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# A Geometric Solution to the Jacobian Problem

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**Annotation** – In this article given a geometric solution to the well-known Jacobian problem. The two-dimensional polynomial Keller map is considered in four-dimensional Euclidean space $R^4$. Used the concept of parallel. A well-known example of Vitushkin is also considered. Earlier it was known that Vitushkin's map has a nonzero constant Jacobian and it is not injective. We will show that the Vitushkin map is not surjective and moreover it has two inverse maps in the domain of its definition.

## 1. Introduction

In works [1], [2], [5], [6] the Jacobian problem is reduced to the injectivity problem of polynomial mapping. And in papers [3], [4] the Jacobian problem is reduced to the reversibility of a polynomial map with a non-constant nilpotent Jacobi matrix.

## 2. Properties of Tangent Spaces

Consider the polynomial mapping

$$F(x,y) = (u,v)$$

where $u = f(x,y), v = g(x,y)$ are polynomials from two variables and their Jacobians

$$f_x(x,y)\cdot g_y(x,y) - f_y(x,y)\cdot g_x(x,y) = 1.$$

Such polynomial maps are called kellerovas. The main result of this paper reads as follows:

**Theorem 1.** Any Keller polynomial map is injective over a field of real numbers *R*.

The proof of the theorem relies on methods of analytic geometry in the four-dimensional space $R^4$, where $R^4$ is the field of real numbers. We define the surface $\pi$ in space $R^4$ as a graph of Keller mapping $F: R^4 \to R^4$

$$\pi = \{(x, y, u, v) \in R^4 \mid u = f(x, y), v = g(x, y)\} \quad (1)$$

Tangent plane $K$ of the surface $\pi$ at the point $(x_0, y_0, u_0, v_0) \in \pi$ is determined by the following equations:

$$K: \begin{cases} u = u_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0), \\ v = v_0 + g_x(x_0, y_0(x - x_0) + g_y(x_0, y_0)(y - y_0). \end{cases}$$

Let's write parametric equations of tangent plane $K$:

$$K: \begin{cases} x = t, \\ y = s, \\ u = f_x(x_0, y_0)\,t + f_y(x_0, y_0)\,s + u_0', \\ v = g_x(x_0, y_0)\,t + g_y(x_0, y_0)\,s + v_0'. \end{cases} \quad (2)$$

where

$$u_0' = u_0 - f_x(x_0, y_0)\,x_0 - f_y(x_0, y_0)\,y_0,$$
$$v_0' = v_0 - g_x(x_0, y_0)\,x_0 - g_y(x_0, y_0)\,y_0.$$

Then

$$K = \langle M, \vec{a}, \vec{b} \rangle = \{X \in R^4 \mid \overrightarrow{MX} = x\vec{a} + y\vec{b}, x, y \in R\},$$

where

$M(0, 0, u_0', v_0')$ is the starting point in K,

$$\vec{a} = (1, 0, f_x(x_0, y_0), g_x(x_0, y_0)),$$
$$\vec{b} = (0, 1, f_y(x_0, y_0), g_y(x_0, y_0)), -$$

the guiding vectors of the plane K. As we seen from the parametric equations of the tangent plane K, the surface $\pi(1)$ at the every point has a two - dimensional tangent plane. Therefore, the surface $\pi(1)$ has a dimension equal to two.

The following Lemma plays a key role in the proof of the theorem.

**Lemma 1.** Any tangential plane $K(2)$ of the surface $\pi(1)$ in space $\square^4$ does not contain a line parallel to coordinate planes $Oxy$ and $Ouv$.

**Proof.** Let $M_1 = (x_1, y_1, u_1, v_1)$ and $M_2 = (x_2, y_2, u_2, v_2)$ two different points of the tangent plane $K$. If the line $(M_1, M_2)$ is parallel to the $Oxy$ plane, then vector $\overrightarrow{M_1M_2} = (x_2 - x_1, y_2 - y_1, u_2 - u_1, v_2 - v_1)$ is expressed linearly via vectors $\vec{e_1} = (1, 0, 0, 0)$ and $\vec{e_2} = (0, 1, 0, 0)$. Then from (2) we get

$$u_2 - u_1 = f_x(x_0, y_0)(x_2 - x_1) + f_y(x_0, y_0)(y_2 - y_1) = 0$$
$$v_2 - v_1 = g_x(x_0, y_0)(x_2 - x_1) + g_y(x_0, y_0)(y_2 - y_1) = 0$$

Since the Jacobian equal to 1, then $x_2 - x_1 = 0, y_2 - y_1 = 0$. Hence $u_2 - u_1 = 0, v_2 - v_1 = 0$, that is $M_1 \equiv M_2$. Contradiction.

