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Original Article

The R_0 and R_1 Properties in Fuzzy Soft Topological Spaces

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Abstaract — The purpose of this paper is to introduce and study some new properties so-called fuzzy soft R_i (for short, FSR_i, $i = 0, 1$) on fuzzy soft spaces by using quasi-coincident relation for fuzzy soft points, we get some characterizations and properties of them. Also, the relationships of these properties in fuzzy soft topologies which are constructed from crisp topology and soft topology over X and vice versa are studied with some illustrative examples.

Keywords — *Fuzzy soft set, Fuzzy soft point, Fuzzy soft quasi-coincident, Fuzzy soft topology.*

1 Introduction

In 1999, Molodtsov [8] introduced the concept of soft set as one of mathematical tools for dealing with uncertainties. The works on the soft set theory have been applied in several directions. Maji et al.[7] introduced the concept of fuzzy soft set with some its properties. Then fuzzy soft theory and its applications have been studied by many authors. Chang [2] introduced the concept of fuzzy topology. Tanay et al.[12] introduced the definition of fuzzy soft topology over a subset of the initial universe set while Roy and Samanta [9] gave the definition of fuzzy soft topology over the initial universe set. In recent time, many of notions and results in fuzzy soft topology have been studied as in [1, 3, 4, 5, 10].

In this paper, we define and study some new properties and results related to fuzzy soft spaces. The main aim of our work is to introduce and study the R_0 and R_1 properties in fuzzy soft topological spaces by using quasi-coincidence for fuzzy soft points. Some characterizations and basic properties of them are studied. Also we, investigate the relationships of these properties in fuzzy soft topologies which are derived from crisp topology and vice versa with some necessary examples.

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2 Definitions and Notions

Throughout this work, X refers to a universe set, E be the set of all parameters for X, $P(X)$ is the power set of X and I^X be the set of all fuzzy subsets of X, where $I = [0, 1]$.

Definition 2.1. [2, 13] A fuzzy set A on X is a set characterized by a membership function $\mu_A : X \longrightarrow I$ whose value $\mu_A(x)$ represents the degree of membership of x in A for $x \in X$. A fuzzy point x_λ $(0 < \lambda \leq 1)$ is a fuzzy set in X given by $x_\lambda(y) = \lambda$ at $x = y$ and $x_\lambda(y) = 0$ otherwise for all $y \in X$. Here x and λ are called support and the value of x_λ, respectively. The set of all fuzzy point in X denoted by $FP(X)$. For $\alpha \in I$, $\underline{\alpha} \in I^X$ refers to the fuzzy constant function where, $\underline{\alpha}(x) = \alpha \;\; \forall x \in X$ and for $x_\lambda \in FP(X)$, O_{x_λ} refers to a fuzzy open set contains x_λ and called fuzzy open neighborhood of x_λ. For $A,\ B \in I^X$, the basic operations for fuzzy sets are given by Zadah [13].

Definition 2.2. [6, 8] A soft set $F_E = (F, E)$ over X with the set E of parameters is a mapping $F : E \longrightarrow P(X)$ the value $F(e)$ is a set called e-element of the soft set for all $e \in E$. Thus a soft set over X can be represented by the set of ordered pairs $F_E = \{(e, F(e)) : \ e \in E, \ F(e) \in P(X) \}$, we denote the family of all soft sets over X by $SS(X, E)$.

Definition 2.3. [6, 11] Let $F_E \in SS(X, E)$ be a soft set over X. Then:

i. F_E is called a null soft set, denoted by $\emptyset_E$, if $F(e) = \emptyset$ for every $e \in E$. And if $F(e) = X$ for all $e \in E$, then F_E is called an universal soft set, denoted by X_E.

ii. If $F(e) = \{x\}$ and $F(e^{'}) = \emptyset$ for all $e^{'} \in E - \{e\}$, then F_E is called a soft point and denoted by x^e. The complement of a soft point x^e is a soft set over X dented by $(x^e)^c$ and given by $(x^e)^c(e) = X - \{x\}$, $(x^e)^c(e^{'}) = X$ for all $e^{'} \in E - \{e\}$. The set of all soft points over X is denoted by $SP(X, E)$.

