CONSTRUCTIVE MATHEMATICAL ANALYSIS
8 (2025), No. SI, pp. 18-38
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

Interchange between Hardy-Lorentz-Karamata spaces of predictable martingales

ZHIWEI HAO, MEI LI, YAO WANG, AND FERENC WEISZ*

ABSTRACT. Applying the martingale transform and K-method of interpolation spaces, we investigate the interchanging relations between Hardy-Lorentz-Karamata spaces of predictable martingales. More precisely, let $0 < p_1 < p_2 < \infty$ and $0 < q_1 \leq q_2 < \infty$ and b be a slowly varying function, it is shown that the elements in Hardy-Lorentz-Karamata spaces $\mathbb{H}_{p_1,q_1,b}$ are none others than the martingale transforms of those in $\mathbb{H}_{p_2,q_2,b}$, where $\mathbb{H}_{p_i,q_i,b} \in \{\mathcal{P}_{p_i,q_i,b}, \mathcal{Q}_{p_i,q_i,b}\}$ for $i = 1, 2$. And it is also proved that a martingale is in $\mathbb{H}_{p,q,b} \in \{\mathcal{P}_{p,q,b}, \mathcal{Q}_{p,q,b}\}$ for $0 < p, q < \infty$, if and only if it is the transform of a martingale from $\mathbb{BMO} \in \{BMO_1, BMO_2\}$.

Keywords: Martingale transform, interpolation space, Hardy-Lorentz-Karamata space.

2020 Mathematics Subject Classification: 46B09, 60B11, 60G46.

1. INTRODUCTION

As is well known, Lorentz spaces play an important role in classical Harmonic analysis [2, 3, 10, 11, 31, 20, 13] and references therein. Lorentz-Karamata spaces, as a generalization of Lorentz spaces and Lorentz-Zygmund spaces, were firstly introduced by Edmunds, Kerman and Pick in [16]. The feature of these spaces is that their construction encapsules both the Lorentz-type structure of function spaces and the concept of so-called slowly varying functions that had been studied by Karamata. We refer the reader to [15, 17, 18, 39, 40] for more information on Lorentz-Karamata spaces.

The definition of martingale was first introduced to probability by Ville in the 1930s, Levy researched the properties of the martingale sequence. Doob systematically studied and summarized the previous results from the perspective of analytics in his famous monograph [14]. Thereafter, some outstanding researchers such as Burkholder, Davis, Gundy [6, 7, 8, 9], Garsia [19] gave a further study in this field. The theory of martingale Hardy spaces, which was regarded as an interdisciplinary field of probability and analysis, flourished over the past few decades. In particular, the analytical properties of these martingale spaces become one of the hot topics. With the development of various function spaces, such as Lebesgue space, Orlicz space, Lorentz space and Lorentz-Karamata space et al., the research of the combinations of these function spaces with the martingale theory has attracted more and more attentions, and some meaningful works were established in [12, 30, 32, 34, 33, 38, 41, 42, 43] and so on.

In 2014, Ho [29] firstly introduced the Karamata theory to martingale spaces and investigated the atomic decompositions, duality and interpolations of martingale Hardy-Lorentz-Karamata spaces. Subsequently, Jiao et al. [35] further studied the duality of the martingale

Received: 10.08.2025; Accepted: 28.11.2025; Published Online: 16.12.2025
*Corresponding author: Ferenc Weisz; weisz@inf.elte.hu
DOI: 10.33205/cma.1818318

Hardy-Lorentz-Karamata spaces via a generalized BMO martingale space, which improved the results in [29]. Wu et al. [44] investigated the modular inequalities in martingale Orlicz-Karamata spaces via the modular atomic decompositions, and Li et al. [36] reconsidered the results in [44] without the constraint that the slowly varying functions b is non-decreasing. We refer to [24, 25, 37] for more information on Lorentz-Karamata martingale spaces.

The main purpose of this paper aims to discuss the interchange between Hardy-Lorentz-Karamata spaces of predictable martingales. Its motivation comes from the classical results of Garsia [19], Chao and Long [12] and Weisz [42]. It is worth mentioning that the main technique we rely on in this paper is martingale transform, which was first introduced by Burkholder [6]. Martingale transform is an analogue of a singular integral in classical harmonic analysis, which is an important tool in stochastic analysis. In addition, we also need the so-called K-method of interpolation space to construct the desired martingale transforms. The results obtained here are regarded as a generalization of the corresponding Hardy-Lorentz predictable martingale spaces by Yu and He [45]. It is shown that the martingale transforms are very useful to research the relations between the "predictable" martingale Hardy-Lorentz spaces, such as $\mathcal{P}_{p,q}$ and $\mathcal{Q}_{p,q}$ (see the definitions in the next section). In [45], the author proved that for $0 < p_1 < p_2 < \infty$ and $0 < q_1 \leq q_2 < \infty$, the elements in Hardy-Lorentz spaces $\mathbb{H}_{p_1,q_1}$ are none others than the martingale transforms of those in Hardy-Lorentz spaces $\mathbb{H}_{p_2,q_2}$, where $\mathbb{H}_{p_i,q_i} \in \{\mathcal{P}_{p_i,q_i}, \mathcal{Q}_{p_i,q_i}\}$ for $i = 1, 2$. We refer to [45] for more information. If $p = q$, then the martingale Hardy-Lorentz spaces $\mathcal{P}_{p,q}$ and $\mathcal{Q}_{p,q}$ return to the martingale Hardy space $\mathcal{P}_p$ and $\mathcal{Q}_p$. The analogue of the interchange between these predictable martingale spaces was obtained by Garsia [19], Weisz [42] and Chao and Long [12], respectively. All of those results can also be found in the monographs of Long [38] and Weisz [43].

The paper is organized as follows. Some preliminaries used in the whole paper will be stated in Section 2. In Section 3, we establish some boundedness of the martingale transform operator on Hardy-Lorentz-Karamata spaces and BMO spaces, which will be used in the next sections. The remaining Sections 4, 5 and 6 are respectively devoted to the characterization by means of martingale transforms about the interchanging relationships between Hardy-Lorentz-Karamata martingale spaces $\mathbb{H}_{p_1,q,b}$ and $\mathbb{H}_{p_2,q,b}$, $\mathbb{H}_{p_1,q_1,b}$ and $\mathbb{H}_{p_2,q_2,b}$, and that between $\mathbb{H}_{p,q,b}$ and BMO, where $\mathbb{H}_{p,q,b} \in \{\mathcal{P}_{p,q,b}, \mathcal{Q}_{p,q,b}\}$. Throughout the paper, we use C to denote a positive constant which may vary from line to line. The symbol $f \lesssim g$ stands for $f \leq Cg$, we write $f \simeq g$ if $f \leq Cg$ and $g \leq Cf$ hold at the same time.

2. Preliminaries

In this section, we present some necessary preliminaries used in the whole paper. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and f be an $\mathcal{F}$-measurable function defined on Ω. The distribution function of f is defined by

$$\lambda_s(f) = \mathbb{P}\big(\{\omega \in \Omega : |f(\omega)| > s\}\big), \ \ (s \geq 0).$$

Denote by f^* the decreasing rearrangement of f defined by

$$f^*(t) = \inf\{s \geq 0 : \lambda_s(f) \leq t\}, \ \ (t \geq 0),$$

with the convention that $\inf \emptyset = \infty$.

Lemma 2.1 ([22, Proposition 1.4.5]). *Let f, g be $\mathcal{F}$-measurable and $0 \leq t,\ t_1,\ t_2 < \infty$. Then the following properties hold:*

(i) $(fg)^*(t_1 + t_2) \leq f^*(t_1)g^*(t_2)$;
(ii) $(|f|^p)^*(t) = (f^*(t))^p$ *when* $0 < p < \infty$.

2.1. **Slowly varying functions.** We recall the definition of slowly varying functions in order to define the Lorentz-Karamata spaces.

Definition 2.1. *A Lebesgue measurable function $b : [1,\infty) \longrightarrow (0,\infty)$ is said to be a slowly varying function if for any given $\epsilon > 0$, the function $t^{\epsilon}b(t)$ is equivalent to a non-decreasing function and the function $t^{-\epsilon}b(t)$ is equivalent to a non-increasing function on $[1,\infty)$.*

The detailed study of Karamata theory, properties and examples of slowly varying functions can be found in [5, 16].

Let b be a slowly varying function on $[1,\infty)$. We denote by γ_b the positive function defined by

$$\gamma_b(t) = b(\max\{t, 1/t\}),\ t \in (0,\infty).$$

Lemma 2.2 ([15]). *Let b be a slowly varying function. Then the following conclusions hold.*

- (i) *For any given $\epsilon > 0$, the function $t^{\epsilon}\gamma_b(t)$ is equivalent to a non-decreasing function and the function $t^{-\epsilon}\gamma_b(t)$ is equivalent to a non-increasing function on $(0,\infty)$.*
- (ii) *If $a > 0$, then for any $t > 0$,*

$$\int_0^t s^{a-1}\gamma_b(s)ds \simeq t^a\gamma_b(t)$$

and

$$\int_t^\infty s^{-a-1}\gamma_b(s)ds \simeq t^{-a}\gamma_b(t).$$

- (iii) *For any $r \in \mathbb{R}$, b^r is a slowly varying function and $\gamma_{b^r} = \gamma_b^r$.*
- (iv) *If ε and r are positive numbers, then there exists positive constants c_ε and C_ε such that*

$$c_\varepsilon \min\{r^\varepsilon, r^{-\varepsilon}\}b(t) \le b(rt) \le C_\varepsilon \max\{r^\varepsilon, r^{-\varepsilon}\}b(t), \quad t > 0.$$

- (v) *For any $a > 0$, denote $b_1(t) = b(t^a)$ on $[1,\infty)$, then b_1 is also a slowly varying function.*

2.2. **Lorentz-Karamata spaces.** We now recall the definition of the Lorentz-Karamata spaces.

Definition 2.2. *Let $0 < p < \infty$, $0 < q \le \infty$ and b be a slowly varying function. The Lorentz-Karamata space, denoted by $L_{p,q,b}$, consists of those measurable functions f with $\|f\|_{p,q,b} < \infty$, where*

$$\|f\|_{p,q,b} = \begin{cases} \left(\int_0^\infty \left(t^{1/p}\gamma_b(t)f^*(t)\right)^q \frac{dt}{t}\right)^{1/q}, & \text{if } 0 < q < \infty, \\ \sup\limits_{t>0} t^{1/p}\gamma_b(t)f^*(t), & \text{if } q = \infty. \end{cases}$$

Remark 2.1.