If the line $(M_1 M_2)$ is parallel to the $Ouv$ plane, then the vector $\overrightarrow{M_1 M_2}$ is expressed linearly via vectors $\vec{e_3} = (0,0,1,0)$ and $\vec{e_4} = (0,0,0,1)$. Then $x_2 - x_1 = 0,$ and $y_2 - y_1 = 0$. Hence $u_2 = u_1$ and $v_2 = v_1$,that is, the points $M_1$ and $M_2$ coincide again. Contradiction.

*Consequence*. Any tangent plane $K(2)$ of the surface $\pi(1)$ in space $R^4$ is not parallel to the coordinate planes $Oxy$ and $Ouv$.

## 3. Proof of Theorem

Let $F(x_1, y_1) = (u_1, v_1) = F(x_2, y_2)$. Then a nonzero vector $\overrightarrow{M_1 M_2} = (x_2 - x_1, y_2 - y_1, 0, 0)$, where $M_1 = (x_1, y_1, u_1, v_1)$ and $M_2 = (x_2, y_2, u_2, v_2)$,is parallel to the coordinate plane $Oxy$. Replacing the mapping $F(x, y)$ with the mapping $F(x + x_1, y + y_1) - F(x_1, y_1)$, we can assume that $M_1 = (x_1, y_1, u_1, v_1)$ coincides with the origin $O(0,0,0,0)$ and the point $M_2 = (x_2, y_2, u_2, v_2)$ coincides with the point $M(a, b, 0, 0)$,where $a = x_2 - x_1, b = y_2 - y_1$. Let

$$\prod = \left\langle O, \overrightarrow{OM}, \vec{e_3} = (0,0,1,0), \vec{e_4} = (0,0,0,1) \right\rangle = \left\{ X \in R^4 \mid \overrightarrow{OX} = x\overrightarrow{OM} + y\vec{e_3} + x\vec{e_4}, x, y, z \in \mathbb{R} \right\}$$

three-dimensional hyperplane in $R^4$. Parametric equations of a plane $\prod$ have the form:

$$\prod : \begin{cases} x = a\tau, \\ y = b\tau, \\ u = p, \\ v = q, \end{cases}$$

where $\tau, p, q \in R$. Parametric equations of a plane $\pi$ have the form:

$$\pi : \begin{cases} x = t, \\ y = s, \\ u = f(t, s), \\ v = g(t, s). \end{cases}$$

where $t, s \in R$. We find the intersection of $\pi \cap \prod$. Have,

$$\pi \cap \prod : \begin{cases} x = at, \\ y = bt, \\ u = f(\text{at}, \text{bt}), \\ v = g(\text{at}, \text{bt}). \end{cases}$$

As you can see, the curve $\pi \cap \prod$ has the following radius-vector

$$r(t) = (at, bt, f(at, bt), g(at, bt))$$

Then the tangent vector to the curve $\pi \cap \prod$ looks like:

$$r'(t) = (a, b, \text{c}(t), \text{d}(t)),$$

where

$$c(t) = f_x(\text{at}, \text{bt})\,\text{a} + f_y(\text{at}, \text{bt})\,\text{b}, \text{d}(t) = g_x(\text{at}, \text{bt})\,\text{a} + \text{g}_y(\text{at}, \text{bt})\,\text{b}.$$