Definition 2.4. [7, 9] A fuzzy soft set $f_E = (f, E)$ over X with the set E of parameters is defined by the set of ordered pairs $f_E = \{(e, f(e)) : e \in E \ , \ f(e) \in I^X\}$. Here f is a mapping given by $f : E \longrightarrow I^X$ and the value $f(e)$ is a fuzzy set called e-element of the fuzzy soft set for all $e \in E$. The family of all fuzzy soft sets over X is denoted by $FSS(X, E)$.

Definition 2.5. [7, 9] Let f_E, g_E are two fuzzy soft sets over X. Then:

i. f_E is called a null fuzzy soft set, denoted by $\tilde{0}_E$ if $f(e) = \underline{0}$ for all $e \in E$. And if $f(e) = \underline{1}$ for all $e \in E$, then f_E is called universal fuzzy soft, denoted by $\tilde{1}_E$.

ii. A fuzzy soft f_E is subset of g_E if $f(e) \leq g(e)$ for all $e \in E$, denoted by $f \sqsubseteq g$.

iii. f_E and g_E are equal if $f_E \sqsubseteq g_E$ and $g_E \sqsubseteq f_E$. It is denoted by $f_E = g_E$.

iv. The complement of f_E is denoted by f^c_E, where $f^c : E \longrightarrow I^X$ is a mapping defined by $f(e)^C = \underline{1} - f(e)$ for all $e \in E$. Clearly, $f^c_E)^c = f_E$.

v. The union of f_E , g_E is a fuzzy soft set h_E defined by $h(e) = f(e) \cup g(e)$ for all $e \in E$. h_E is denoted by $f_E \sqcup g_E$.

vi. The intersection of f_E and g_E is a fuzzy soft set l_E defined by, $l(e) = f(e) \cap g(e)$ for all $e \in E$. l_E is denoted by $f_E \sqcap g_E$.

Definition 2.6. [1] A fuzzy soft point x_α^e over X is a fuzzy soft set over X defined as follows:
$x_\alpha^e\left(e'\right) = \begin{cases} x_\alpha & if \quad e' = e \\ \underline{0} & if \quad e' \in E - \{e\} \end{cases}$ where,
x_α is the fuzzy point in X with support x and value α, $\alpha \in (0, 1]$. The set of all fuzzy soft points in X is denoted by $FSP(X, E)$. The fuzzy soft point x_α^e is called belongs to a fuzzy soft set f_E , denoted by $x_\alpha^e \widetilde{\in} f_E$ iff $\alpha \leq f(e)(x)$. Every non-null fuzzy soft set f_E can be expressed as the union of all the fuzzy soft points belonging to f_E. The complement of a fuzzy soft point x_α^e is a fuzzy soft set over X.

Definition 2.7. [1, 9] Let X be a universe set, E be a fixed set of parameters and δ be the family of fuzzy soft sets over X, then δ is said to be a fuzzy soft topology on X iff:

i. $\tilde{0}_E$, $\tilde{1}_E$ belong to δ,

ii. The union of any number of fuzzy soft sets in δ is in δ,

iii. The intersection of any two fuzzy soft sets in δ is in δ.

In this case, (X, δ, E) is called a fuzzy soft topological space. The members of δ are called fuzzy soft open sets in X, denoted by $FSO(X, \delta, E)$. A fuzzy soft set f_E over X is called fuzzy soft closed in X iff $f_E^c \in \delta$, the set of all fuzzy soft closed sets over X, denoted by $FSC(X, \delta, E)$.

Notation.[10] Let (X, δ, E) be a fuzzy soft topological space. For $x_\alpha^e \in FSP(X, E)$ the fuzzy soft set $O_{x_\alpha^e}$ refers to a fuzzy soft open set contains x_α^e and $O_{x_\alpha^e}$ is called a fuzzy soft open neighborhood of x_α^e. The fuzzy soft open neighborhood system of x_α^e denoted by, $N_E(x_\alpha^e)$ is the family of all its fuzzy soft open neighborhoods.
In general for, $f_E \in FSS(X, E)$ the notation O_{f_E} refers to a fuzzy soft open set contains f_E and is called a fuzzy soft open neighborhood of f_E.

Definition 2.8. [1, 9] Let (X, δ, E) be a fuzzy soft topological space and $f_E \in FSS(X, E)$. Then:

i. The fuzzy soft interior of f_E is the fuzzy soft set denoted by f_E° and given by $f_E^\circ = \sqcup\{g_E : \ g_E \in \delta \text{ and } \ g_E \sqsubseteq f_E\}$, that is f_E° is a fuzzy soft open set. Indeed it is the largest fuzzy soft open set contained in f_E.

ii. The fuzzy soft closure of f_E is the fuzzy soft set denoted by $\overline{f_E}$ and given by $\overline{f_E} = \sqcap\{g_E : g_E \in \delta^c \text{ and } f_E \sqsubseteq g_E\}$, that is $\overline{f_E}$ is a fuzzy soft closed set. Clearly, $\overline{f_E}$ is the smallest fuzzy soft closed set over X which contains f_E.