(1) *It will be convenient for us to use an equivalent quasi-norm definition for $\|f\|_{p,q,b}$, namely*

$$\|f\|_{p,q,b} \approx \begin{cases} \left(\int_0^\infty \left(t\mathbb{P}(|f| > t)^{1/p}\gamma_b\big(\mathbb{P}(|f| > t)\big)\right)^q \frac{dt}{t}\right)^{1/q}, & \text{if } 0 < q < \infty, \\ \sup\limits_{t>0} t\mathbb{P}(|f| > t)^{1/p}\gamma_b(\mathbb{P}(|f| > t)), & \text{if } q = \infty. \end{cases}$$

(2) *The Lorentz-Karamata space is a rearrangement-invariant (r.-i.) quasi-Banach function space (see [28]). When $1 \le p, q < \infty$, $L_{p,q,b}$ is a Banach space (see [15]). Note that if $b \equiv 1$, the space $L_{p,q,b}$ is the classical Lorentz space $L_{p,q}$. Also, if $p = q$ and $b \equiv 1$, then the space $L_{p,q,b}$ is the usual Lebesgue space L_q.*

Lemma 2.3 ([40, Proposition 3.16]). *Let b_1, b_2 be two slowly varying functions such that $b_2 \lesssim b_1$, let $0 < p \leqslant \infty$ and suppose $0 < q < r \leqslant \infty$. Then*

$$\|\cdot\|_{p,r,b_2} \lesssim \|\cdot\|_{p,q,b_1}.$$

Lemma 2.4 ([15, Theorem 3.4.48]). *Set $p_1, p_2 \in (0,\infty)$, $q_1, q_2 \in (0,\infty]$ with $p_2 < p_1$ and let b_1, b_2 be slowly varying functions. Then*

$$L_{p_1,q_1,b_1} \subset L_{p_2,q_2,b_2}.$$

Lemma 2.5. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s > 0$ and b be a slowly varying function. For any $f \in L_{sp,sq,b^{1/s}}$, we have $\||f|^s\|_{p,q,b} = \|f\|^s_{sp,sq,b^{1/s}}$.*

Proof. For any $s > 0$ and $0 < q < \infty$, it follows from Lemma 2.2 (iii) and Lemma 2.1 (ii) that

$$\begin{aligned}
\||f|^s\|_{p,q,b} &= \Big(\int_0^\infty \big(t^{1/p}\gamma_b(t)(|f|^s)^*(t)\big)^q \frac{dt}{t}\Big)^{1/q} \\
&= \Big(\int_0^\infty \big(t^{1/p}\gamma_b(t)(f^*)^s(t)\big)^q \frac{dt}{t}\Big)^{1/q} \\
&= \bigg(\Big(\int_0^\infty \big(t^{1/sp}\gamma_b^{1/s}(t)f^*(t)\big)^{sq} \frac{dt}{t}\Big)^{1/sq}\bigg)^s \\
&= \bigg(\Big(\int_0^\infty \big(t^{1/sp}\gamma_{b^{1/s}}(t)f^*(t)\big)^{sq} \frac{dt}{t}\Big)^{1/sq}\bigg)^s \\
&= \|f\|^s_{sp,sq,b^{1/s}}.
\end{aligned}$$

For $q = \infty$, we have

$$\begin{aligned}
\||f|^s\|_{p,\infty,b} &= \sup_{t>0} t^{1/p}\gamma_b(t)(|f|^s)^*(t) = \sup_{t>0} t^{1/p}\gamma_b(t)(f^*)^s(t) \\
&= \Big(\sup_{t>0} t^{1/sp}\gamma_b^{1/s}(t)f^*(t)\Big)^s = \Big(\sup_{t>0} t^{1/sp}\gamma_{b^{1/s}(t)}f^*(t)\Big)^s \\
&= \|f\|^s_{sp,\infty,b^{1/s}}.
\end{aligned}$$

The proof is complete. □

Hölder's inequality for Hardy-Lorentz-Karamata space is given below.

Lemma 2.6. *Let $0 < p,\ q,\ p_1,\ q_1,\ p_2,\ q_2 < \infty$, b, b_1, b_2 be slowly functions and let $\frac{1}{p_1} = \frac{1}{p} + \frac{1}{p_2}$, $\frac{1}{q_1} = \frac{1}{q} + \frac{1}{q_2}$ and $b_1(t) = b(t)b_2(t)$. Then*

$$\|fg\|_{p_1,q_1,b_1} \lesssim \|f\|_{p,q,b}\|g\|_{p_2,q_2,b_2}.$$

Proof. By Lemma 2.1 (i), we have

$$\begin{aligned}
\|fg\|_{p_1,q_1,b_1} &= \left[\int_0^\infty \Big(t^{1/p_1}\gamma_{b_1}(t)(fg)^*(t)\Big)^{q_1} \frac{dt}{t}\right]^{1/q_1} \\
&\leq \left[\int_0^\infty \Big(t^{1/p_1}\gamma_{b_1}(t)f^*(t/2)g^*(t/2)\Big)^{q_1} \frac{dt}{t}\right]^{1/q_1} \\
&= \left[\int_0^\infty \Big((2t)^{1/p_1}\gamma_{b_1}(2t)f^*(t)g^*(t)\Big)^{q_1} \frac{dt}{t}\right]^{1/q_1}.
\end{aligned}$$

Next, from Lemma 2.2 (iv), we get

$$\begin{aligned}
\|fg\|_{p_1,q_1,b_1} &\lesssim \left[\int_0^\infty \left(t^{1/p_1}\gamma_{b_1}(t)f^*(t)g^*(t)\right)^{q_1}\frac{dt}{t}\right]^{1/q_1}\\
&= \left[\int_0^\infty t^{q_1/p_1}\left(\gamma_b(t)\right)^{q_1}\left(\gamma_{b_2}(t)\right)^{q_1}\left(f^*(t)\right)^{q_1}\left(g^*(t)\right)^{q_1}\frac{dt}{t}\right]^{1/q_1}\\
&= \left[\int_0^\infty t^{q_1/p}\left(\gamma_b(t)\right)^{q_1}\left(f^*(t)\right)^{q_1}t^{q_1/p_2}\left(\gamma_{b_2}(t)\right)^{q_1}\left(g^*(t)\right)^{q_1}\frac{dt}{t}\right]^{1/q_1}.
\end{aligned}$$

Since $\frac{q_1}{q}+\frac{q_1}{q_2}=1$, by Hölder's inequality, we have

$$\begin{aligned}
\|fg\|_{p_1,q_1,b_1} &\lesssim \left[\int_0^\infty t^{q_1/p}\left(\gamma_b(t)\right)^{q_1}\left(f^*(t)\right)^{q_1}t^{q_1/p_2}\left(\gamma_{b_2}(t)\right)^{q_1}\left(g^*(t)\right)^{q_1}\frac{dt}{t}\right]^{1/q_1}\\
&\lesssim \Bigg[\left[\int_0^\infty \left(t^{q_1/p}\left(\gamma_b(t)\right)^{q_1}\left(f^*(t)\right)^{q_1}\right)^{q/q_1}\frac{dt}{t}\right]^{q_1/q}\\
&\quad\times\left[\int_0^\infty \left(t^{q_1/p_2}\left(\gamma_{b_2}(t)\right)^{q_1}\left(g^*(t)\right)^{q_1}\right)^{q_2/q_1}\frac{dt}{t}\right]^{q_1/q_2}\Bigg]^{1/q_1}\\
&= \left[\int_0^\infty \left(t^{1/p}\gamma_b(t)f^*(t)\right)^{q}\frac{dt}{t}\right]^{1/q}\left[\int_0^\infty \left(t^{1/p_2}\gamma_{b_2}(t)g^*(t)\right)^{q_2}\frac{dt}{t}\right]^{1/q_2}.
\end{aligned}$$

Thus

$$\|fg\|_{p_1,q_1,b_1} \lesssim \|f\|_{p,q,b}\|g\|_{p_2,q_2,b_2},$$

which completes the proof. □

2.3. **Hardy-Lorentz-Karamata martingale spaces.** Now, we introduce some standard notations which can be referred to [19, 41, 43]. Let $\{\mathcal{F}_n\}_{n\geq 1}$ be a non-decreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}=\sigma(\cup_{n\geq 1}\mathcal{F}_n)$. The expectation operator and the conditional expectation operator relative to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. A sequence of measurable functions $f=(f_n)_{n\geq 1}\subset L_1(\Omega)$ is called a martingale with respect to $\{\mathcal{F}_n\}_{n\geq 1}$ if $\mathbb{E}_n f_{n+1}=f_n$ for every $n\geq 1$. Denote by $\mathcal{M}$ the set of all martingales $f=(f_n)_{n\geq 1}$ relative to $\{\mathcal{F}_n\}_{n\geq 1}$. For $f\in\mathcal{M}$, we define its martingale difference by $d_if=f_i-f_{i-1}$ $(i\geq 1)$, with convention $f_0=0$, $\mathcal{F}_0=\{\emptyset,\Omega\}$. Let $\mathcal{T}$ be the set of all stopping times relative to $\{\mathcal{F}_n\}_{n\geq 1}$. For $f\in\mathcal{M}$ and $\tau\in\mathcal{T}$, the stopped martingale $f^\tau=(f_n^\tau)_{n\geq 1}$ is defined by

$$f_n^\tau=\sum_{i=1}^n \chi_{\{\tau\geq i\}}d_if.$$

Moreover, if $f_n\in L_p$ for $n\geq 1$, then $f=(f_n)_{n\geq 1}$ is called an L_p-martingale with respect to $\{\mathcal{F}_n\}_{n\geq 1}$. In this case, we set

$$\|f\|_p=\sup_{n\geq 1}\|f_n\|_p.$$

If $\|f\|_p<\infty$, then f is called a bounded L_p-martingale and it is denoted by $f\in L_p$.

Define the maximal function, the square function and the conditional square function of a martingale f, respectively, as follows:

$$M_n(f) = \sup_{1\le i\le n} |f_i|, \quad M(f) = \sup_{i\ge 1} |f_i|;$$

$$S_n(f) = \left(\sum_{i=1}^{n} |d_i f|^2\right)^{1/2}, \quad S(f) = \left(\sum_{i=1}^{\infty} |d_i f|^2\right)^{1/2};$$

$$s_n(f) = \left(\sum_{i=1}^{n} \mathbb{E}_{i-1}|d_i f|^2\right)^{1/2}, \quad s(f) = \left(\sum_{i=1}^{\infty} \mathbb{E}_{i-1}|d_i f|^2\right)^{1/2}.$$

Let $0 < p < \infty$, $0 < q \le \infty$ and b be a slowly varying function. We usually define the Hardy-Lorentz-Karamata martingale spaces as below:

$$H^M_{p,q,b} = \{f \in \mathcal{M} : \|f\|_{H^M_{p,q,b}} = \|M(f)\|_{p,q,b} < \infty\};$$
$$H^S_{p,q,b} = \{f \in \mathcal{M} : \|f\|_{H^S_{p,q,b}} = \|S(f)\|_{p,q,b} < \infty\};$$
$$H^s_{p,q,b} = \{f \in \mathcal{M} : \|f\|_{H^s_{p,q,b}} = \|s(f)\|_{p,q,b} < \infty\}.$$

Denote by $\Lambda_{p,q,b}$ the collection of all sequences $\rho = (\rho_n)_{n\ge 0}$ of non-decreasing, non-negative and adapted functions with $\rho_\infty = \lim_{n\to\infty} \rho_n \in L_{p,q,b}$. A martingale $f = (f_n)_{n\ge 1}$ is said to have predictable control in $L_{p,q,b}$ if there exists a sequence $\rho = (\rho_n)_{n\ge 0}$ such that

$$|f_n| \le \rho_{n-1} \quad \text{and} \quad \rho \in \Lambda_{p,q,b},\ n \ge 1.$$

The other two types of Hardy-Lorentz-Karamata spaces for predictable martingales, denoted by $\mathcal{Q}_{p,q,b}$ and $\mathcal{P}_{p,q,b}$, are defined respectively as follows:

$$\mathcal{Q}_{p,q,b} = \{f \in \mathcal{M} : \exists\, \rho = (\rho_n)_{n\ge 0} \in \Lambda_{p,q,b} \text{ s.t. } S_n(f) \le \rho_{n-1}\}$$

with $\|f\|_{\mathcal{Q}_{p,q,b}} = \inf_\rho \|\rho_\infty\|_{p,q,b}$;

$$\mathcal{P}_{p,q,b} = \{f \in \mathcal{M} : \exists\, \rho = (\rho_n)_{n\ge 0} \in \Lambda_{p,q,b} \text{ s.t. } |f_n| \le \rho_{n-1}\}$$

with $\|f\|_{\mathcal{P}_{p,q,b}} = \inf_\rho \|\rho_\infty\|_{p,q,b}$.

