I_{\mathcal{H}}$ and $0_{\mathcal{H}}$ denote, respectively, the identity and the zero operator on $\mathcal{H}$. We also write $\mathbf{M}_d(\mathbb{C})$ for the algebra of $d \times d$ complex matrices, with I_d and 0_d denoting, respectively, the identity and the zero matrix in $\mathbb{C}^d$.

2.2. **Some known results.** Let $(\gamma_n)_{n\geq 0}$ be a sequence of complex numbers determined by the initial conditions $\gamma_0 = \alpha_0, \quad \gamma_1 = \alpha_1, \quad \ldots, \quad \gamma_{r-1} = \alpha_{r-1}$, and satisfying the linear recurrence relation of order $r \geq 2$:

$$\gamma_{n+r} = a_{r-1}\gamma_{n+r-1} + a_{r-2}\gamma_{n+r-2} + \cdots + a_0\gamma_n, \quad \forall n \geq 0, \tag{2.4}$$

where $a_i \in \mathbb{C}$ are fixed coefficients with $a_{r-1} \neq 0$.

Sequences defined by (2.4), usually referred to as r-generalized Fibonacci sequences, have been extensively studied in the literature. They exhibit a rich algebraic and analytic structure, with connections to the theory of characteristic polynomials, generating functions, special functions, and also to moment problems in the scalar setting (see, for example, [2, 7, 19, 21, 22]). The recurrence relation (2.4) is governed by the characteristic polynomial

$$P(X) = X^r - a_{r-1}X^{r-1} - a_{r-2}X^{r-2} - \cdots - a_0,$$

whose spectral properties completely determine the behavior of the sequence. In particular, the sequence admits a closed-form representation of Binet type:

$$\gamma_n = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \beta_{i,j} n^j \right) \lambda_i^n,$$

where $\lambda_1, \ldots, \lambda_s$ denote the distinct roots of P, m_i their respective multiplicities, and the coefficients $\beta_{i,j}$ are uniquely determined by the initial conditions.

An alternative description of the general term, involving combinatorial coefficients, is given in [19]. More precisely, for every $n \geq r$,

$$\gamma_n = \rho(n,r)W_0 + \rho(n-1,r)W_1 + \cdots + \rho(n-r+1,r)W_{r-1}, \tag{2.5}$$

where

$$W_s = a_{r-1}\gamma_s + a_{r-2}\gamma_{s+1} + \cdots + a_s\gamma_{r-1}, \quad 0 \leq s \leq r-1,$$

and

$$\rho(n,r) = \sum_{\substack{\mathbf{k}\in\mathbb{Z}_+^r \\ \langle \mathbf{k},\mathbf{d}\rangle = n-r}} \binom{|\mathbf{k}|}{\mathbf{k}} \mathbf{a}^{\mathbf{k}},$$

with the conventions $\rho(r,r) = 1$ and $\rho(n,r) = 0$ for $n < r$. The relationship between the combinatorial coefficients in (2.5) and the spectral data of the characteristic polynomial P was further analyzed in [5], where refined connections with binomial identities and generalizations of Binet's formula are established. These representations provide useful tools in number theory, combinatorics, and the analysis of recursive algorithms.

In the matrix case, there is no direct analog of the Binet-type formula due to the noncommutativity of matrices and the lack of simultaneous diagonalizability. However, in [3], the authors generalized the combinatorial formula (2.5) to the algebra of matrices, relying on the fact that pairwise commuting symmetric matrices are simultaneously diagonalizable in the same basis. Using the Cayley–Hamilton Theorem, they derived, as an application, explicit formulas for A^n ($n \geq r$) and e^{tA} for every $r \times r$ matrix A, expressed in terms of the coefficients of its characteristic polynomial and the matrices A^j, where $0 \leq j \leq r-1$. More specifically, consider a family $\mathbf{A} = \{A_0, A_1, \ldots, A_{r-1}\}$ of $d \times d$ symmetric matrices that are pairwise commuting, with $A_{r-1} \neq 0_d$. Let $(Y_n)_{n \geq 0} \subseteq \mathbf{M}_d(\mathbb{C})$ be the matrix-valued sequence defined recursively by

$$\begin{cases} Y_i = V_i, & \text{for } i = 0, \ldots, r-1, \\ Y_n = A_0 Y_{n-1} + A_1 Y_{n-2} + \cdots + A_{r-1} Y_{n-r}, & \text{for } n \geq r, \end{cases}$$

where $\{V_0, \ldots, V_{r-1}\} \subseteq \mathbf{M}_d(\mathbb{C})$ is a prescribed set of initial matrices. Then, for every $n \geq r$, the sequence satisfies the recurrence relation

$$Y_n = \sum_{s=0}^{r-1} \rho(n-s, r)\, W_s,$$

where

$$W_s = \sum_{j=s}^{r-1} A_j V_{s+r-1-j}, \quad (s = 0, 1, \ldots, r-1),$$

and the matrix coefficients $\rho(n, r)$ are given by

$$\rho(r, r) = I_d, \quad \rho(p, r) = 0_d \quad (\text{for } p < r),$$

and, for all $n \geq r$,

$$\rho(n, r) = \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^r \\ \langle \mathbf{k}, \mathbf{d} \rangle = n-r}} \binom{|\mathbf{k}|}{\mathbf{k}} \mathbf{A}^{\mathbf{k}},$$

with

$$\mathbf{A}^{\mathbf{k}} = A_0^{k_0} A_1^{k_1} \cdots A_{r-1}^{k_{r-1}}.$$

To extend the previous theorem to the infinite-dimensional case, we need some tools from spectral theory for commuting n-tuples of operators. Let $\mathbf{A} = (A_1, A_2, \ldots, A_n)$ be a family of pairwise commuting self-adjoint operators on a Hilbert space $\mathcal{H}$. The following spectral theorem for commuting n-tuples of self-adjoint operators asserts:

Theorem 2.1. [23, Theorem 5.23] *There exists a unique spectral measure E defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ such that, for each $k = 1, \ldots, n$,*

$$A_k = \int_{\sigma(\mathbf{A})} t_k \, dE(t_1, \ldots, t_n),$$

where $\sigma(\mathbf{A}) \subseteq \mathbb{R}^n$ is the joint spectrum of the n-tuple $\mathbf{A}$, i.e., the support of the spectral measure E. Moreover, this joint spectrum satisfies

$$\sigma(\mathbf{A}) \subseteq \sigma(A_1) \times \cdots \times \sigma(A_n).$$

A useful characterization of n-tuples of pairwise commuting operators can be viewed as an extension of the joint spectral theorem. This characterization parallels the classical single-operator case established by Halmos [15], which asserts that every cyclic self-adjoint operator is unitarily equivalent to a multiplication operator. The proof we present here follows the same strategy as in Halmos's approach.

Theorem 2.2. *Let $\mathbf{A} = (A_1, A_2, \ldots, A_n)$ be a family of pairwise commuting self-adjoint operators on $\mathcal{H}$, and assume that $\mathbf{A}$ admits a (joint) cyclic vector. Then, there exists a measure space (X, μ), a unitary operator $U : L^2(X, \mu) \to \mathcal{H}$, and measurable real-valued functions $a_k : X \to \mathbb{R}$ such that, for each $k = 1, \ldots, n$, $U^{-1} A_k U = M_{a_k}$, where M_{a_k} is the multiplication operator defined by $(M_{a_k} f)(x) = a_k(x) f(x)$, for all $f \in L^2(X, \mu)$ and almost every $x \in X$.*

Proof. Let E denote the joint spectral measure associated with the n-tuple $(A_1, \ldots, A_n)$, as provided by Theorem 2.1. For $\xi \in \mathcal{H}$, define a positive finite scalar measure μ_ξ on $\mathbb{R}^n$ by

$$\mu_\xi(\Delta) := \langle E(\Delta)\xi, \xi \rangle, \quad \Delta \in \mathcal{B}(\mathbb{R}^n).$$

Define a linear map on the space of complex polynomials in n variables, $\mathcal{P}(\mathbb{R}^n)$, by

$$U_0 : \mathcal{P}(\mathbb{R}^n) \longrightarrow \mathcal{H}, \qquad U_0(p) := p(A_1, \ldots, A_n)\, \xi.$$

For any polynomial $p \in \mathcal{P}(\mathbb{R}^n)$, we have

$$\begin{aligned} \|U_0(p)\|^2 &= \langle p(A_1, \ldots, A_n)\, \xi,\, p(A_1, \ldots, A_n)\, \xi \rangle \\ &= \langle (\overline{p}p)(A_1, \ldots, A_n)\, \xi,\, \xi \rangle \\ &= \int_{\mathbb{R}^n} |p(x_1, \ldots, x_n)|^2 \, d\mu_\xi(x_1, \ldots, x_n). \end{aligned}$$

Hence U_0 is an isometry if we identify the polynomial p with the function $(x_1, \ldots, x_n) \mapsto p(x_1, \ldots, x_n)$ in $L^2(\mathbb{R}^n, \mu_\xi)$. Polynomials in n variables are dense in $L^2(\mathbb{R}^n, \mu_\xi)$ because continuous functions with compact support are dense, and they can be approximated by polynomials on the support of μ_ξ. Thus, the domain of U_0 is dense in $L^2(\mathbb{R}^n, \mu_\xi)$. By continuity, U_0 extends to an isometry $U : L^2(\mathbb{R}^n, \mu_\xi) \longrightarrow \mathcal{H}$.

If ξ is a cyclic vector for the n-tuple $(A_1, \ldots, A_n)$, then the image of U is the closure of $\{\, p(A_1, \ldots, A_n)\, \xi : p \in \mathcal{P}(\mathbb{R}^n) \,\}$, which equals $\mathcal{H}$ by the cyclicity assumption. Therefore, U is a unitary operator.