Have $r'(t) = \overrightarrow{OM} + c(t)\overrightarrow{e_3} + d(t)\overrightarrow{e_4}$, where the vector $\overrightarrow{OM}$ is perpendicular to the vector $c(t)\overrightarrow{e_3} + d(t)\overrightarrow{e_4}$ . We find the outer product of vectors $\overrightarrow{OM}$ and $r'(t)$. Have

$$\overrightarrow{OM} \wedge r'(t) = 0 \cdot \overrightarrow{e_1} \wedge \overrightarrow{e_2} + a \cdot c(t)\overrightarrow{e_1} \wedge \overrightarrow{e_3} + a \cdot d(t)\overrightarrow{e_1} \wedge \overrightarrow{e_4} + b \cdot c(t)\overrightarrow{e_2} \wedge \overrightarrow{e_3} + b \cdot d(t)\overrightarrow{e_2} \wedge \overrightarrow{e_4} + 0 \cdot \overrightarrow{e_3} \wedge \overrightarrow{e_4}.$$

Have $\left| r'(t) \wedge \overrightarrow{OM} \right|^2 = (a^2 + b^2)(c(t)^2 + d(t)^2)$.

On the other hand $\left| r'(t) \wedge \overrightarrow{OM} \right| = \left| r'(t) \right| \cdot \left| \overrightarrow{OM} \right| \cdot \sin(\alpha(t))$, where $\alpha(t)$ is the angle between the vectors $r'(t)$ and $\overrightarrow{OM}$ . Here the area of a parallelogram is understood as a focused area. In the vicinity of the point $O(0,0,0,0)$ $\sin \alpha(t)$ is positive , and in vicinity of the points $M(a, \text{b}, 0, 0)$ $\sin \alpha(t)$ is negative or vice versa. Here we assume that map F between the points O and M has no zeros. Then at some $\text{t} \in [0,1] \sin(\alpha(t))$ has zero value. Then, $a^2 + b^2 = 0$ or $c(t)^2 + d(t)^2 = 0$ . Contradiction. The theorem is proved.

## 3. Vitushkin Example

Is considered the following well-known example of Vitushkin:

$$u(x, y) = x^2 y^6 + 2xy^2$$

$$v(x, y) = xy^3 + \frac{1}{y}, y \neq 0$$

The map $F: R^2 \to R^2$ is defined as

$$F(x, y) = (u, v),\ y \neq 0 .$$

Vitushkin's map is not injective, namely $F(-3,-1) = F(1,1) = (3,2)$ and has a constant nonzero Jacobian: $J(F) = -2$. Since $\lim_{(x,y)\to(x,+0)} v(x, y) = +\infty$ and $\lim_{(x,y)\to(x,-0)} v(x, y) = -\infty$ , domain of the Vitushkin map is divided into two parts, with the points $(-3,-1,3,2)$ and $(1,1,3,2)$ lying in different parts of the domin. Namely, these points lie in different sides of the hyperplane $y = 0$ of dimension three.

**Theorem 2.** *Vitushkin's map not surjective and has two reverse-mapping.*

**Proof**. $v^2 - u = \dfrac{1}{y^2} > 0$, that is, $u < v^2$. Hence, the upper part of the three-dimensional paraboloid $u = v^2$ has no inverse image. Consider the following maps:

$$G_+(x, y) = \left( \frac{x\left(y^2 - x\right)^{\frac{3}{2}}}{\sqrt{y^2 - x} + y}, \frac{1}{\sqrt{y^2 - x}} \right), y > 0, y^2 > x,$$

$$G_-(x, y) = \left( \frac{x\left(y^2 - x\right)^{\frac{3}{2}}}{\sqrt{y^2 - x} - y}, -\frac{1}{\sqrt{y^2 - x}} \right), y < 0, y^2 > x,$$

An immediate check indicates that

$$F \circ G_+ = E = G_+ \circ F \text{ , for } y > 0, y^2 > x$$

and

$$F \circ G_- = E = G_- \circ F \text{ , } y > 0, y^2 > x .$$

Thus, the Vitushkin's map has the following four properties:

1. The Vitushkin's map has a nonzero constant Jacobian,

$$J(F) = -2;$$

2. The Vitushkin's map is not injective,

$$F(-3,-2) = (3,2) = F(1,1);$$

3. The Vitushkin's map not surjective,

$$v^2 - u = \frac{1}{y^2} > 0 \text{ , } (u,v) = F(x, y);$$

4. The Vitushkin map has two inverse mappings,

$$F^{-1} = G_{+}, y > 0, y^2 > x,$$
$$F^{-1} = G_{-}, y < 0, y^2 > x.$$

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