For the above definitions, it is clear that $\mathcal{Q}_{p,q,b} \subset H^S_{p,q,b}$ and $\mathcal{P}_{p,q,b} \subset H^M_{p,q,b}$. Furthermore, if $b \equiv 1$, we obtain the definitions of $H^M_{p,q}$, $H^S_{p,q}$, $H^s_{p,q}$, $\mathcal{Q}_{p,q}$ and $\mathcal{P}_{p,q}$, respectively; see [34]. In addition, if $p = q$ and $b \equiv 1$, we obtain the martingale Hardy spaces H^M_p, H^S_p, H^s_p, $\mathcal{Q}_p$ and $\mathcal{P}_p$, respectively; see Weisz [41, 43].

Remark 2.2. *It is obvious that the "*$\inf$*" taken in the $\mathcal{Q}_{p,q,b}$ and $\mathcal{P}_{p,q,b}$ quasi-norms are attainable. Indeed, let $\lambda^k = (\lambda^k_n)_{n\ge 0}$ be a predictable majorant sequence of $(f_n)_{n\ge 1}$ for every $k \in \mathbb{N}$ such that $\|\lambda^k_\infty\| \to \|f\|_{\mathcal{P}_{p,q,b}}$ as $k \to \infty$. Setting $\lambda_n = \inf_k \lambda^k_n$ for all $n \ge 0$, it is clear that $\lambda = (\lambda_n)_{n\ge 0}$ is a predictable majorant sequence of $(f_n)_{n\ge 1}$ and $\|f\|_{\mathcal{P}_{p,q,b}} = \|\lambda_\infty\|_{p,q,b}$. Such a sequence $\lambda = (\lambda_n)_{n\ge 0}$ will be called the predictable least majorant of $(f_n)_{n\ge 1}$ for $f \in \mathcal{P}_{p,q,b}$. The proof of $\mathcal{Q}_{p,q,b}$ is similar.*

We introduce the following definition which will be frequently used in the sequel.

Definition 2.3. *Let $0 < p < \infty$, $0 < q \le \infty$ and b be a slowly varying function. Define the following class of processes $v = (v_n)_{n\ge 0}$ adapted to $\{\mathcal{F}_n\}_{n\ge 1}$ (for each $n \ge 1$, v_n is measurable with respect to $\mathcal{F}_n$; for convention, $v_0 = v_1$) such that*

$$V_{p,q,b} = \{v = (v_n)_{n\ge 0} : \|v\|_{V_{p,q,b}} = \|M(v)\|_{p,q,b} < \infty\},$$

where $M(v) = \sup_{n\geq 0}|v_n|$. *The martingale transform operator* T_v *for given martingale* f *and* $v \in V_{p,q,b}$ *is defined by* $T_v(f) = \big(T_v(f_n)\big)_{n\geq 1}$, *where*

$$T_v(f_n) = \sum_{k=1}^{n} v_{k-1} d_k f, \quad n \geq 1.$$

The following results are useful in this paper.

Lemma 2.7 ([1, Abel's lemma]). *For two arbitrary sequences* $(f_k)_{k\geq 1}$ *and* $(g_k)_{k\geq 1}$, *we have*

$$\sum_{k=m}^{n-1}(f_{k+1}-f_k)g_k = \sum_{k=m}^{n-1} f_{k+1}(g_k - g_{k+1}) + f_n g_n - f_m g_m.$$

Lemma 2.8 ([43, Corollary 2.64]). *The spaces* $\mathcal{P}_p$ *and* $\mathcal{Q}_p$ *are equivalent for all* $0 < p < \infty$, *more exactly,*

$$C_p^{-1}\|f\|_{\mathcal{P}_p} \leq \|f\|_{\mathcal{Q}_p} \leq C_p\|f\|_{\mathcal{P}_p},$$

where C_p *is a positive constant only depend on* p.

Lemma 2.9 ([26, Theorem 3.2]). *Let* $1 < p < \infty$, $0 < q \leq \infty$ *and* b *be a slowly varying function. Then we have*

$$\|f\|_{p,q,b} \leq \|M(f)\|_{p,q,b} \lesssim \|f\|_{p,q,b}, \quad \forall\, f = (f_n)_{n\geq 1} \in L_{p,q,b}.$$

Lemma 2.10. *Let* $0 < \theta < 1$, $0 < p_0 < p_1 \leq \infty$, $0 < q \leq \infty$ *and* b *be a slowly varying function. Then we have*

$$(\mathcal{P}_{p_0}, \mathcal{P}_{p_1})_{\theta,q,b} = \mathcal{P}_{p,q,b_\alpha}$$

and

$$(\mathcal{Q}_{p_0}, \mathcal{Q}_{p_1})_{\theta,q,b} = \mathcal{Q}_{p,q,b_\alpha},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$ *and* $b_\alpha(t) = b(t^{1/\alpha})$.

Proof. The case $p_1 < \infty$ was proved in [29, Theorem 7.12]. The case $p_1 = \infty$ was given in [27, Theorem 6.6]. □

The following lemma was proved by Gustavsson in [23].

Lemma 2.11. *Let* (X_0, X_1) *be a compatible couple of (quasi)-normed spaces,* $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, q_1 \leq \infty$ *and* b_0, b_1, $\bar{b}$ *be slowly varying functions. If* $0 < \eta < 1$, *we have*

$$\big((X_0, X_1)_{\theta_0,q_0,b_0}, (X_0, X_1)_{\theta_1,q_1,b_1}\big)_{\eta,q,b} = (X_0, X_1)_{\bar{\theta},q,\bar{b}},$$

where $\bar{\theta} = (1-\eta)\theta_0 + \eta\theta_1$ *and* $\bar{b}(t) = b_0^{1-\eta}(t) b_1^{\eta}(t) b\big(t^{\theta_1-\theta_0} b_0(t)/b_1(t)\big)$, $\forall\, t > 0$.

By applying Lemma 2.10 and Lemma 2.11, we have the following lemma.

Lemma 2.12. *Suppose that* $0 < \eta < 1$, $0 < p_1 < p_2 < \infty$, $0 < p < \infty$, $0 < q_1, q_2, q \leq \infty$ *and* b, b_1, b_2 *are slowly varying functions. If* $\frac{1}{p} = \frac{1-\eta}{p_1} + \frac{\eta}{p_2}$, *then*

$$(\mathcal{P}_{p_1,q_1,b_1}, \mathcal{P}_{p_2,q_2,b_2})_{\eta,q,b} = \mathcal{P}_{p,q,b_\eta},$$
$$(\mathcal{Q}_{p_1,q_1,b_1}, \mathcal{Q}_{p_2,q_2,b_2})_{\eta,q,b} = \mathcal{Q}_{p,q,b_\eta},$$

where $b_\eta(t) = b_1^{1-\eta}(t) b_2^{\eta}(t) b\big(t^{1/p_1-1/p_2} b_1(t)/b_2(t)\big)$.

Proof. Take $0 < s < p_1$ and put $\eta_0 = 1 - s/p_1$, $\eta_1 = 1 - s/p_2$, $X_0 = \mathcal{P}_s$, $X_1 = \mathcal{P}_\infty$, $B_1(t) = b_1(t^s)$, $B_2(t) = b_2(t^s)$ for all $t > 0$. Then, by applying Lemma 2.10 and Lemma 2.11, we have

$$(\mathcal{P}_{p_1,q_1,b_1}, \mathcal{P}_{p_2,q_2,b_2})_{\eta,q,b} = \big((\mathcal{P}_s, \mathcal{P}_\infty)_{\eta_0,q_1,B_1}, (\mathcal{P}_s, \mathcal{P}_\infty)_{\eta_1,q_2,B_2}\big)_{\eta,q,b}$$
$$= (\mathcal{P}_s, \mathcal{P}_\infty)_{\eta',q,b_{\eta'}},$$

where $\eta' = (1-\eta)\eta_0 + \eta\eta_1$ and

$$b_{\eta'}(t) = B_1^{1-\eta}(t) B_2^{\eta}(t) b\big(t^{\eta_1 - \eta_0} B_1(t)/B_2(t)\big), \qquad \forall\, t > 0. \tag{2.1}$$

Firstly, one can verify that p satisfies the following equality

$$\eta' = (1-\eta)(1 - s/p_1) + \eta(1 - s/p_2)$$
$$= 1 - \left(\frac{1-\eta}{p_1} + \frac{\eta}{p_2}\right)s = 1 - \frac{s}{p}.$$

Moreover, since $\eta_1 - \eta_0 = s/p_1 - s/p_2$, we see from (2.1) that

$$b_{\eta'}(t^{1/s}) = b_1^{1-\eta}(t) b_2^{\eta}(t) b\big(t^{1/p_1 - 1/p_2} b_1(t)/b_2(t)\big) = b_\eta(t)$$

for all $t > 0$. Therefore, form Lemma 2.10, we have $(\mathcal{P}_s, \mathcal{P}_\infty)_{\eta',q,b_{\eta'}} = \mathcal{P}_{p,q,b_\eta}$ and the result follows. The proof of $\mathcal{Q}_{p,q,b}$ is similar. □

Lemma 2.13 ([29, Theorem 7.14]). *Let $0 < p < \infty$, $0 < q \leq \infty$ and b be a slowly varying function. Then*

$$\mathcal{P}_{p,q,b} = \mathcal{Q}_{p,q,b}$$

with equivalent (quasi)-norms, namely, $\|f\|_{\mathcal{P}_{p,q,b}} \simeq \|f\|_{\mathcal{Q}_{p,q,b}}$.

Therefore, the results obtained in the following sections for the type of spaces $\mathcal{P}_{p,q,b}$ are also effective for the type of spaces $\mathcal{Q}_{p,q,b}$. Thus, for the sake of simplicity, in the following sections, we will only state and prove our results with respect to the space $\mathcal{P}_{p,q,b}$.

3. Boundedness of martingales transform operators

Let $0 < p, q < \infty$, b be a slowly varying function and $v \in V_{p,q,b}$. The boundedness of the martingale transform operator T_v on Hardy-Lorentz-Karamata spaces and BMO spaces will be investigated in this section.

The spaces of martingales with bounded mean oscillation are defined for $1 \leq p < \infty$ by

$$BMO_p = \left\{ f = (f_n)_{n\geq 1} : \|f\|_{BMO_p} = \sup_{n\geq 1} \left\| \mathbb{E}_{n-1}|f - f_{n-1}|^p \right\|_\infty^{1/p} < \infty \right\}.$$

It is well known that BMO_p spaces are all equivalent for $1 \leq p < \infty$ by the John-Nirenberg theorem.

The subsequent Lemma ensures that distributional inequality (the so called good λ-inequality) can be switched to the norm inequality for $L_{p,q,b}$ (see proof in [29]).