Definition 2.9. [4] Let (X, δ, E) be a fuzzy soft topological space and $Y \subseteq X$. Let h_E^Y be a fuzzy soft set over (Y, E) such that $h_E^Y : E \longrightarrow I^Y$ such that $h_E^Y(e) \in I^Y$,
$h_E^Y(e)(x) = \begin{cases} 1 & \text{if} \quad x \in Y \\ 0 & \text{if} \quad x \notin Y \end{cases}$.
Let $\delta_Y = \{h_E^Y \sqcap g_E : g_E \in \delta\}$, then the fuzzy soft topology δ_Y on (Y, E) is called

fuzzy soft subspace topology for (Y, E) and (Y, δ_Y, E) is called a fuzzy soft subspace of (X, δ, E). If $h_E^Y \in \delta$ (resp. $h_E^Y \in \delta^c$), then (Y, δ_Y, E) is called fuzzy open (resp. closed) soft subspace of (X, δ, E).

Definition 2.10. [10] For $A \subseteq X$. The soft characteristic of A, denoted by $\widetilde{\chi}_A$ is a fuzzy soft set $\widetilde{\chi}_A : E \longrightarrow I^X$ defined by, $\widetilde{\chi}_A(e) = \chi_A \ \forall e \in E,$ where χ_A is the characteristic of A. *i.e.* $\widetilde{\chi}_A = \{(e, \chi_A) : e \in E\}$, where $\chi_A : X \longrightarrow \{0, 1\}$.

Definition 2.11. [10] Let $f_E \in FSS(X, E)$. Then the soft support of f_E, denoted by $Ssup(f_E)$ is a soft set given by, $Ssup(f_E) = \{(e, S(f(e)) : e \in E\}$, where $S(f(e))$ is the support of fuzzy set $f(e)$, which is given by the set $S(f(e)) = \{x \in X : f(e)(x) > 0\} \subseteq X$.

Definition 2.12. [1] The fuzzy soft sets f_E and g_E in (X, E) are called fuzzy soft quasi-coincident, denoted by $f_E q g_E$ iff there exist $e \in E$, $x \in X$ such that $f(e)(x) + g(e)(x) > 1$. If f_E is not fuzzy soft quasi-coincident with g_E, then we write $f_E \tilde{q} g_E$, that is $f_E \tilde{q} g_E$ iff $f(e)(x) + g(e)(x) \leq 1$, *i.e.* $f(e)(x) \leq g^c(e)(x)$ for all $x \in X$ and $e \in E$.
A fuzzy soft point x_α^e is said to be soft quasi-coincident with f_E, denoted by $x_\alpha^e q f_E$ iff there exists $e \in E$ such that $\alpha + f_E(e)(x) > 1$.

Proposition 2.13. [1, 10] Let x_α^e, $y_\beta^e \in FSP(X, E)$, f_E, g_E, $h_E \in FSS(X, E)$ and $\{f_{iE} : i \in J\} \subseteq FSS(X, E)$. Then we have:

1. $f_E \tilde{q} g_E \Longleftrightarrow f_E \sqsubseteq g_E^c$,
2. $f_E \sqcap g_E = \widetilde{0}_E \Longrightarrow f_E \tilde{q} g_E$,
3. $f_E \tilde{q} g_E$, $h_E \sqsubseteq g_E \Longrightarrow f_E \tilde{q} h_E$,
4. $f_E q g_E \Longleftrightarrow x_\alpha^e q g_E,$ *for some* $x_\alpha^e \widetilde{\in} f_E$,
5. $x_\alpha^e \tilde{q} f_E \Longleftrightarrow x_\alpha^e \widetilde{\in} f_E^c$,
6. $f_E \sqsubseteq g_E \Longleftrightarrow (x_\alpha^e q f_E \Longrightarrow x_\alpha^e q g_E$ for all $x_\alpha^e)$,
7. $f_E \tilde{q} f_E^c$,
8. If $x_\alpha^e q(\sqcap_{i \in J} f_{iE})$, then $x_\alpha^e q f_{iE}$ for all $i \in j$,
9. $x \neq y \Longrightarrow x_\alpha^e \tilde{q} y_\beta^e$ for all $\alpha, \beta \in I$,
10. $x_\alpha^e \tilde{q} y_\beta^e \Longleftrightarrow x \neq y$ *or* $(x = y$ *and* $\alpha + \beta \leq 1)$.