For any polynomial $p \in \mathcal{P}(\mathbb{R}^n)$ and for each $j \in \{1, \ldots, n\}$, we have

$$\begin{aligned} U^{-1} A_j U(p) &= U^{-1}\big(A_j\, (p(A_1, \ldots, A_n)\, \xi)\big) \\ &= U^{-1}\big((x_j \cdot p)(A_1, \ldots, A_n)\, \xi\big) \\ &= x_j \cdot p. \end{aligned}$$

Since the polynomials form a dense subspace of $L^2(\mathbb{R}^n, \mu_\xi)$, this equality extends by continuity to all of $L^2(\mathbb{R}^n, \mu_\xi)$. Thus, denoting by M_{x_j} the multiplication operator by the j-th coordinate, we obtain

$$U^{-1} A_j U = M_{x_j} \text{ for every } j = 1, \ldots, n.$$

□

Remark 2.1. *In the previous theorem, when no single cyclic vector exists, we construct a decomposition into cyclic subspaces. That is, there exists a family of closed subspaces $\{H_m\}_{m \in \mathbb{N}}$ such that*

$$\mathcal{H} = \bigoplus_{n \in \mathbb{N}} H_m,$$

where each H_m is reducing for the algebra $\mathcal{A} := C^*(A_1, \ldots, A_n)$ and admits a cyclic vector $\xi_m \in H_m$. Take a dense sequence $(e_k)_{k \geq 1}$ in $\mathcal{H}$, since $\mathcal{H}$ is separable Hilbert space. Inductively define

$$H_1 := \overline{\mathcal{A}e_1}, \quad H_m := \overline{\mathcal{A}e_{k_m}}, \ m \geq 2,$$

where e_{k_m} is the first vector not in $H_1 \oplus \cdots \oplus H_{m-1}$. Each H_m is nonzero, reducing, and cyclic, and the sum is dense:

$$\mathcal{H} = \overline{\bigoplus_{m \in \mathbb{N}} H_m} = \bigoplus_{m \in \mathbb{N}} H_m.$$

For each H_m with cyclic vector ξ_m, define the scalar measure μ_m and unitary $U_m : L^2(\mathbb{R}^n, \mu_m) \to H_m$ as in the cyclic case. Let $\mu := \sum_{m \in \mathbb{N}} \mu_m$ be the joint spectral measure. There is a natural isomorphism

$$\bigoplus_{m \in \mathbb{N}} L^2(\mathbb{R}^n, \mu_m) \simeq L^2(\mathbb{R}^n, \mu),$$

given by $\Phi(f_1 \oplus f_2 \oplus \dots) := \sum_{m \in \mathbb{N}} f_m$, where the sum is well-defined μ-almost everywhere because the measures μ_m are mutually singular on their supports. Define

$$V : \bigoplus_{m \in \mathbb{N}} L^2(\mathbb{R}^n, \mu_m) \longrightarrow \mathcal{H}, \quad V(f_1 \oplus f_2 \oplus \dots) := \sum_{m \in \mathbb{N}} U_m f_m.$$

Then the map

$$U = V \circ \Phi^{-1} : L^2(\mathbb{R}^n, \mu) \longrightarrow \mathcal{H},\ U(f) := \sum_{m \in \mathbb{N}} U_m f_m, \text{ with } \Phi^{-1}(f) = (f_m)_{m \in \mathbb{N}}$$

is unitary. Moreover, for each $k = 1, \dots, n$,

$$U^{-1} A_k U = M_{x_k} \quad \text{on } L^2(\mathbb{R}^n, \mu).$$

3. An Operator-valued Generalization of a Combinatorial Formula

In the following, we consider a sequence of vectors $(u_n)_{n \geq 0} \subseteq \mathcal{H}$ and a sequence of operators $(T_n)_{n \geq 0} \subseteq \mathbf{B}(\mathcal{H})$, satisfying, respectively, the following vector recurrence relation and the operator recurrence relation of order $r \geq 2$:

$$u_{n+r} = A_0 u_n + A_1 u_{n+1} + \cdots + A_{r-1} u_{n+r-1}, \quad \forall n \geq 0, \tag{3.6}$$

$$T_{n+r} = A_0 T_n + A_1 T_{n+1} + \cdots + A_{r-1} T_{n+r-1}, \quad \forall n \geq 0, \tag{3.7}$$

where the coefficients $\mathbf{A} = (A_0, A_1, \dots, A_{r-1})$ are bounded, self-adjoint, and pairwise commuting operators, with $A_{r-1} \neq 0_{\mathcal{H}}$.

These recurrence relations play a central role in the study of discrete-time linear systems in both finite- and infinite-dimensional settings. The vector recurrence (3.6) generalizes classical linear recurrences for scalar and vector sequences, whereas the operator recurrence (3.7) provides a natural framework for analyzing sequences of operators in Hilbert spaces.