Lemma 3.14 ([29, Proposition 4.3]). *Let $\alpha > 1$, $\beta > 0$, $0 < p < \infty$, $0 < q \leq \infty$, b be a slowly varying function and F, G be locally integrable functions. If there exist $\epsilon_{\alpha,\beta}, k_{\alpha,\beta} > 0$ satisfying $\lim_{\beta\to 0} \epsilon_{\alpha,\beta} = 0$ and*

$$\mathbb{P}(F > \alpha\lambda) \leq \epsilon_{\alpha,\beta}\mathbb{P}(F > \lambda) + k_{\alpha,\beta}\mathbb{P}(G > \beta\lambda), \quad \forall\, \lambda > 0, \tag{3.2}$$

then

$$\|F\|_{p,q,b} \leq C\|G\|_{p,q,b}$$

for some $C > 0$ independent of F and G.

Remark 3.3. *As usual, we say that the pair (F, G) satisfies the "good λ-inequality" if F and G satisfy the inequality (3.2).*

The generalization of the famous Burkholder-Gundy inequalities for martingale Hardy-Lorentz-Karamata spaces is given below.

Lemma 3.15 ([30, Theorem 3.2])**.** *Let $1 < p < \infty$, $0 < q \leq \infty$ and b be a slowly varying function. Then for every martingale $f = (f_n)_{n\geq 1}$,*

$$C_1\|M(f)\|_{p,q,b} \leq \|S(f)\|_{p,q,b} \leq C_2\|M(f)\|_{p,q,b}$$

for some constant $C_2 \geq C_1 > 0$.

Theorem 3.1. *Let $0 < p, q < \infty$, b, b_1, b_2 be slowly varying functions and $v \in V_{p,q,b}$. If $0 < p_1, q_1 < \infty$, $\frac{1}{p_1} = \frac{1}{p} + \frac{1}{p_2}$, $\frac{1}{q_1} = \frac{1}{q} + \frac{1}{q_2}$ and $b_1(t) = b(t)b_2(t)$, then T_v is of types $(H^S_{p_2,q_2,b_2}, H^S_{p_1,q_1,b_1})$ and $(\mathcal{P}_{p_2,q_2,b_2}, \mathcal{P}_{p_1,q_1,b_1})$ (resp.$(\mathcal{Q}_{p_2,q_2,b_2}, \mathcal{Q}_{p_1,q_1,b_1})$) with $\|T_v\| \lesssim \|v\|_{V_{p,q,b}}$.*

Proof. For every $v \in V_{p,q,b}$ and $f \in H^S_{p_2,q_2,b_2}$, using the pointwise estimation, we obtain

$$S(T_v(f)) \leq M(v) \cdot S(f), \qquad a.e.$$

Combining this with Lemma 2.6, we have

$$\|T_v(f)\|_{H^S_{p_1,q_1,b_1}} = \|S(T_v(f))\|_{p_1,q_1,b_1} \leq \|M(v) \cdot S(f)\|_{p_1,q_1,b_1} \lesssim \|M(v)\|_{p,q,b}\|S(f)\|_{p_2,q_2,b_2} \approx \|v\|_{V_{p,q,b}}\|f\|_{H^S_{p_2,q_2,b_2}}. \tag{3.3}$$

This implies that T_v is of type $(H^S_{p_2,q_2,b_2}, H^S_{p_1,q_1,b_1})$ with $\|T_v\| \leq C\|v\|_{V_{p,q,b}}$.

For every $v \in V_{p,q,b}$ and $f \in \mathcal{P}_{p_1,q_1,b_1}$. Let $\lambda = (\lambda_n)_{n\geq 0}$ be the non-decreasing, predictable least majorant of $f = (f_n)_{n\geq 1}$. Then, for all $n \geq 1$, we have

$$|d_n f| = |f_n - f_{n-1}| \leq |f_n| + |f_{n-1}| \leq \lambda_{n-1} + \lambda_{n-2} \leq 2\lambda_{n-1} \tag{3.4}$$

and

$$|d_n(T_v(f))| = |v_{n-1}d_n f| \leq 2M_{n-1}(v)\lambda_{n-1} \triangleq \rho_{n-1}. \tag{3.5}$$

Therefore,

$$|T_v(f_n)| = |T_v(f_{n-1}) + d_n(T_v(f))| \leq M_{n-1}(T_v(f)) + \rho_{n-1}, \quad \forall\, n \geq 1. \tag{3.6}$$

Since the sequence $\big(M_n(T_v(f)) + \rho_n\big)_{n\geq 0}$ is adapted to $\{\mathcal{F}_n\}_{n\geq 0}$, it follows from (3.6) that $\big(M_n(T_v(f))+\rho_n\big)_{n\geq 0}$ is a non-decreasing, predictable least majorant of martingale $\big(T_v(f_n)\big)_{n\geq 1}$. By Lemma 2.6, (3.5) and the definition of ρ_n, we get

$$\begin{aligned}\|\rho_\infty\|_{p_1,q_1,b_1} &\approx \|M(v)\lambda_\infty\|_{p_1,q_1,b_1} \\ &\lesssim \|M(v)\|_{p,q,b}\|\lambda_\infty\|_{p_2,q_2,b_2} \\ &\lesssim \|v\|_{V_{p,q,b}}\|f\|_{\mathcal{P}_{p_2,q_2,b_2}}.\end{aligned} \tag{3.7}$$

As in [7], the pairs $\big(S(f), M(f) + \lambda_\infty\big)$ and $\big(M(T_v(f)), S(T_v(f)) + \rho_\infty\big)$ satisfy the good λ-inequality. Hence, by Lemma 3.14, we obtain

$$\begin{aligned}\|S(f)\|_{p_2,q_2,b_2} &\lesssim \|M(f) + \lambda_\infty\|_{p_2,q_2,b_2} \\ &\lesssim \|M(f)\|_{p_2,q_2,b_2}\| + \|\lambda_\infty\|_{p_2,q_2,b_2} \\ &\lesssim \|f\|_{\mathcal{P}_{p_2,q_2,b_2}}\end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \|M(T_v(f))\|_{p_1,q_1,b_1} &\lesssim \|S(T_v(f)) + \rho_\infty\|_{p_1,q_1,b_1} \\ &\lesssim \|S(T_v(f))\|_{p_1,q_1,b_1} + \|\rho_\infty\|_{p_1,q_1,b_1}. \end{aligned} \tag{3.9}$$

Combining (3.3), (3.7), (3.8) and (3.9), we get

$$\begin{aligned} \|M(T_v(f))\|_{p_1,q_1,b_1} &\lesssim \|S(T_v(f))\|_{p_2,q_2,b_2} + \|\rho_\infty\|_{p_1,q_1,b_1} \\ &\lesssim \|v\|_{V_{p,q,b}} \big(\|S(f)\|_{p_2,q_2,b_2} + \|f\|_{\mathcal{P}_{p_2,q_2,b_2}}\big) \\ &\lesssim \|v\|_{V_{p,q,b}} \big(\|f\|_{\mathcal{P}_{p_2,q_2,b_2}} + \|f\|_{\mathcal{P}_{p_2,q_2,b_2}}\big) \\ &\lesssim \|v\|_{V_{p,q,b}} \|f\|_{\mathcal{P}_{p_2,q_2,b_2}}. \end{aligned} \tag{3.10}$$

Moreover, it follows from (3.6), (3.7) and (3.10) that

$$\begin{aligned} \|T_v(f)\|_{\mathcal{P}_{p_1,q_1,b_1}} &\le \|M(T_v(f)) + \rho_\infty\|_{p_1,q_1,b_1} \\ &\lesssim \|M(f)\|_{p_1,q_1,b_1} + \|\rho_\infty\|_{p_1,q_1,b_1} \\ &\lesssim \|v\|_{V_{p,q,b}} \|f\|_{\mathcal{P}_{p_2,q_2,b_2}}, \end{aligned} \tag{3.11}$$

which implies T_v is of type $(\mathcal{P}_{p_2,q_2,b_2}, \mathcal{P}_{p_1,q_1,b_1})$ with $\|T_v\| \lesssim \|v\|_{V_{p,q,b}}$. □

Corollary 3.1. *Let $0 < p, q < \infty$, b be a slowly varying function, $v \in V_{p,q,b}$, $0 < p_1 < p_2 < \infty$ with $\frac{1}{p_1} = \frac{1}{p} + \frac{1}{p_2}$. Then T_v is of type $(\mathcal{P}_{p_2,q,b}, \mathcal{P}_{p_1,q,b})$ (resp. $(\mathcal{Q}_{p_2,q,b}, \mathcal{Q}_{p_1,q,b})$) with $\|T_v\| \lesssim \|v\|_{V_{p,q,b}}$.*

Proof. Since $\frac{1}{q} = \frac{1}{2q} + \frac{1}{2q}$, it follows from Lemma 2.3 and Theorem 3.1 that

$$\begin{aligned} \|T_v(f)\|_{\mathcal{P}_{p_1,q,b}} &\lesssim \|v\|_{V_{p,2q,b}} \|f\|_{\mathcal{P}_{p_2,2q,b}} \\ &\lesssim \|v\|_{V_{p,q,b}} \|f\|_{\mathcal{P}_{p_2,q,b}}. \end{aligned}$$

The proof is complete. □

Lemma 3.16 ([38, Fefferman's inequality]). *Let $f = (f_n)_{n\ge1} \in H_1^S$ and $\varphi = (\varphi_n)_{n\ge1} \in BMO_2$. Then*

$$\big|E(f_n\varphi_n)\big| \le \sqrt{\frac{2}{p}} \|f\|_{H_1^S} \|\varphi\|_{BMO_2}.$$

Lemma 3.17. *Let $1 < p, q < \infty$, b be a slowly varying function and $v \in V_{p,q,b}$. Then T_v is of types $(BMO_2, H^S_{p,q,b})$ and $(BMO_2, H^M_{p,q,b})$ with $\|T_v\| \lesssim \|v\|_{V_{p,q,b}}$.*

Proof. Note that T_v is selfadjoint in the sense that for nice martingales f and g (for example, both f and g are in L_2), we have $E\big(gT_v(f)\big) = E\big(fT_v(g)\big)$ (see [12] as well as [42]). We know that $BMO_2 \subset L_2$, $H^S_{p,q,b} \subset L_2$ when $2 < p < \infty$, and L_2 is dense in $H^S_{p,q,b}$ for $1 < p \le 2$. Then for any martingale $f \in BMO_2$ and $g \in H^S_{p,q,b}$, it follows from Lemma 3.16 that

$$|E(gT_v(f))| = |E(fT_v(g))| \le \sqrt{2}\|f\|_{BMO_2} \|T_v(g)\|_{H_1^S}. \tag{3.12}$$

Set $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. For any $v \in V_{p,q,b}$, by Theorem 3.1, we have

$$\|T_v(g)\|_{H_1^S} \lesssim \|v\|_{V_{p,q,b}} \|g\|_{H^S_{p',q',1/b}}. \tag{3.13}$$

Both (3.12) and (3.13) give that

$$|E(gT_v(f))| \lesssim \|v\|_{V_{p,q,b}} \|f\|_{BMO_2} \|g\|_{H^S_{p',q',1/b}}. \tag{3.14}$$

Now, from Lemma 3.15, we have $H^S_{p',q',1/b} \approx H^M_{p',q',1/b} \approx L_{p',q',1/b}$. Then (3.14) implies that $T_v(f) \in (H^M_{p',q',1/b})' \approx (L_{p',q',1/b})' \approx L_{p,q,b} \approx H^S_{p,q,b}$ and

$$\|T_v(f)\|_{H^S_{p,q,b}} \lesssim \|v\|_{V_{p,q,b}} \|f\|_{BMO_2},$$

which means that T_v is of type $(BMO_2, H^S_{p,q,b})$ with $\|T_v\| \lesssim \|v\|_{V_{p,q,b}}$. Since $H^S_{p,q,b} \approx H^M_{p,q,b}$ for $1 < p, q < \infty$, T_v is also of type $(BMO_2, H^M_{p,q,b})$. □

In fact, the above lemma still holds for the exponents p and q in a larger range.