Lemma 2.14. [10] Let (X, δ, E) be a fuzzy soft topological space and $x_\alpha^e \in FSP(X, E)$. Then:

i. $g_E q f_E \Longleftrightarrow g_E q \overline{f_E}$ for all $g_E \in FSO(X, \delta, E)$,

ii. $x_\alpha^e q \overline{f_E} \Longleftrightarrow O_{x_\alpha^e} q f_E$ for all $O_{x_\alpha^e} \in N_E(x_\alpha^e)$.

Theorem 2.15. [10]

i. Let (X, τ) be a crisp topological space, then the family $\delta_\tau = \{ \widetilde{\chi}_A : A \in \tau\}$ forms a fuzzy soft topology on X induced by τ,

ii. Every fuzzy soft topological space (X, δ, E) defines a crisp topology on X in the form $\tau_\delta = \{A \subseteq X : \widetilde{\chi}_A \in \delta\}$ which is induced by δ.

Theorem 2.16. [10]

i. Let (X, τ^*, E) be a soft topological space, then the collection $\delta_{\tau^*} = \{f_E \in FSS(X, E) : Ssup(f_E) \in \tau^*\}$ defines the fuzzy soft topology on X which is induced by τ^*.

ii. Let (X, δ, E) be a fuzzy soft topological space, then the family $\tau^*_\delta = \{Ssup(f_E) : f_E \in \delta\}$ defines the soft topology on X which is induced by δ.

Proposition 2.17. [10] Let (X, τ) be a topological space, (X, τ^*, E) be a soft topological space and (X, δ, E) be a fuzzy soft topological space. Then:

i. $\underline{\alpha}_E \in \delta_{\tau^*}$ for all $\alpha \in I$,

ii. $F_E \in \tau^* \Longrightarrow \widetilde{\chi}_{F_E} \in \delta_{\tau^*}$, in particular $\delta_\Delta \subseteq \delta_{\tau^*}$.

3 Fuzzy Soft R_i -Spaces, $i = 0$,1.

Definition 3.1. A fuzzy soft topological space (X, δ, E) is said to be:

i. Fuzzy soft R_0 (FSR_0, for short) iff for every x^e_α , $y^e_\beta \in FSP\,(X, E)$ with $x^e_\alpha \tilde{q} \overline{y^e_\beta}$ implies $\overline{x^e_\alpha}\, \tilde{q} y^e_\beta$.

ii. Fuzzy soft R_1 (FSR_1, for short) iff for every x^e_α , $y^e_\beta \in FSP(X, E)$ with $x^e_\alpha \tilde{q} \overline{y^e_\beta}$ implies there exist $O_{x^e_\alpha}$ and $O_{y^e_\beta} \in \delta$ such that $O_{x^e_\alpha} \tilde{q} O_{y^e_\beta}$.

In the following we get some characteristics of $FSR_i -$ spaces, $i = 0, 1$.

Theorem 3.2. Let (X, δ, E) be a fuzzy soft topological space. Then the following items are equivalent:

i. (X, δ, E) is FSR_0.

ii. $\overline{x^e_\alpha} \sqsubseteq O_{x^e_\alpha}$ for all $O_{x^e_\alpha} \in \delta$.

iii. $\overline{x^e_\alpha} \sqsubseteq \sqcap\{O_{x^e_\alpha} : O_{x^e_\alpha} \in \delta\}$ for all $x^e_\alpha \in FSP(X, E)$.