We first consider the vector recurrence (3.6) as a preliminary step to study the operator recurrence (3.7), since (3.7) is more general. Indeed, once a solution of (3.6) is known, the operator case can be addressed by evaluating each operator T_n on vectors $h \in \mathcal{H}$, i.e., $u_n = T_n h$. This observation reduces the study of the operator sequence to a family of vector sequences, which can then be analyzed using classical techniques. By introducing the operator companion matrix

$$\mathbf{B} = \begin{pmatrix} A_{r-1} & A_{r-2} & \cdots & A_1 & A_0 \\ I_{\mathcal{H}} & 0 & \cdots & 0 & 0 \\ 0 & I_{\mathcal{H}} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & I_{\mathcal{H}} & 0 \end{pmatrix} \in \mathbf{B}(\mathcal{H}^{(r)}), \tag{3.8}$$

we can rewrite the vector recurrence (3.6) as a first-order system in $\mathcal{H}^{(r)}$:

$$Y_{n+1} = \mathbf{B} Y_n, \quad Y_n := \begin{pmatrix} u_{n+r-1} \\ u_{n+r-2} \\ \vdots \\ u_n \end{pmatrix} \in \mathcal{H}^{(r)}.$$

This formulation naturally expresses the evolution of the sequence through the powers of $\mathbf{B}$:

$$Y_n = \mathbf{B}^n Y_0, \quad Y_0 := \begin{pmatrix} u_{r-1} \\ u_{r-2} \\ \vdots \\ u_0 \end{pmatrix}.$$

Thus, the companion operator matrix $\mathbf{B}$ provides a natural framework linking higher-order recurrences to first-order dynamical systems, while $\mathbf{B}^n$ plays a central role in deriving explicit combinatorial and spectral formulas for both scalar and operator sequences [1, 4, 6].

However, computing the powers of the companion operator matrix $\mathbf{B}$ directly is highly nontrivial: there is no general algorithm for $\mathbf{B}^n$ when the entries are noncommutative operators, especially in infinite-dimensional Hilbert spaces. Therefore, it is essential to develop a method to represent the vectors u_n explicitly in terms of the initial data $u_0, \ldots, u_{r-1}$. Once such a representation is established, the corresponding powers $\mathbf{B}^n$ can be deduced indirectly, providing a systematic approach to the study of both the vector and operator recurrences. The following remark illustrates a key aspect of our approach.

Remark 3.2. *Let $(u_n)_{n\geq 0} \subseteq \mathcal{H}$ be a sequence satisfying the operator-valued recurrence relation (3.6), and let*

$$f_n := U^{-1} u_n \in L^2(X, \mu), \quad n \geq 0,$$

as in Theorem 2.2. Then, the sequence $(f_n)_{n\geq 0} \subseteq L^2(X, \mu)$ satisfies the scalar recurrence relation

$$f_{n+r}(x) = a_0(x)\, f_n(x) + a_1(x)\, f_{n+1}(x) + \cdots + a_{r-1}(x)\, f_{n+r-1}(x) \tag{3.9}$$

for μ-almost every $x \in X$.

From the scalar combinatorial formula (2.5), we have the following lemma.

Lemma 3.1. *The general solution of (3.9) satisfies, for every $n \geq r$,*

$$f_n = \sum_{s=0}^{r-1} \rho(n-s, r; \cdot)\, w_s, \quad \mu\text{-a.e.},$$

where

$$w_s := \sum_{j=s}^{r-1} a_j\, f_{s+r-1-j}, \quad s = 0, \ldots, r-1,$$

and the coefficients $\rho(m, r)$ are given by the combinatorial formula

$$\rho(n, r; \cdot) := \sum_{\substack{\mathbf{k} \in \mathbb{Z}_+^r \\ \langle \mathbf{k}, \mathbf{d} \rangle = n-r}} \binom{|\mathbf{k}|}{\mathbf{k}} a_0^{k_0} a_1^{k_1} \cdots a_{r-1}^{k_{r-1}},$$

with the conventions

$$\rho(r, r; \cdot) = 1, \qquad \rho(m, r, ; \cdot) = 0 \quad \text{for } m < r.$$

We derive the following result.

Theorem 3.3. *Let $(u_n)_{n\geq 0} \subseteq \mathcal{H}$ be a sequence satisfying the vector-valued recurrence relation (3.6). Then, for all $n \geq r$, the following explicit expression holds:*

$$u_n = \sum_{s=0}^{r-1} \rho(n-s, r; \mathbf{A})\, W_s,$$

where:

- *the operator coefficients $\rho(n, r; \mathbf{A})$ are given by*

$$\rho(m, r; \mathbf{A}) := \sum_{\substack{\mathbf{k}\in\mathbb{Z}_+^r \\ \langle \mathbf{k},\mathbf{d}\rangle = m-r}} \binom{|\mathbf{k}|}{\mathbf{k}} \mathbf{A}^{\mathbf{k}}, \tag{3.10}$$

with the multi-index notation $\mathbf{A}^{\mathbf{k}} = A_0^{k_0} \cdots A_{r-1}^{k_{r-1}}$, and the conventions

$$\rho(r, r; \mathbf{A}) = I_{\mathcal{H}} \quad \text{and} \quad \rho(m, r; \mathbf{A}) = 0_{\mathcal{H}} \quad \text{for } m < r. \tag{3.11}$$

- *The vectors $W_s \in \mathcal{H}$ are defined by*

$$W_s := \sum_{j=s}^{r-1} A_j\, u_{s+r-1-j} \quad \text{for } s = 0, \ldots, r-1.$$