Theorem 3.2. *Let $0 < p, q < \infty$ and b be a slowly varying function. If $v \in V_{p,q,b}$, then T_v is of types $(BMO_2, H^S_{p,q,b})$ and $(BMO_2, H^M_{p,q,b})$ with $\|T_v\| \lesssim \|v\|_{V_{p,q,b}}$.*

Proof. Take some $p_1 > \max\{1, p, \frac{p}{q}\}$ and set $q_1 = \frac{p_1}{p} q$. Then $q_1 > \max\{1, q\}$. Since $p_1 > p$ and $q_1 > q$, set $0 < p_2, q_2 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. In addition, let b_1, b_2 be slowly varying functions satisfying $b(t) = b_1(t) b_2(t)$ with $b(t) = b_1^{1/r}(t) = b_2^{1/(1-r)}(t)$.

For every $v \in V_{p,q,b}$, we suppose $v^{(1)} = \left(v_n^{(1)}\right)_{n\ge 0}$ with $v_n^{(1)} = v_n^r$ and $v^{(2)} = \left(v_n^{(2)}\right)_{n\ge 0}$ with $v_n^{(2)} = v_n^{1-r}$ for all $n \ge 0$, where $0 < r = \frac{p}{p_1} < 1$. Thus, $v = v^{(1)} \cdot v^{(2)}$. With the help of Lemma 2.5, we conclude that

$$\begin{aligned} \|v^{(1)}\|_{V_{p_1,q_1,b_1}} &= \|M(v^{(1)})\|_{p_1,q_1,b_1} = \|M(v^r)\|_{p_1,q_1,b_1} \\ &= \|(M(v))^r\|_{p_1,q_1,b_1} = \|M(v)\|^r_{rp_1,rq_1,b_1^{1/r}} \\ &= \|M(v)\|^r_{p,q,b} = \|v\|^r_{V_{p,q,b}} < \infty \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \|v^{(2)}\|_{V_{p_2,q_2,b_2}} &= \|M(v^{(2)})\|_{p_2,q_2,b_2} = \|M(v^{1-r})\|_{p_2,q_2,b_2} \\ &= \|(M(v))^{1-r}\|_{p_2,q_2,b_2} = \|M(v)\|^{1-r}_{(1-r)p_2,(1-r)q_2,b_2^{1/(1-r)}} \\ &= \|M(v)\|^{1-r}_{p,q,b} = \|v\|^{1-r}_{V_{p,q,b}} < \infty. \end{aligned} \tag{3.16}$$

This implies that $v^{(1)} \in V_{p_1,q_1,b_1}$ and $v^{(2)} \in V_{p_2,q_2,b_2}$.

Since $1 < p_1, q_1 < \infty$, it follows from Lemma 3.17 that

$$\|T_{v^{(1)}}(f)\|_{H^S_{p_1,q_1,b_1}} \lesssim \|v^{(1)}\|_{V_{p_1,q_1,b_1}} \cdot \|f\|_{BMO_2} < \infty, \ f \in BMO_2, \tag{3.17}$$

which means that $\|T_{v^{(1)}}(f)\| \in H^S_{p_1,q_1,b_1}$.

It is easy to observe that $b(t) = b_1(t) b_2(t)$ and $v^{(2)} \in V_{p_2,q_2,b_2}$. Applying Theorem 3.1, we obtain $T_{v^{(2)}}(f)$ is of type $(H^S_{p_1,q_1,b_1}, H^S_{p,q,b})$ with $\|T_{v^{(2)}}\| \lesssim \|v^{(2)}\|_{V_{p_2,q_2,b_2}}$. Therefore, combining (3.15), (3.16) and (3.17), we get

$$\begin{aligned} \|T_v(f)\|_{H^S_{p,q,b}} &= \|T_{v^{(1)}}(T_{v^{(2)}}(f))\|_{H^S_{p,q,b}} \\ &\lesssim \|v^{(2)}\|_{V_{p_2,q_2,b_2}} \|T_{v^{(1)}}(f)\|_{H^S_{p_1,q_1,b_1}} \\ &\lesssim \|v^{(2)}\|_{V_{p_2,q_2,b_2}} \|v^{(1)}\|_{V_{p_1,q_1,b_1}} \|f\|_{BMO_2} \\ &\lesssim \|v\|^{1-r}_{V_{p,q,b}} \|v\|^r_{V_{p,q,b}} \|f\|_{BMO_2} = C \|v\|_{V_{p,q,b}} \|f\|_{BMO_2}, \end{aligned}$$

which implies that T_v is of type $(BMO_2, H^S_{p,q,b})$ with $\|T_v\| \lesssim \|v\|_{Vp,q,b}$. The type $(BMO_2, H^M_{p,q,b})$ of T_v can be proved similarly. □

Theorem 3.3. *Set $0 < p, q < \infty$, b be a slowly varying function and $v \in V_{p,q,b}$. Then T_v is of type $(BMO_1, \mathcal{P}_{p,q,b})$ (resp. $(BMO_2, \mathcal{Q}_{p,q,b})$) with $\|T_v\| \lesssim \|v\|_{V_{p,q,b}}$.*

Proof. Given any $f \in BMO_2$. For all $n \geq 1$, $|d_n f| = |f_n - f_{n-1}| \leq \|f\|_{BMO_2}$. Thus, we obtain

$$\begin{aligned} |T_v(f_n)| &= |T_v(f_{n-1}) + d_n(T_v(f))| \\ &\leq M_{n-1}(T_v(f)) + |d_n(T_v(f))| \\ &= M_{n-1}(T_v(f)) + |v_{n-1} d_n f| \\ &\leq M_{n-1}(T_v(f)) + M_{n-1}(v)\|f\|_{BMO_2}. \end{aligned}$$

Thus $\big(M_n(T_v(f))+M_n(v)\|f\|_{BMO_2}\big)_{n\geq 0}$ is a predictable and non-decreasing majorant of $\big(T_v(f_n)\big)_{n\geq 1}$. By Theorem 3.2, we get that

$$\begin{aligned} \|T_v(f_n)\|_{\mathcal{P}_{p,q,b}} &\leq \|M(T_v(f)) + M(v)\|f\|_{BMO_2}\|_{p,q,b} \\ &\lesssim \|M(T_v(f))\|_{p,q,b} + \|M(v)\|_{p,q,b}\|f\|_{BMO_2} \\ &\lesssim \|v\|_{V_{p,q,b}}\|f\|_{BMO_2}, \end{aligned}$$

which proves that $T_v(f) = \big(T_v(f_n)\big)_{n\geq 1} \in \mathcal{P}_{p,q,b}$ and T_v is of type $(BMO_2, \mathcal{P}_{p,q,b})$ with $\|T_v\| \lesssim \|v\|_{V_{p,q,b}}$. Since BMO_p spaces are equivalent for $1 \leq p < \infty$ by John-Nirenberg's theorem, we obtain that T_v is also of type $(BMO_1, \mathcal{P}_{p,q,b})$. □

4. Relations between $\mathcal{P}_{p_1,q,b}$ and $\mathcal{P}_{p_2,q,b}$ (resp. $\mathcal{Q}_{p_1,q,b}$ and $\mathcal{Q}_{p_2,q,b}$)

Set $0 < p_1 < p_2 < \infty$ and $0 < q < \infty$. From Lemma 2.4, we know that $L_{p_2,q,b} \subset L_{p_1,q,b}$, then it is obvious that $\mathcal{P}_{p_2,q,b} \subset \mathcal{P}_{p_1,q,b}$ (resp. $\mathcal{Q}_{p_2,q,b} \subset \mathcal{Q}_{p_1,q,b}$). In this section, we will prove that any martingale $f \in \mathcal{P}_{p_1,q,b}$ (resp. $\mathcal{Q}_{p_1,q,b}$) can be represented as a martingale transform of some element in $f \in \mathcal{P}_{p_2,q,b}$ (resp. $\mathcal{Q}_{p_2,q,b}$).

Firstly, let (A_0, A_1) be a compatible couple of quasi-normed spaces A_0 and A_1, that is, we assume that both A_0 and A_1 are continuously, embedded in some common quasi-normed spaces. Then the spaces $A_0 + A_1$ are defined as the set of all f, which can be written as $f = f^0 + f^1$ with $f^0 \in A_0$ and $f^1 \in A_1$. The interpolation spaces between A_0 and A_1 are defined by means of the so-called K-functional $K(t, f; A_0, A_1)$, which can be found in [4].

Definition 4.4. *Let (A_0, A_1) be a compatible couple of quasi-normed spaces A_0 and A_1. For any $f \in A_0 + A_1$, the K-functional is defined as*

$$K(t, f; A_0, A_1) = \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1\},$$

where the infimum is taken over all $f = f_0 + f_1$ with $f_i \in A_i$, $i = 0, 1$.

Definition 4.5 ([21])**.** *Let $0 < \theta < 1$, $0 < q \leq \infty$ and let b be a slowly varying function. Suppose that A_0 and A_1 are two quasi-normed spaces. The space $(A_0, A_1)_{\theta,q,b}$ consist of all $f \in A_0 + A_1$ such that $\|f\|_{(A_0,A_1)_{\theta,q,b}} < \infty$, where*

$$\|f\|_{(A_0,A_1)_{\theta,q,b}} = \begin{cases} \left(\displaystyle\int_0^\infty \big(t^{-\theta}\gamma_b(t)K(f,t)\big)^q \frac{dt}{t}\right)^{1/q}, & \text{if } 0 < q < \infty, \\ \sup\limits_{t>0} t^{-\theta}\gamma_b(t)K(f,t), & \text{if } q = \infty. \end{cases}$$

Theorem 4.4. *Let $0 < p, q < \infty$, $0 < p_1 < p_2 < \infty$ with $\frac{1}{p_1} = \frac{1}{p} + \frac{1}{p_2}$, $0 < \alpha, \alpha' < \infty$ and b, b_0 be slowly varying functions. Then for each martingale $f = (f_n)_{n\geq 1} \in \mathcal{P}_{p_1,q,b_\alpha}$ (resp. $\mathcal{Q}_{p_1,q,b_\alpha}$), there exist a martingale $g = (g_n)_{n\geq 1} \in \mathcal{P}_{p_2,q,b_{\alpha'}}$ (resp. $\mathcal{Q}_{p_2,q,b_{\alpha'}}$) and multiplier sequence $v = (v_n)_{n\geq 0} \in V_{p,\frac{p_2 q}{p_2 - p_1},b_0}$ such that f is the martingale transform of g by T_v, that is, $f = T_v(g)$.*

Proof. First of all, we construct an appropriate martingale $g = (g_n)_{n\geq 1} \in \mathcal{P}_{p_2,q,b_{\alpha'}}$, which is associated with the martingale $f = (f_n)_{n\geq 1} \in \mathcal{P}_{p_1,q,b_\alpha}$.