Proof. i $\Longrightarrow$ ii) Let (X, δ, E) be FSR_0 and $y^e_\beta q \overline{x^e_\alpha}$, then $x^e_\alpha q \overline{y^e_\beta}$ implies $y^e_\beta q O_{x^e_\alpha}$ $\forall\, O_{x^e_\alpha}$. Hence $\overline{x^e_\alpha} \sqsubseteq O_{x^e_\alpha}$ $\forall\, O_{x^e_\alpha}$ (by 6) of Proposition 2.13).
ii $\Longrightarrow$ iii) Obvious.
iii $\Longrightarrow$ i) Let $\overline{x^e_\alpha} \sqsubseteq \sqcap\{O_{x^e_\alpha} : O_{x^e_\alpha} \in N_E(x^e_\alpha)\} \sqsubseteq O_{x^e_\alpha}$ $\forall\, O_{x^e_\alpha}$. Now let x^e_α , $y^e_\beta \in FSP(X, E)$ with $x^e_\alpha \tilde{q} \overline{y^e_\beta}$, then $x^e_\alpha \in \overline{y^e_\beta}^c = O_{x^e_\alpha}$ and so, by hypothesis $\overline{x^e_\alpha} \sqsubseteq O_{x^e_\alpha} = \overline{y^e_\beta}^c = (y^e_\beta)^{c\circ} \sqsubseteq (y^e_\beta)^c \Longrightarrow \overline{x^e_\alpha} \tilde{q} y^e_\beta$. Hence (X, δ, E) is FSR_0.

Theorem 3.3. Let (X, δ, E) be a fuzzy soft topological space and $f_E \in FSC(X, \delta, E)$. Then the following items are equivalent:

i. (X, δ, E) is FSR_0.

ii. $x^e_\alpha \tilde{q} f_E$ implies there exists $O_{f_E} \in \delta$ contains f_E such that $x^e_\alpha \tilde{q} O_{f_E}$.

iii. $x^e_\alpha \tilde{q} f_E \Longrightarrow \overline{x^e_\alpha} \tilde{q} f_E$.

iv. $x^e_\alpha \tilde{q} \overline{y^e_\beta} \Longrightarrow \overline{x^e_\alpha} \tilde{q} \overline{y^e_\beta}$.

Proof. i $\Longrightarrow$ ii) Let (X, δ, E) be FSR_0 ,$f_E \in FSC(X, \delta, E)$ and $x^e_\alpha \tilde{q} f_E$, then $x^e_\alpha \in f^c_E = O_{x^e_\alpha} \Longrightarrow \overline{x^e_\alpha} \sqsubseteq f^c_E = O_{x^e_\alpha}$ (by Theorem 3.2) $\Longrightarrow f_E \sqsubseteq \overline{x^e_\alpha}^c = O_{f_E}$. Since $x^e_\alpha \sqsubseteq \overline{x^e_\alpha}$, then $\overline{x^e_\alpha}^c \sqsubseteq (x^e_\alpha)^c$. Hence $x^e_\alpha \tilde{q} \overline{x^e_\alpha}^c = O\ _{f_E}$.
ii) $\Longrightarrow$ iii) Let $x^e_\alpha \tilde{q} f_E$, then by hypothesis there exists O_{f_E} such that $x^e_\alpha \tilde{q} O_{f_E} \Longrightarrow \overline{x^e_\alpha} \tilde{q} f_E$ (by ii. of Lemma 2.14).
iii $\Longrightarrow$ iv) it is clear.
iv $\Longrightarrow$ i) Let x^e_α , $y^e_\beta \in FSP(X, E)$ with $x^e_\alpha \tilde{q} \overline{y^e_\beta} \Longrightarrow \overline{x^e_\alpha} \tilde{q} \overline{y^e_\beta}$ (by given). Since $y^e_\beta \sqsubseteq \overline{y^e_\beta}$, then $\overline{x^e_\alpha} \tilde{q} y^e_\beta$. Hence (X, δ, E) is FSR_0.

Theorem 3.4. Every $FSR_1 - space$ is a $FSR_0 - space$.

Proof. Obvious.

Corollary 3.5. Let (X, δ, E) be a fuzzy soft topological space. Then (X, δ, E) is FSR_1 if and only if for all x^e_α , $y^e_\beta \in FSP(X, E)$ with $x^e_\alpha \tilde{q} \overline{y^e_\beta}$ implies there exist $O_{\overline{x^e_\alpha}}$, $O_{\overline{y^e_\beta}} \in \delta$ such that $O_{\overline{x^e_\alpha}} \tilde{q} O_{\overline{y^e_\beta}}$.

Proof. Follows from the above theorem and from ii. of Theorem 3.2 .

Theorem 3.6. Every subspace (Y, δ_Y, E) of a $FSR_i - space$ (X, δ, E) is a $FSR_i - space$, $i = 0, 1$.