Proof. Under the assumption, the sequence $f_n = U^{-1}u_n$ satisfies equation (3.9). From Lemma 3.1, we have for every $n \geq r$:

$$f_n = \sum_{s=0}^{r-1} \rho(n-s, r; \cdot)\, w_s, \quad \mu\text{-a.e}\ ,$$

where

$$w_s := \sum_{j=s}^{r-1} a_j\, f_{s+r-1-j}, \quad s = 0, \ldots, r-1,$$

and the coefficients $\rho(n, r; \cdot)$ are defined by the combinatorial formula:

$$\rho(n, r; \cdot) := \sum_{\substack{\mathbf{k}\in\mathbb{Z}_+^r \\ \langle \mathbf{k},\mathbf{d}\rangle = n-r}} \binom{|\mathbf{k}|}{\mathbf{k}} a_0^{k_0} a_1^{k_1} \cdots a_{r-1}^{k_{r-1}},$$

with the conventions

$$\rho(r, r; \cdot) = 1, \quad \text{and} \quad \rho(m, r; \cdot) = 0 \quad \text{for } m < r.$$

Now, define the vectors in $\mathcal{H}$:

$$W_s := \sum_{j=s}^{r-1} A_j u_{s+r-1-j} = \sum_{j=s}^{r-1} A_j U f_{s+r-1-j}.$$

Applying U^{-1}, we get

$$U^{-1}W_s = \sum_{j=s}^{r-1} M_{a_j} f_{s+r-1-j} = w_s.$$

Hence, for every $n \geq r$,

$$f_n = \sum_{s=0}^{r-1} \rho(n-s, r; \cdot) U^{-1} W_s.$$

Applying U to both sides yields

$$u_n = Uf_n = \sum_{s=0}^{r-1} U\rho(n-s,r;\cdot)U^{-1}W_s = \sum_{s=0}^{r-1} \rho(n-s,r;\mathbf{A})\,W_s,$$

where the operator $\rho(n-s,r;\mathbf{A})$ is defined by functional calculus as

$$\begin{aligned}\rho(n-s,r;\mathbf{A}) &:= U\rho(n-s,r;\cdot)U^{-1}\\ &= \rho(n,r;\cdot)\\ &= \sum_{\substack{\mathbf{k}\in\mathbb{Z}_+^r\\ \langle\mathbf{k},\mathbf{d}\rangle=n-r}} \binom{|\mathbf{k}|}{\mathbf{k}} Ua_0^{k_0}a_1^{k_1}\cdots a_{r-1}^{k_{r-1}}U^{-1}\\ &= \sum_{\substack{\mathbf{k}\in\mathbb{Z}_+^r\\ \langle\mathbf{k},\mathbf{d}\rangle=n-r}} \binom{|\mathbf{k}|}{\mathbf{k}} \mathbf{A}^{\mathbf{k}}\end{aligned}$$

and $\rho(r,r;\mathbf{A}) = I_{\mathcal{H}}$, $\rho(m,r;\mathbf{A}) = 0_{\mathcal{H}}$ for $m < r$. This concludes the proof.

□

As a corollary, we obtain the following result, which extends [3, Proposition 2.1] to the setting of operator algebras on infinite-dimensional Hilbert spaces.

Theorem 3.4. *Let $(T_n)_{n\geq 0} \subseteq \mathcal{B}(\mathcal{H})$ be a sequence of bounded operators satisfying the operator-valued linear recurrence relation (3.7) of order r. Then, for all $n \geq r$, the following explicit expression holds:*

$$T_n = \sum_{s=0}^{r-1} \rho(n-s,r;\mathbf{A})\,W_s,$$

where

- *The operator coefficients $\rho(m,r;\mathbf{A})$ are defined as in (3.10) and (3.11).*
- *The operators $W_s \in \mathbf{B}(\mathcal{H})$ are defined by*

$$W_s := \sum_{j=s}^{r-1} A_j\,T_{s+r-1-j} \quad \textit{for } s = 0,\ldots,r-1.$$

Proof. For each $h \in \mathcal{H}$, we apply Theorem 3.3 to the sequence $(u_n)_{n\geq 0}$ defined by $u_n := T_n h$. □

3.1. **Computation of powers of the operator-valued companion matrix.** Let $(u_n)_{n\geq 0} \subseteq \mathcal{H}$ satisfy the recurrence (3.6). To compute the powers of the operator-valued companion matrix $\mathbf{B}$, which is the block companion matrix given in (3.8), recall the state vector

$$Y_n := \begin{pmatrix} u_{n+r-1}\\ u_{n+r-2}\\ \vdots\\ u_n\end{pmatrix} \in \mathcal{H}^{(r)}$$

satisfying

$$Y_n = \mathbf{B}^n\,Y_0, \quad n \in \mathbb{N}.$$

When the operators $A_0, \dots, A_{r-1}$ commute, Theorem 3.3 provides an explicit combinatorial formula for the solution of the system:

$$u_n = \sum_{s=0}^{r-1} \rho(n-s, r; \mathbf{A})\, W_s,$$

where

$$W_s := \sum_{j=s}^{r-1} A_j\, u_{s+r-1-j}, \qquad s = 0, \dots, r-1.$$

Theorem 3.5 (Entries of $\mathbf{B}^n$). *Under the previous notations, for all $n \geq 0$, the entries of $\mathbf{B}^n = (B_{i,k}^{(n)})_{0 \leq i,k \leq r-1}$ are given by*

$$B_{i,k}^{(n)} = \sum_{s=0}^{r-1} \rho(n+i-s, r; \mathbf{A})\, C_{s,k},$$

where

$$C_{s,k} = \begin{cases} A_{s+k}, & 0 \leq s+k \leq r-1, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. The general solution of the recurrence is

$$u_n = \sum_{s=0}^{r-1} \rho(n-s, r; \mathbf{A})\, W_s.$$

Setting $k = j - s$ gives $W_s = \sum_{k=0}^{r-1} C_{s,k}\, u_{r-1-k}$ with

$$C_{s,k} = \begin{cases} A_{s+k}, & 0 \leq s+k \leq r-1, \\ 0, & \text{otherwise}. \end{cases}$$