Taking a suitable number p_1' such that $0 < p_1' < \min\{\frac{p_1}{p_2}, p_1\}$ and set $\theta = 1 - \frac{p_1'}{p_1}$, $\frac{1}{\alpha} = \frac{1}{p_1'}$ with $b_\alpha(t) = b(t^{1/\alpha})$, by applying Lemma 2.10, we obtain

$$(\mathcal{P}_{p_1'}, \mathcal{P}_\infty)_{\theta,q,b} = \mathcal{P}_{p_1,q,b_\alpha}. \tag{4.18}$$

Then, for any $f = (f_n)_{n\geq 1} \in \mathcal{P}_{p_1,q,b_\alpha}$, we have the following decomposition

$$f = f^0 + f^1 \quad \text{with} \quad f^0 \in \mathcal{P}_{p_1'} \quad \text{and} \quad f^1 \in \mathcal{P}_\infty. \tag{4.19}$$

Denote by $\lambda = (\lambda_n)_{n\geq 0}$ the non-decreasing and predictable least majorant of the martingale $f^0 = (f_n^0)_{n\geq 1}$. We pick β such that $\beta = \frac{p_2 - p_1}{p_2} = 1 - \frac{p_1}{p_2}$ and $v_n = \max\{\lambda_n^\beta, 1\}$ for all $n \geq 0$, then the sequence $v = (v_n)_{n\geq 0}$ is adapted to $\{\mathcal{F}_n\}_{n\geq 1}$. Now, we define the martingale $g = (g_n)_{n\geq 1}$ by

$$g_n = \sum_{k=1}^n v_{k-1}^{-1} d_k f, \quad n \geq 1. \tag{4.20}$$

By (4.19) and (4.20), it is easy to know that the martingale g has also the corresponding decomposition $g = g^0 + g^1$ with $g^0 = (g_n^0)_{n\geq 1}$ and $g^1 = (g_n^1)_{n\geq 1}$ such that

$$g_n = g_n^0 + g_n^1,\ g_n^0 = \sum_{k=1}^n v_{k-1}^{-1} d_k f^0,\ g_n^1 = \sum_{k=1}^n v_{k-1}^{-1} d_k f^1, \quad n \geq 1. \tag{4.21}$$

Taking $p_2' = \frac{p_2}{p_1} p_1'$, then we conclude that $0 < p_1' < p_2' < 1$ and $\frac{1}{p_2} = \frac{1-\theta}{p_2'}$. Therefore, according to Lemma 2.10 again, we also get

$$(\mathcal{P}_{p_2'}, \mathcal{P}_\infty)_{\theta,q,b} = \mathcal{P}_{p_2,q,b_{\alpha'}}, \tag{4.22}$$

where $\alpha' = p_2'$.

For any $n \geq 1$, from Lemma 2.7, g_n^0 can be written in the form of

$$\begin{aligned} g_n^0 &= \sum_{k=1}^n (f_k^0 - f_{k-1}^0) v_{k-1}^{-1} \\ &= f_n^0 \cdot v_n^{-1} - f_1^0 \cdot v_0^{-1} + \sum_{k=1}^n f_k^0 (v_{k-1}^{-1} - v_k^{-1}) \\ &= f_n^0 \cdot v_{n-1}^{-1} + \sum_{k=1}^{n-1} f_k^0 (v_{k-1}^{-1} - v_k^{-1}). \end{aligned} \tag{4.23}$$

On the one hand, since $\beta = 1 - \frac{p_1}{p_2} > 0$ and $0 < v_n^{-1} = \frac{1}{\max\{\lambda_n^\beta, 1\}} = \min\{\lambda_n^{-\beta}, 1\} \leq \lambda_n^{-\beta}$ for all $n \geq 0$, the sequence $(v_n^{-1})_{n\geq 1}$ is non-increasing. Then $v_{k-1}^{-1} - v_k^{-1} \geq 0$ for all $k \geq 1$. From (4.23) and the definition of $(\lambda_n)_{n\geq 1}$, for any $n \geq 1$, we have

$$g_n^0 \leq |f_n^0| \cdot v_{n-1}^{-1} + \sum_{k=1}^{n-1} |f_k^0|(v_{k-1}^{-1} - v_k^{-1}) \tag{4.24}$$
$$\leq \lambda_{n-1} \cdot \lambda_{n-1}^{-\beta} + \sum_{k=1}^{n-1} \lambda_{k-1}(v_{k-1}^{-1} - v_k^{-1})$$
$$= \lambda_{n-1}^{\frac{p_1'}{p_2'}} + \sum_{k=1}^{n-1} \lambda_{k-1}(v_{k-1}^{-1} - v_k^{-1}).$$

Since $(\lambda_n^{-\beta})_{n\geq 0}$ is decreasing, without loss of generality, we can suppose that there exists a number $m \geq 1$ such that $\lambda_{m-1}^{-\beta} > 1$ and $\lambda_m^{-\beta} \leq 1$. Then we get

$$\sum_{k=1}^{n-1} \lambda_{k-1}(v_{k-1}^{-1} - v_k^{-1}) = \sum_{k=1}^{n-1} \lambda_{k-1}(1-1) = 0, \qquad n < m+1,$$

and

$$\sum_{k=1}^{n-1} \lambda_{k-1}(v_{k-1}^{-1} - v_k^{-1}) = \sum_{k=m}^{n-1} \lambda_{k-1}(\lambda_{k-1}^{-\beta} - \lambda_k^{-\beta})$$
$$\leq \sum_{k=1}^{n-1} \lambda_{k-1}\left(\lambda_{k-1}^{\frac{p_1'}{p_2'}-1} - \lambda_k^{\frac{p_1'}{p_2'}-1}\right), \qquad n \geq m+1.$$

As a result, we have

$$\sum_{k=1}^{n-1} \lambda_{k-1}(v_{k-1}^{-1} - v_k^{-1}) \leq \sum_{k=1}^{n-1} \lambda_{k-1}\left(\lambda_{k-1}^{\frac{p_1'}{p_2'}-1} - \lambda_k^{\frac{p_1'}{p_2'}-1}\right), \qquad n \geq 1. \tag{4.25}$$

From (4.24) and (4.25), we have

$$|g_n^0| \leq \lambda_{n-1}^{\frac{p_1'}{p_2'}} + \sum_{k=1}^{n-1} \lambda_{k-1}\left(\lambda_{k-1}^{\frac{p_1'}{p_2'}-1} - \lambda_k^{\frac{p_1'}{p_2'}-1}\right) \tag{4.26}$$
$$= \sum_{k=1}^{n} \lambda_{k-1}^{\frac{p_1'}{p_2'}-1}(\lambda_{k-1} - \lambda_{k-2})$$
$$\leq \int_0^{\lambda_{n-1}} t^{\frac{p_1'}{p_2'}-1}\frac{dt}{t} = \frac{p_2'}{p_1'}\lambda_{n-1}^{\frac{p_1'}{p_2'}}, \qquad \forall n \geq 1,$$

which means that the sequence $\left(\frac{p_2'}{p_1'}\lambda_n^{\frac{p_1'}{p_2'}}\right)_{n\geq 0}$ is a non-decreasing and predictable majorant of the martingale $g^0 = (g_n^0)_{n\geq 1}$. Moreover, by (4.26) we obtain that

$$\|g^0\|_{\mathcal{P}_{p_2'}} \leq \frac{p_2'}{p_1'}\left(E\left(\lambda_\infty^{\frac{p_1'}{p_2'}}\right)^{p_2'}\right)^{\frac{1}{p_2'}} = \frac{p_2'}{p_1'}\|f^0\|_{\mathcal{P}_{p_1'}}^{\frac{p_1'}{p_2'}} \tag{4.27}$$
$$\leq C_1 \max\{1, \|f^0\|_{\mathcal{P}_{p_1'}}\}.$$

On the other hand, denote by $\hat{\lambda} = (\hat{\lambda}_n)_{n\geq 0}$ the non-decreasing and predictable least majorant of the martingale $f^1 = (f_n^1)_{n\geq 1}$. Then $|\sum_{k=1}^n d_k f^1| = |f_n^1| \leq \hat{\lambda}_{n-1} \leq \hat{\lambda}_\infty$ for all $n \geq 1$ and

$\|f^1\|_{\mathcal{P}_\infty} = \|\hat{\lambda}_\infty\|_\infty$. Since $0 < v_k^{-1} = \min\{\lambda_k^{-\beta}, 1\} \le 1$ and the sequence $(v_n^{-1})_{n\ge 0}$ is decreasing, according to Lemma 2.7, we have

$$|g_n^1| = \Big|\sum_{k=1}^{n} v_{k-1}^{-1} d_k f^1\Big| = \Big|f_n^1 \cdot v_{n-1}^{-1} + \sum_{k=1}^{n-1} f_k^1 (v_{k-1}^{-1} - v_k^{-1})\Big|$$
$$\le 3 \sup_{1\le k\le n} |v_{k-1}^{-1}| \cdot \sup_{0\le m\le n} \Big|\sum_{k=1}^{m} d_k f^1\Big| \le 3\hat{\lambda}_\infty, \quad \forall n \ge 1.$$

This implies that $\|g^1\|_{\mathcal{P}_\infty} \le 3\|f^1\|_{\mathcal{P}_\infty}$.

According to (4.27), it is easy to know that $\|g^0\|_{\mathcal{P}_{p_2'}} \le C_1$ when $\|f^0\|_{\mathcal{P}_{p_1'}} < 1$. Set $\tilde{f} = 1 + f^1$, since $f^1 \in \mathcal{P}_\infty$, we have

$$\begin{aligned}\|\tilde{f}\|_{\mathcal{P}_{p_1,q,b_\alpha}} &= \|1 + f^1\|_{\mathcal{P}_{p_1,q,b_\alpha}} \le C_2(1 + \|f^1\|_{\mathcal{P}_{p_1,q,b_\alpha}}) \\ &\le C_2(1 + \|f^1\|_{\mathcal{P}_\infty}) \le \infty,\end{aligned}$$

which implies that the martingale $\tilde{f} \in \mathcal{P}_{p_1,q,b_\alpha}$. Then we have

$$\|g^0\|_{\mathcal{P}_{p_2'}} + t\|g^1\|_{\mathcal{P}_\infty} \le C_1\|1\|_{\mathcal{P}_{p_1'}} + 3t\|f^1\|_{\mathcal{P}_\infty}, \qquad t > 0.$$

Therefore, we get

$$K(t, g; \mathcal{P}_{p_2'}, \mathcal{P}_\infty) \le C_3 K(t, \tilde{f}; \mathcal{P}_{p_1'}, \mathcal{P}_\infty), \qquad t > 0,$$

then we obtain

$$\|g\|_{\mathcal{P}_{p_2,q,b_{\alpha'}}} = \|g\|_{(\mathcal{P}_{p_2'},\mathcal{P}_\infty)} \le C\|\tilde{f}\|_{(\mathcal{P}_{p_2'},\mathcal{P}_\infty)} = C\|\tilde{f}\|_{\mathcal{P}_{p_1,q,b_\alpha}} < \infty,$$

which yields $g \in \mathcal{P}_{p_2,q,b_{\alpha'}}$ when $\|f^0\|_{\mathcal{P}_{p_1'}} < 1$.

