MATI, ISSN:2636-7785
http://dergipark.gov.tr/mati

MATI **1** (2019), 21–24.

ON SIZE, ORDER, DIAMETER AND VERTEX-CONNECTIVITY

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Let G be a finite connected graph. We give an asymptotically sharp upper bound on the size of G in terms of its order, diameter and vertex-connectivity. The result is a strengthening of an old classical theorem of Ore [**5**] if vertex-connectivity is prescribed and constant.

1. Introduction

Let G be a finite connected graph with vertex set $V(G)$ and edge set $E(G)$. We denote the order of G by n and the size by m. The *distance*, $d_G(u, v)$, between vertices u and v in G is the length of a shortest $u-v$ path in G. The *eccentricity* of a vertex $v \in V(G)$ is the maximum distance between v and any other vertex in G. The *degree*, deg v, of a vertex v of G is the number of edges incident with it, and the diameter of G, d, is $\max\{d_G(u, v) : u, v \in V\}$, whilst the radius of G, r, is the minimum value of the eccentricities of vertices of G. The vertex-connectivity $\kappa(G)$ of G is defined as the minimum number of vertices whose deletion from G results in a disconnected or trivial graph. We say that G is k-vertex-connected, or simply k-connected, if $\kappa(G) \geq k$.

The diameter, apart from being an interesting graph theoretical parameter, plays an important role in analysing communication networks (see for example [**1**]). In such networks the time delay or signal disgradation for sending a message from one point to another is often proportional to the distance between the two points. The diameter can be used to indicate the worst case performance.

Several bounds on the size of a graph in terms of other graph parameters, for example, order and radius [**3, 6**], order and degree set [**7**], and order and domination

2010 Mathematics Subject Classification. 05C12.
Keywords and Phrases. size, vertex-connectivity, diameter.

number [**2**] have been investigated. An upper bound on the size in terms of order and diameter was determined by Ore [**5**] as early as 1968. Several authors [**6, 7**] have presented simple and short proofs to Ore's theorem. Recently one of the present authors [**4**] reported on the size, order, diameter and minimum degree. In this note we present an upper bound on the size in terms of order, diameter and vertex-connectivity. The bound, for a fixed vertex-connectivity κ, is a strengthening of Ore's theorem [**5**], which we state below

Theorem 1. *Let G be a connected graph of order n, diameter d and size m. Then* $m \leq \frac{1}{2}(n-d-1)(n-d+4)+d = \frac{1}{2}(n-d)^2 + O(n)$.

2. Results

Let G be a finite connected graph of order n, size m and diameter d. From now onwards $v_0 \in V(G)$ is a fixed vertex of eccentricity d and for each $i = 0, 1, 2, 3, \ldots, d$,

$$N_i := \{x \in V(G) | d_G(x, v_0) = i\}.$$

The following result is a strengthening of Ore's theorem if vertex-connectivity is prescribed and constant.

Theorem 2. *Let G be a κ-connected graph of order n, diameter d and size m. Then*

$$m \leq \frac{1}{2}(n - \kappa d)^2 + O(n)$$

and the bound, for fixed κ, is asymptotically tight.

Proof. Assume the notation for v_0 and N_i as above. Note that $|N_i| \geq \kappa$, for all $i = 1, 2, \ldots, d-1$. For each N_i, $i = 1, 2, \ldots, d-1$, choose any κ vertices and let this set be $N_i' = \{u_{i1}, u_{i2}, \ldots, u_{i\kappa}\}$. For each $j = 1, 2, \ldots, \kappa$, let $P_j := \{u_{1j}, u_{2j}, u_{3j}, \ldots, u_{d-1j}\}$ and $N = \cup_{j=1}^{\kappa} P_j$. Then,

$$|N| = \kappa(d-1). \tag{1}$$

Claim 1. *Let N be as above. Then* $\sum_{x \in N} \deg x \leq O(n)$.

Proof of Claim 1: First consider P_j. Partition P_j as follows:
$P_j = U_1 \cup U_2 \cup U_3$, where

$$U_1 = \{u_{1j}, u_{4j}, u_{7j}, \ldots\},$$

$$U_2 = \{u_{2j}, u_{5j}, u_{8j}, \ldots\},$$

and

$$U_3 = \{u_{3j}, u_{6j}, u_{9j}, \ldots\}.$$

Note that for any $x, y \in U_i, i = 1, 2, 3$ we have $N[x] \cap N[y] = \emptyset$. It follows that

$$n \geq |\cup_{x \in U_i} N[x]| = \sum_{x \in U_i} \deg x + |U_i|, \text{for} \quad i = 1, 2, 3.$$

Therefore,

$$\begin{aligned} 3n &\geq \sum_{x \in U_1} \deg x + \sum_{x \in U_2} \deg x + \sum_{x \in U_3} \deg x + |U_1| + |U_2| + |U_3| \\ &= \sum_{x \in P_j} \deg x + |P_j| \end{aligned}$$

Thus, $\sum_{x \in P_j} \deg x \leq 3n - |P_j|$. We conclude that

$$\begin{aligned} \sum_{x \in N} \deg x &= \sum_{j=1}^{\kappa} (\sum_{x \in P_j} \deg x) \\ &\leq \sum_{j=1}^{\kappa} (3n - |P_j|) \\ &\leq 3n\kappa - |N| \\ &= O(n), \end{aligned}$$

as required.

Now let $Q = V - N$. Then from (1)

$$|Q| = n - \kappa(d-1). \tag{2}$$

Claim 2. *Let $x \in Q$. Then* $\deg x \leq n - \kappa d + O(1)$.

Proof of Claim2: Let $x \in Q$. Then x can only be adjacent to vertices from at most 3 of the sets $N_i, i = 1, 2, 3, \ldots, d-1$. Hence x is adjacent to at most 3κ vertices from N. It follows that

$$\begin{aligned} \deg x &\leq |Q| + 3\kappa \\ &= n - \kappa(d-1) + 3\kappa \\ &= n - \kappa d + 4\kappa, \end{aligned}$$

as desired.

By Claim 2, and from (2), we have

$$\begin{aligned} \sum_{x \in Q} \deg x &\leq \sum_{x \in Q} (n - \kappa d + O(1)) \\ &\leq (n - \kappa(d-1))\,(n - \kappa d + O(1)) \\ &= (n - \kappa d)^2 + O(n). \end{aligned}$$

Combining this and Claim 1, we get

$$\begin{aligned}\sum_{x\in V}\deg x &= \sum_{x\in N}\deg x + \sum_{x\in Q}\deg x\\ &\leq (n-\kappa d)^2 + O(n).\end{aligned}$$

It follows, by the Handshaking Lemma that

$$m = \frac{1}{2}\sum_{x\in V}\deg x \leq \frac{1}{2}(n-\kappa d)^2 + O(n).$$

To see that the bound is asymptotically sharp, consider the graph $G_{n,d,\kappa} = G_0 + G_1 + \ldots + G_\kappa$ where $G_i = K_\kappa$ for $i = 0, 1, 2, 3, \cdots, d-1$ and $G_d = K_{n-\kappa d}$. □

Using the counting technique employed in Theorem 2, we obtain the following theorem which is an improvement of Vizing's Theorem [**8**] if vertex-connectivity is prescribed.

Theorem 3. *Let G be a κ-connected graph of order n, radius r and size m. Then*

$$m \leq \frac{1}{2}(n-2r\kappa)^2 + O(n).$$

Moreover, this inequality is, for a fixed κ, asymptotically tight. □

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