Then, for $i = 0, \dots, r-1$,

$$u_{n+i} = \sum_{k=0}^{r-1} \Big(\sum_{s=0}^{r-1} \rho(n+i-s, r; \mathbf{A})\, C_{s,k} \Big) u_k,$$

which yields the entries of $\mathbf{B}^n$ as

$$B_{i,k}^{(n)} = \sum_{s=0}^{r-1} \rho(n+i-s, r; \mathbf{A})\, C_{s,k}.$$

□

Remark 3.3. *This result generalizes both versions: the matrix case* [1, Theorem 4.1] *for constant matrix coefficients, and the scalar case given by Chen-Louck's Theorem* [6, Theorem 3.1].

4. The Case of Scalar Coefficients

We now focus on the recurrence relations given respectively by (3.6) and (3.7) where the coefficients are scalar multiples of the identity, i.e.,

$$A_k = a_k I_{\mathcal{H}}, \quad a_k \in \mathbb{R}, \quad k = 0, \dots, r-1,$$

so that the recurrence takes the form

$$u_{n+r} = a_0 u_n + a_1 u_{n+1} + \cdots + a_{r-1} u_{n+r-1}, \quad \forall n \geq 0, \tag{4.12}$$

$$T_{n+r} = a_0 T_n + a_1 T_{n+1} + \cdots + a_{r-1} T_{n+r-1}, \quad \forall n \geq 0. \tag{4.13}$$

In this case, if $\lambda_1, \dots, \lambda_s$ are the distinct roots of the associated characteristic polynomial $P(X) = X^r - a_{r-1}X^{r-1} - \cdots - a_0$, with respective multiplicities $m_1, \dots, m_s$, we can write a general Binet-type formula for the sequence $(u_n)_{n\geq 0}$, analogous to the classical scalar case, as follows:

Theorem 4.6 (Explicit Binet Formula for Vector-Valued Recurrence). *Let $(u_n)_{n\geq 0} \subseteq \mathcal{H}$ satisfy the scalar-coefficient recurrence* (4.12). *Then there exist unique vectors $v_{i,j} \in \mathcal{H}$, indexed by $1 \leq i \leq s$ and $0 \leq j \leq m_i - 1$, such that*

$$u_n = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} v_{i,j}\, n^j\, \lambda_i^n,$$

where the vectors $v_{i,j}$ are determined by the initial values $u_0, \dots, u_{r-1}$.

Proof. For every nonzero $x \in \mathcal{H}$, the scalar sequence $(\langle u_n, x\rangle)_{n\geq 0}$ satisfies the scalar recurrence relation associated with (3.6). Then, there exist unique scalars $\beta_{i,j}(x) \in \mathbb{C}$, indexed by $1 \leq i \leq s$ and $0 \leq j \leq m_i - 1$, such that

$$\langle u_n, x\rangle = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} \beta_{i,j}(x)\, n^j\, \lambda_i^n.$$

By uniqueness, the mapping $\beta_{i,j} : \mathcal{H} \to \mathbb{C}$ is a linear form, then from the Riesz Representation Theorem, there exists a unique vector $v_{i,j} \in \mathcal{H}$ such that

$$\beta_{i,j}(x) = \langle v_{i,j}, x\rangle \quad \text{(for all } x \in \mathcal{H}).$$

In particular, for all $x \in \mathcal{H}$, we have

$$\langle u_n, x\rangle = \left\langle \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} v_{i,j}\, n^j\, \lambda_i^n,\, x \right\rangle,$$

which implies the following identity in $\mathcal{H}$:

$$u_n = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} v_{i,j}\, n^j\, \lambda_i^n.$$

□

As a corollary, we obtain the following theorem for operators.

Theorem 4.7 (Explicit Binet Formula for an operator-valued recurrence). *Let $(T_n)_{n\geq 0} \subseteq \mathbf{B}(\mathcal{H})$ satisfy the scalar-coefficient recurrence* (4.13). *Then, there exist unique operators $S_{i,j} \in \mathcal{H}$, indexed by $1 \leq i \leq s$ and $0 \leq j \leq m_i - 1$, such that*

$$T_n = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} S_{i,j}\, n^j\, \lambda_i^n,$$

where the operators $S_{i,j} \in \mathcal{B}(\mathcal{H})$ are uniquely determined by the initial values $T_0, \ldots, T_{r-1}$.