Next, if $\|f^0\|_{\mathcal{P}_{p_1'}} \ge 1$, it follows from (4.27) that $\|g^0\|_{\mathcal{P}_{p_2'}} \le C_1\|f^0\|_{\mathcal{P}_{p_1'}}$. Then we have

$$\|g^0\|_{\mathcal{P}_{p_2'}} + t\|g^1\|_{\mathcal{P}_\infty} \le \max\{C_1, 3\}(\|f^0\|_{\mathcal{P}_{p_1'}} + t\|f^1\|_{\mathcal{P}_\infty}), \qquad t > 0.$$

Hence

$$K(t, g; \mathcal{P}_{p_2'}, \mathcal{P}_\infty) \le C_2 K(t, f; \mathcal{P}_{p_1'}, \mathcal{P}_\infty), \qquad t > 0.$$

Consequently,

$$\begin{aligned}\|g\|_{\mathcal{P}_{p_2,q,b_{\alpha'}}} &= \|g\|_{(\mathcal{P}_{p_2'},\mathcal{P}_\infty)_{\theta,q,b}} \\ &= \Big(\int_0^\infty \big(t^{-\theta}\gamma_b(t) K(t, g; \mathcal{P}_{p_2'}, \mathcal{P}_\infty)\big)^q \frac{dt}{t}\Big)^{1/q} \\ &\le C_2\Big(\int_0^\infty \big(t^{-\theta}\gamma_b(t) K(t, f; \mathcal{P}_{p_1'}, \mathcal{P}_\infty)\big)^q \frac{dt}{t}\Big)^{1/q} \\ &= C_2\|f\|_{(\mathcal{P}_{p_1'},\mathcal{P}_\infty)_{\theta,q,b}} = C_2\|f\|_{\mathcal{P}_{p_1,q,b_\alpha}} < \infty.\end{aligned}$$

As a result, $g \in \mathcal{P}_{p_2,q,b_{\alpha'}}$ when $\|f^0\|_{\mathcal{P}_{p_1'}} \ge 1$. In conclusion, we get $g \in \mathcal{P}_{p_2,q,b_{\alpha'}}$ whether $\|f^0\|_{\mathcal{P}_{p_1'}} < 1$ or $\|f^0\|_{\mathcal{P}_{p_1'}} \ge 1$.

Secondly, we verify that the multiplier sequence $v = (v_n)_{n\ge 0} \in V_{p, \frac{p_2 q}{p_2 - p_1}, b_0}$. Applying Minkowski's inequality, we obtain

$$\begin{aligned}\|v\|_{V_{p,\frac{p_2 q}{p_2-p_1},b_0}} &= \|M(v)\|_{p,\frac{1}{\beta}q,b_0} = \|\max\{\lambda_\infty^\beta, 1\}\|_{p,\frac{1}{\beta}q,b_0} \\ &\le C_1(\|\lambda_\infty^\beta\|_{p,\frac{1}{\beta}q,b_0} + 1).\end{aligned}$$

Let $b_0^{1/\beta}(t)=b_\alpha(t)$, from Lemma 2.5 and (4.19), we easily conlude that

$$\begin{aligned}\|\lambda_\infty^\beta\|_{p,\frac{1}{\beta}q,b_0} &= \|\lambda_\infty\|^\beta_{\beta p,q,b_0^{1/\beta}} = \|\lambda_\infty\|^\beta_{p_1,q,b_\alpha}\\ &= \|f^0\|^\beta_{\mathcal{P}_{p_1,q,b_\alpha}} = \|f-f^1\|^\beta_{\mathcal{P}_{p_1,q,b_\alpha}}\\ &\le C_2(\|f\|_{\mathcal{P}_{p_1,q,b_\alpha}}+\|f^1\|_{\mathcal{P}_{p_1,q,b_\alpha}})^\beta\\ &\le C_2(\|f\|_{\mathcal{P}_{p_1,q,b_\alpha}}+\|f^1\|_{\mathcal{P}_\infty})^\beta<\infty.\end{aligned}$$

Hence

$$\|\lambda_\infty^\beta\|_{p,\frac{1}{\beta}q,b_0}\le C_1(C_2(\|f\|_{p_1,q,b_\alpha}+\|f\|_{\mathcal{P}_\infty})^\beta+1)<\infty.$$

This implies that $v=(v_n)_{n\ge0}\in V_{p,\frac{p_2q}{p_2-p_1},b_0}$.

Finally, according to (4.20), we get

$$f=T_v(g)\quad\text{with}\quad f_n=\sum_{k=1}^n v_{k-1}^{-1}d_kg,\qquad n\ge1.$$

It is clearly shown that f is the martingale transform of g by T_v. The proof is complete. □

5. Relations between $\mathcal{P}_{p_1,q_1,b}$ and $\mathcal{P}_{p_2,q_2,b}$ (resp. $\mathcal{Q}_{p_1,q_1,b}$ and $\mathcal{Q}_{p_2,q_2,b}$)

We will extend our discussion to the relationship between $\mathcal{P}_{p_1,q_1,b}$ and $\mathcal{P}_{p_2,q_2,b}$ (resp. $\mathcal{Q}_{p_1,q_1,b}$ and $\mathcal{Q}_{p_2,q_2,b}$) in this section, where $0<p_1<p_2<\infty$ and $0<q_1<q_2<\infty$.

Theorem 5.5. *Let $0<p_1<p_2<\infty$, $0<q_1<q_2<\infty$, $0<p,q<\infty$, $\frac{1}{p_1}=\frac{1}{p}+\frac{1}{p_2}$, $\frac{1}{q_1}=\frac{1}{q}+\frac{1}{q_2}$, $0<\alpha<\infty$ and b,b' be slowly varying functions. Then for each martingale $f=(f_n)_{n\ge1}\in\mathcal{P}_{p_1,q_1,b_\alpha}$ (resp. Q_{p_1,q_1,b_α}), there exist a martingale $\tilde{g}=(_n)_{n\ge1}\in\mathcal{P}_{p_2,q_2,b'_\alpha}$ (resp. $\mathcal{Q}_{p_2,q_2,b'_\alpha}$) and multiplier sequence $v=(v_n)_{n\ge0}\in V_{p,q,b}$ such that f is the martingale transform of $\tilde{g}$ by T_v, that is, $f=T_v(\tilde{g})$.*

Proof. Let $0<p_1<p_2<\infty$, $0<q_1<q_2<\infty$, $\frac{1}{q_1}=\frac{1}{q}+\frac{1}{q_2}$ and b be a slowly varying function with $b_\alpha(t)=b(t^{1/\alpha})$. For any $0<\eta<1$, select positive numbers $p_1'(\ne p_1)$, p_1'' and p_2'' such that $\frac{1}{p_1}=\frac{1-\eta}{p_1'}+\frac{\eta}{p_1''}$ and $0<p_2''<\min\{\frac{p_1'}{p_2'},p_1'\}$. It is easy to check that $p_1'\ne p_1''$. Set $q'=\frac{p_1'q}{p}$ and $q''=\frac{p_1''q}{p}$, then $0<q',q''<\infty$. By Lemma 2.12, we get

$$(\mathcal{P}_{p_1',q',b_\alpha'},\mathcal{P}_{p_1'',q'',b_\alpha''})_{\eta,q_1,b}=\mathcal{P}_{p_1,q_1,b_\alpha},$$

where $b_\alpha(t)=b_\alpha'^{1-\eta}(t)b_\alpha''^{\eta}(t)b(t^{1/p_1'-1/p_1''}b_\alpha'(t)/b_\alpha''(t))$. Therefore, for any martingale $f=(f_n)_{n\ge1}\in\mathcal{P}_{p_1,q_1,b_\alpha}$, we have a suitable decomposition as below

$$f=f^0+f^1,\quad\text{where}\quad f^0\in\mathcal{P}_{p_1',q',b_\alpha'}\quad\text{and}\quad f^1\in\mathcal{P}_{p_1'',q'',b_\alpha''}.$$

Since $\frac{1}{p_1}-\frac{1}{p_2}=\frac{1}{p}>0$ and $\frac{1}{p_1}=\frac{1-\eta}{p_1'}+\frac{\eta}{p_1''}$, we can also take two positive numbers $p_2'>p_1'$ and $p_2''>p_1''$ such that

$$\frac{1}{p_1}-\frac{1}{p_2}=\frac{1}{p_1'}-\frac{1}{p_2'}=\frac{1}{p_1''}-\frac{1}{p_2''}=\frac{1}{p}\quad\text{and}\quad p_2''=\frac{p_2'}{p_1'}\cdot p_1''.$$

Then we obtain $\frac{p'_1p'_2}{p'_2-p'_1}=p$ and $\frac{p'_2q'}{p'_2-p'_1}=\frac{p'_2}{p'_2-p'_1}\cdot\frac{p'_1q}{p}=q$, $\frac{p''_1p''_2}{p''_2-p''_1}=p$ and $\frac{p''_2q''}{p''_2-p''_1}=\frac{p''_2}{p''_2-p''_1}\cdot\frac{p''_1q}{p}=q$. We can also get

$$\begin{aligned}\frac{1}{p_1}-\frac{1}{p_2}&=(1-\eta)\Big(\frac{1}{p'_1}-\frac{1}{p'_2}\Big)+\eta\Big(\frac{1}{p''_1}-\frac{1}{p''_2}\Big)\\&=\Big(\frac{1-\eta}{p'_1}+\frac{\eta}{p''_1}\Big)-\Big(\frac{1-\eta}{p'_2}+\frac{\eta}{p''_2}\Big)\\&=\frac{1}{p_1}-\Big(\frac{1-\eta}{p'_2}+\frac{\eta}{p''_2}\Big),\end{aligned}$$

which implies $\frac{1}{p_2}=\frac{1-\eta}{p'_2}+\frac{\eta}{p''_2}$. Applying Theorem 4.4 to $f^0\in\mathcal{P}_{p'_1,q',b'_\alpha}$ and $f^1\in\mathcal{P}_{p''_1,q'',b''_\alpha}$, respectively, it is easy to find that there exist two sequences $v'=(v'_n)_{n\geq 0}$ and $v''=(v''_n)_{n\geq 0}$ such that

$$v'\in V_{\frac{p'_1p'_2}{p'_2-p'_1},\frac{p'_2q'}{p'_2-p'_1},b_0}=V_{p,q,b_0}\quad\text{and}\quad v''\in V_{\frac{p''_1p''_2}{p''_2-p''_1},\frac{p''_2q''}{p''_2-p''_1},b'_0}=V_{p,q,b'_0},$$

where $\beta=1-\frac{p'_1}{p'_2}$, $b_0(t)=b_\alpha^\beta(t)$ and $b'_0(t)=b_{\alpha'}^\beta(t)$. Furthermore, there exist two martingales $g^0\in\mathcal{P}_{p'_2,q',b'_{\alpha'}}$ and $g^1\in\mathcal{P}_{p''_2,q'',b''_{\alpha'}}$, respectively, such that

$$g_n^0=\sum_{k=1}^n v'^{-1}_{k-1}d_kf^0\quad\text{and}\quad g_n^1=\sum_{k=1}^n v''^{-1}_{k-1}d_kf^1,\quad n\geq 1.$$

Let $v_k=\max\{v'_k,v''_k\}$, or equivalently $v_k^{-1}=\min\{v'^{-1}_k,v''^{-1}_k\}$ for all $k\geq 0$ and let b be a slowly varying function with $b=\max\{b_0,b'_0\}$. Then it is obvious that $v=(v_k)_{k\geq 0}\in V_{p,q,b}$.