Proof. Fix any $h \in \mathcal{H}$, and define the sequence of vectors $u_n := T_n(h) \in \mathcal{H}$. Then, by Theorem 4.6, there exist unique vectors $\beta_{i,j}(h) \in \mathcal{H}$ (for $1 \leq i \leq s$, $0 \leq j \leq m_i - 1$) such that $T_n(h) = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} \beta_{i,j}(h)\, n^j\, \lambda_i^n$. For each pair (i,j), define the operator $S_{i,j} : \mathcal{H} \to \mathcal{H}$ by $S_{i,j}(h) := \beta_{i,j}(h)$. By uniqueness, each $S_{i,j}$ is a bounded linear operator, since it arises from a linear combination of the bounded operators $T_0, \ldots, T_{r-1}$ and the roots λ_i. $\square$

Under the assumptions of the previous theorem, if the characteristic polynomial has simple roots $\{\lambda_1, \ldots, \lambda_r\}$, then:

Corollary 4.1. *The sequence $(T_n)_{n\geq 0}$ given by (4.13) consists of the moments of a finitely atomic representing operator measure $E = \sum_{i=1}^{r} S_i\, \delta_{\lambda_i}$, where each $S_i \in \mathbf{B}(\mathcal{H})$ and δ_{λ_i} is the Dirac measure concentrated at the atom λ_i. That is, $T_n = \sum_{i=1}^{r} S_i\, \lambda_i^n = \int_{\mathbb{R}} t^n\, dE(t), \quad \forall n \geq 0.$*

Remark 4.4. *Under the previous notation, the sequence $(T_n)_{n\geq 0}$ is an operator moment sequence on $\{\lambda_1, \ldots, \lambda_r\}$ (i.e., the operators S_i are positive on $\mathcal{H}$) if and only if the representing operator measure is positive. This is equivalent to the positive semidefiniteness of the local moment matrix*

$$\big(\langle T_{i+j}x, x\rangle\big)_{0\leq i,j\leq r-1} \in \mathbf{M}_r(\mathbb{R})$$

for every $x \in \mathcal{H}$ (see [9, Theorem 6.10]).

4.1. **Algebraic operators.**

4.1.1. *Powers of algebraic operators.* Recall that a bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ is said to be algebraic if there exists a non-zero polynomial $P \in \mathbb{C}[X]$ such that $P(T) = 0_{\mathcal{H}}$. Let T be an algebraic operator and let P be the associated monic minimal polynomial of degree r, given by

$$P(X) = X^r - a_{r-1}X^{r-1} - \cdots - a_1 X - a_0,$$

with real coefficients $a_0, \ldots, a_{r-1} \in \mathbb{R}$. Then T satisfies the polynomial identity

$$P(T) = 0_{\mathcal{H}} \quad \Longleftrightarrow \quad T^{n+r} = a_{r-1}T^{n+r-1} + \cdots + a_1 T^{n+1} + a_0 T^n, \quad \forall n \geq 0.$$

This identity induces an operator-valued linear recurrence relation of order r for the sequence of powers $(T^n)_{n\geq 0}$, with operator coefficients $A_k = a_k I_{\mathcal{H}}$ for $k = 0, \ldots, r-1$. This recurrence can be directly identified with the scalar case by applying the scalar product.

Corollary 4.2. *Under the above assumptions, if the characteristic polynomial*

$$P(X) = X^r - a_{r-1}X^{r-1} - \cdots - a_1 X - a_0$$

has roots λ_i with multiplicities m_i, then for every $n \geq r$, the powers of T admit the two following representations:

$$\begin{aligned} T^n &= \sum_{s=0}^{r-1} \rho(n-s, r)\, W_s \quad \textit{(Combinatorial representation)} \\ &= \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} C_{i,j}\, n^j\, \lambda_i^n, \quad \textit{(Binet-type representation)} \end{aligned}$$

where the scalar coefficients $\rho(m,r)$ are defined as in (2.5), the operators $W_s \in \mathcal{B}(\mathcal{H})$ are given by

$$W_s := \sum_{j=s}^{r-1} a_j\, T^{s+r-1-j}, \quad s = 0, \dots, r-1,$$

and the operators $C_{i,j} \in \mathcal{B}(\mathcal{H})$ depend only on the initial terms $T^0, \dots, T^{r-1}$.

4.1.2. *Exponentials of algebraic operators.* In order to compute the exponential of algebraic operators, we need the Binet expression and some tools from the theory of Bell polynomials. Recall that the Bell number B_n counts the number of partitions of a set with n elements. Dobinski's formula provides an explicit representation of the Bell numbers:

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

The Stirling number of the second kind with parameters j and k, denoted $S(j,k)$, enumerates the number of partitions of a set with j elements into k disjoint, nonempty subsets. In particular, the Bell numbers can be expressed as

$$B_n = \sum_{k=0}^{n} S(n,k).$$

The numbers $S(n,k)$ are also called Stirling partition numbers. More generally, the n-th Bell polynomial is defined by

$$B_n(x) = \sum_{k=0}^{n} S(n,k)\, x^k. \tag{4.14}$$

These numbers and polynomials have many remarkable properties and appear in several combinatorial identities. A comprehensive reference is [8]. More recently, the authors in [20] generalized the Bell numbers and polynomials to the so-called r-Bell numbers and polynomials, as follows:

Theorem 4.8 (Dobinski's formula for r-Bell numbers [20, Theorem 5.1])**.** *The r-Bell polynomials satisfy the identity*

$$B_{n,r}(x) = \frac{1}{e^x} \sum_{k=0}^{\infty} \frac{(k+r)^n}{k!}\, x^k.$$

Consequently, the r-Bell numbers are given by

$$B_{n,r} = \frac{1}{e} \sum_{k=0}^{\infty} \frac{(k+r)^n}{k!}.$$

Proposition 4.1 (Exponential of T)**.** *Under the above assumptions, the exponential of T can be expressed explicitly as*

$$e^T = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} C_{i,j}\, e^{\lambda_i} P_j(\lambda_i),$$

where $P_j(\lambda)$ is the Bell polynomial given explicitly by (4.14).

Proof. By definition, the exponential of T is given by the series

$$e^T := \sum_{n=0}^{\infty} \frac{T^n}{n!}.$$

Substituting the Binet-type formula for T^n yields

$$e^T = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} C_{i,j} \sum_{n=0}^{\infty} \frac{n^j \lambda_i^n}{n!}.$$

Thus, it remains to compute the scalar series $\sum_{n=0}^{\infty} \frac{n^j \lambda^n}{n!}$. This is a special case of Theorem 4.8 with $r = 0$: for each $j \geq 0$,

$$\sum_{n=0}^{\infty} \frac{n^j \lambda^n}{n!} = e^{\lambda} P_j(\lambda).$$

Hence, we obtain the explicit formula

$$e^T = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} C_{i,j}\, e^{\lambda_i} P_j(\lambda_i),$$

which completes the proof. □

Remark 4.5. *These results illustrate the interplay between combinatorial techniques and operator theory, providing effective tools for analyzing operator sequences in Hilbert spaces.*

5. Continuous-Time Operator Recurrence

We consider next an application to the continuous-time analogue of the discrete operator recurrence studied in this paper. Such systems are classical in the study of linear evolution equations in Hilbert spaces. More precisely $u : \mathbb{R} \longrightarrow \mathcal{H}$, satisfying a linear operator differential equation of order r:

$$\frac{d^r}{dt^r}u(t) = A_0 u(t) + A_1 \frac{d}{dt}u(t) + \cdots + A_{r-1}\frac{d^{r-1}}{dt^{r-1}}u(t), \tag{5.15}$$

where $A_0, \ldots, A_{r-1} \in \mathcal{B}(\mathcal{H})$ are bounded operator coefficients.

As in the discrete case, we introduce the state vector in decreasing order of derivatives:

$$Y(t) := \begin{pmatrix} \frac{d^{r-1}}{dt^{r-1}}u(t) \\ \frac{d^{r-2}}{dt^{r-2}}u(t) \\ \vdots \\ u(t) \end{pmatrix} \in \mathcal{H}^{(r)},$$

and rewrite (5.15) as a first-order operator differential system:

$$\frac{d}{dt}Y(t) = \mathbf{B}\, Y(t), \quad Y(0) = Y_0 \in \mathcal{H}^{(r)},$$

where $\mathbf{B} \in \mathcal{B}(\mathcal{H}^{(r)})$ is the operator-valued companion matrix given by (3.8). Formally, the solution is given by the exponential of the operator matrix:

$$Y(t) = e^{t\mathbf{B}}\, Y_0.$$

In other words: $u(t) = \begin{bmatrix} 0 & 0 & \cdots & I_{\mathcal{H}} \end{bmatrix} e^{t\mathbf{B}}\, Y_0$, where $\begin{bmatrix} 0 & \cdots & I_{\mathcal{H}} \end{bmatrix}$ is the projection from $\mathcal{H}^{(r)}$ onto the last component. From this formulation naturally raises the following question.

Open question 2. *How can we extend the combinatorial techniques and explicit formulas for powers of the companion matrix, developed in the discrete-time commuting case, to the continuous-time operator setting? In particular, can we obtain a closed-form expression for $e^{t\mathbf{B}}$?*

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Raúl E. Curto
The University of Iowa
Department of Mathematics
14 MacLean Hall, Iowa City, Iowa 52246, U.S.A.
ORCID: 0000-0002-1776-5080
Email address: raul-curto@uiowa.edu

Abderrazzak Ech-charyfy
Mohammed V University in Rabat
Laboratory of Mathematical Analysis and Applications
Rabat, Morocco
ORCID: 0009-0004-8273-5496
Email address: abderrazzak.echcharyfy@gmail.com, abderrazzak_echcharyfy@um5.ac.ma

Kaissar Idrissi
Mohammed V University in Rabat
Laboratory of Mathematical Analysis and Applications
Rabat, Morocco
ORCID: 0000-0002-3279-4613
Email address: kaissar.idrissi@fsr.um5.ac.ma

El Hassan Zerouali
Mohammed V University in Rabat
Laboratory of Mathematical Analysis and Applications
Rabat, Morocco
ORCID: 0000-0001-6240-7859
Email address: elhassan.zerouali@fsr.um5.ac.ma