We define that

$$\tilde{g}_n^0=\sum_{k=1}^n v^{-1}_{k-1}d_kf^0\quad\text{and}\quad \tilde{g}_n^1=\sum_{k=1}^n v^{-1}_{k-1}d_kf^1,\quad n\geq 1.$$

Since $0<v_k^{-1}\leq v'^{-1}_k$ and $0<v_k^{-1}\leq v''^{-1}_k$, we have

$$\|\tilde{g}_n^0\|_{\mathcal{P}_{p'_2,q',b'_{\alpha'}}}\leq\|g_n^0\|_{\mathcal{P}_{p'_2,q',b'_{\alpha'}}},$$

$$\|\tilde{g}_n^1\|_{\mathcal{P}_{p''_2,q'',b''_{\alpha'}}}\leq\|g_n^1\|_{\mathcal{P}_{p''_2,q'',b''_{\alpha'}}}.$$

This means that $\tilde{g}_n^0=(\tilde{g}_n^0)_{n\geq1}\in\mathcal{P}_{p'_2,q',b'_{\alpha'}}$ and $\tilde{g}_n^1=(\tilde{g}_n^1)_{n\geq1}\in\mathcal{P}_{p''_2,q'',b''_{\alpha'}}$.

For all $n\geq 1$, we set $\tilde{g}=(\tilde{g}_n)_{n\geq1}$ with

$$\begin{aligned}\tilde{g}_n=g_n^0+g_n^1&=\sum_{k=1}^n v^{-1}_{k-1}d_kf^0+\sum_{k=1}^n v^{-1}_{k-1}d_kf^1\\&=\sum_{k=1}^n v^{-1}_{k-1}(d_kf^0+d_kf^1)=\sum_{k=1}^n v^{-1}_{k-1}d_kf.\end{aligned}$$

Then we obtain $\tilde{g}=T_{v^{-1}}(f)$ or equivalently $f=T_v(\tilde{g})$.

Using Lemma 2.12, we get

$$(\mathcal{P}_{p'_2,q',b'_{\alpha'}},\mathcal{P}_{p''_2,q'',b''_{\alpha'}})_{\eta,q_2,b'}=\mathcal{P}_{p_2,q_2,b'_\alpha},$$

then $\tilde{g}=(\tilde{g}_n)_{n\geq1}\in\mathcal{P}_{p_2,q_2,b'_\alpha}$.

Hence, similarly to the proof of Theorem 4.4, we can show that $f = T_v(\tilde{g})$ with

$$f_n = \sum_{k=1}^{n} v_{k-1} d_k \tilde{g}$$

for all $k \geq 1$. This completes the proof. $\square$

6. Relations between $\mathcal{P}_{p,q,b}$ and BMO_1 (resp. $\mathcal{Q}_{p,q,b}$ and BMO_2)

Theorem 6.6. *Let $0 < p, q < \infty$, b be a slowly varying function and $f = (f_n)_{n\geq 1} \in \mathcal{P}_{p,q,b}$ (resp. $\mathcal{Q}_{p,q,b}$). Then there exist a martingale $g = (g_n)_{n\geq 1} \in BMO_1$ (resp. BMO_2) with $\|g\|_{BMO_1} \leq 4$ (resp. $\|g\|_{BMO_2} \leq \sqrt{2}$), and a non-decreasing and non-negative sequence $v = (v_n)_{n\geq 1} \in V_{p,q,b}$ with $\|v\|_{V_{p,q,b}} \leq C\|f\|_{\mathcal{P}_{p,q,b}}$ (resp. $\|v\|_{V_{p,q,b}} \leq C\|f\|_{\mathcal{Q}_{p,q,b}}$) such that f is the martingale transform of g by T_v, that is, $f = T_v(g)$.*

Conversely, for any $v \in V_{p,q,b}$ and $g \in BMO_1$ (resp. $g \in BMO_2$), the martingale transform $f = T_v(g)$ belongs to $\mathcal{P}_{p,q,b}$ (resp. $\mathcal{Q}_{p,q,b}$) and $\|f\|_{\mathcal{P}_{p,q,b}} \leq C\|v\|_{V_{p,q,b}}\|g\|_{BMO_1}$ (resp. $\|f\|_{\mathcal{Q}_{p,q,b}} \leq C\|v\|_{V_{p,q,b}}\|g\|_{BMO_2}$).

Proof. By using Theorem 3.3, we can prove the converse assertion immediately.

For any martingale $f = (f_n)_{n\geq 1} \in \mathcal{P}_{p,q,b}$, denote by $\lambda = (\lambda_n)_{n\geq 0}$ the non-decreasing and predictable least majorant of the sequence $(f_n)_{n\geq 1}$. Taking a suitable position number $p_0 < \min\{p, q, 1\}$, define the sequence $v = (v_k)_{k\geq 0}$ and martingale $g = (g_n)_{n\geq 1}$ by

$$v_k = \sup_{m\leq k} \big(E(\lambda_\infty^{p_0}|\mathcal{F}_m)\big)^{\frac{1}{p_0}}; \quad g_n = \sum_{k=1}^{n} v_{k-1}^{-1} d_k f, \quad k \geq 1, \;\; n \geq 1.$$

Then we have $f = T_v(g)$ with $f_n = \sum_{k=1}^{n} v_{k-1} d_k g$ for all $n \geq 1$. What we need to prove is that $v \in V_{p,q,b}$ and $g \in BMO_1$.

It is easy to check that the sequence $\big(E(\lambda_\infty^{p_0}|\mathcal{F}_n)\big)_{n\geq 0}$ is a martingale. In what follows, for the sake of convenience in writing, we will still denote by $\lambda_\infty^{p_0}$ the martingale $\big(E(\lambda_\infty^{p_0}|\mathcal{F}_n)\big)_{n\geq 0}$, that is, we write as usual $\lambda_\infty^{p_0} = \big(E(\lambda_\infty^{p_0}|\mathcal{F}_n)\big)_{n\geq 0}$. Then we obtain

$$\begin{aligned} M(v) &= \sup_{k<\infty} |v_k| = \sup_{k<\infty} \sup_{m\leq k} \big(E(\lambda_\infty^{p_0}|\mathcal{F}_m)\big)^{1/p_0} \\ &\leq \big(\sup_{k<\infty} E(\lambda_\infty^{p_0}|\mathcal{F}_m)\big)^{1/p_0} = \big(M(\lambda_\infty^{p_0})\big)^{1/p_0}. \end{aligned}$$

Since $\frac{p}{p_0} > 1$ and $\frac{q}{p_0} > 1$, applying Lemma 2.9 and Lemma 2.5, we have

$$\begin{aligned} \|v\|_{V_{p,q,b}} &= \|M(v)\|_{p,q,b} \leq \|(M(\lambda_\infty^{p_0}))^{1/p_0}\|_{p,q,b} \\ &= \|M(\lambda_\infty^{p_0})\|^{1/p_0}_{\frac{p}{p_0},\frac{q}{p_0},b^{p_0}} \leq C\|\lambda_\infty^{p_0}\|^{1/p_0}_{\frac{p}{p_0},\frac{q}{p_0},b^{p_0}} \\ &= C\|\lambda_\infty^{p_0}\|_{p,q,b} = C\|f\|_{\mathcal{P}_{p,q,b}} < \infty. \end{aligned}$$

This proves $v \in V_{p,q,b}$. For any $N > n \geq 1$, by Abel's rearrangement, we get

$$\begin{aligned} |\sum_{k=n}^{N} d_k g| &= |g_N - g_{n-1}| = |\sum_{k=n}^{N} v_{k-1}^{-1} d_k f| \\ &= |v_{N-1}^{-1} \cdot (f_N - f_{n-1}) + \sum_{k=n}^{N-1} (f_k - f_{n-1}) \cdot (v_{k-1}^{-1} - v_k^{-1})| \\ &\leq 2 v_{N-1}^{-1} \lambda_{N-1} + 2 \sum_{k=n}^{N-1} \lambda_{k-1} \cdot (v_{k-1}^{-1} - v_k^{-1}) \\ &= 2 v_{n-1}^{-1} \lambda_{n-1} + 2 \sum_{k=m}^{N-1} v_k^{-1} \cdot (\lambda_k - \lambda_{k-1}). \end{aligned}$$

In addition, using Cauchy-Schwarz's inequality, for any $k \geq 0$, we obtain

$$1 = \Big[E\big(\lambda_\infty^{\frac{p_0}{2}} \lambda_\infty^{-\frac{p_0}{2}} |\mathcal{F}_k \big) \Big]^2 \leq E(\lambda_\infty^{p_0} |\mathcal{F}_k) E(\lambda_\infty^{-p_0} |\mathcal{F}_k) \leq v_k^{p_0} \cdot E(\lambda_\infty^{-p_0} |\mathcal{F}_k).$$

With the help of Jensen's inequality, we write

$$v_k^{-1} \leq E(\lambda_\infty^{-p_0} |\mathcal{F}_k)^{1/p_0} \leq E(\lambda_\infty^{-1} |\mathcal{F}_k).$$

Therefore, we have

$$\begin{aligned} E\bigg(|\sum_{k=n}^{N} d_k g| \bigg| \mathcal{F}_n \bigg) &\leq 2E[\lambda_{n-1} E(\lambda_\infty^{-1} |\mathcal{F}_n)] + 2E\bigg[\sum_{k=n}^{N-1} (\lambda_k - \lambda_{k-1}) \cdot E(\lambda_\infty^{-1} |\mathcal{F}_k) \big| \mathcal{F}_n \bigg] \\ &= 2 + 2E\bigg[\sum_{k=n}^{N-1} \lambda_\infty^{-1} \cdot (\lambda_k - \lambda_{k-1}) \big| \mathcal{F}_n \bigg] \leq 4. \end{aligned}$$

From the above inequality, we conclude

$$\|g\|_{BMO_1} = \sup_{n \geq 1} \bigg\| \bigg[E\bigg(|\sum_{k=n}^{\infty} d_k g|^p \bigg| \mathcal{F}_n \bigg) \bigg]^{1/p} \bigg\|_\infty \leq 4.$$

This proves that $g \in BMO_1$. Then the proof is completed. □

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ZHIWEI HAO
HUNAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
SCHOOL OF MATHEMATICS AND STATISTICS
XIANGTAN 411201, PEOPLE'S REPUBLIC OF CHINA
ORCID: 0000-0001-5553-3398
Email address: haozhiwei@hnust.edu.cn

MEI LI
HUNAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
SCHOOL OF MATHEMATICS AND STATISTICS
XIANGTAN 411201, PEOPLE'S REPUBLIC OF CHINA
ORCID: 0009-0004-3829-745X
Email address: limei@mail.hnust.edu.cn

YAO WANG
HUNAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
SCHOOL OF MATHEMATICS AND STATISTICS
XIANGTAN 411201, PEOPLE'S REPUBLIC OF CHINA
ORCID: 0009-0001-9707-4035
Email address: wangyao@mail.hnust.edu.cn

FERENC WEISZ
EÖTVÖS L. UNIVERSITY
DEPARTMENT OF NUMERICAL ANALYSIS
PÁZMÁNY P. SÉTÁNY 1/C, HUNGARY
ORCID: 0000-0002-7766-2745
Email address: weisz@inf.elte.hu