Proof. As a sample we prove the case $i = 1$.
Let x^e_α, y^e_β are fuzzy soft points in (Y, E) with $x^e_\alpha \tilde{q} \overline{y^e_\beta}$. Then x^e_α, y^e_β also in (X, E) with $x^e_\alpha \tilde{q} \overline{y^e_\beta}$. Since (X, δ, E) is FSR_1, then there exist $O_{x^e_\alpha}$, $O_{y^e_\beta} \in \delta$ such that $O_{x^e_\alpha} \tilde{q} O_{y^e_\beta}$ and so, there exist $O^*_{x^e_\alpha} = O_{x^e_\alpha} \sqcap h^Y_E \in \delta_Y$, $O^*_{y^e_\beta} = O_{y^e_\beta} \sqcap h^Y_E \in \delta_Y$ such that $O^*_{x^e_\alpha} \tilde{q} O^*_{y^e_\beta}$.
Hence (Y, δ_Y, E) is FSR_1

Lemma 3.7. Let (X, τ) and (X, τ^*, E) be a topological space and a soft topological space respectively, then we have:

i. $\overline{\widetilde{\chi}}^{\delta_\tau}_{\{x\}} = \widetilde{\chi}_{\overline{\{x\}}^\tau}$ for all $x \in X$.

ii. $\overline{\widetilde{\chi}}^{\delta_{\tau^*}}_{\{x^e\}} = \widetilde{\chi}_{\overline{\{x^e\}}^{\tau^*}}$ for all $x^e \in SP(X, E)$.

Proof. Straightforward.

In the following, we introduce some relationships for FSR_i-axioms, $i = 0, 1$ in fuzzy soft topologies and that on crisp and soft topologies.

Theorem 3.8. Let (X, τ) be a topological space. Then (X, δ_τ, E) is a FSR_i-space if and only if (X, τ) is an $R_i -$ space, $i = 0, 1$.

Proof. 1) For the case $i = 0$. Let (X, δ_τ, E) be FSR_0 and $x \in \overline{y}$. Then $x_1^e \widetilde{\in} \overline{y_1^e}$ and $x_1^e q \overline{y_1^e}$ (by i. of the above lemma). Since (X, δ_τ, E) is FSR_0, then $x_1^e q O_{y_1^e} \Longrightarrow y_1^e q \overline{x_1^e}$ (by ii. of Lemma 2.14). Thus $y_1^e \widetilde{\in} \overline{x_1^e}$ and so, $y \in \overline{x}$ (by i. of the above lemma). Hence (X, τ) is an $R_0 -$ space.
Conversely, let (X, τ) be R_0 and $x_\alpha^e,\ y_\beta^e \in FSP(X, E)$ with $x_\alpha^e q \overline{y_\beta^e}$, in particular $x_1^e q \overline{y_1^e} \Longrightarrow x_1^e \not\sqsubseteq \overline{y_1^e}^c \Longrightarrow x_1^e \sqsubseteq \overline{y_1^e} \Longrightarrow x \in \overline{y}$(by i. of the above lemma). Since (X, τ) is R_0, then $y \in \overline{x} \Longrightarrow y_1^e \widetilde{\in} \overline{x_1^e} = \overline{x_\alpha^e} \Longrightarrow y_\beta^e \sqsubseteq y_1^e \widetilde{\notin} \overline{x_1^e}^c = \overline{x_\alpha^e}^c$(by i. of the above lemma)$\Longrightarrow y_\beta^e q \overline{x_\alpha^e}$. Hence we obtain the result.

2) For the case $i = 1$. Let (X, τ) be R_1 and $x_\alpha^e,\ y_\beta^e \in FSP(X, E)$ with $x_\alpha^e \tilde{q} \overline{y_\beta^e}$, in particular $x_1^e \tilde{q} \overline{y_1^e} \Longrightarrow x_1^e \sqsubseteq \overline{y_1^e}^c \Longrightarrow x_1^e \not\sqsubseteq \overline{y_1^e} \Longrightarrow x \notin \overline{y} \Longrightarrow \overline{x} \neq \overline{y}$, then there exist $O_x,\ O_y \in \tau$ such that $O_x \cap O_y = \emptyset$. Take $O_{x_\alpha^e} = \widetilde{\chi}_{O_x} \in \delta_\tau$ and $O_{y_\beta^e} = \widetilde{\chi}_{O_y} \in \delta_\tau$, then $O_{x_\alpha^e} \tilde{q} O_{y_\beta^e}$. Hence (X, δ_τ, E) is FSR_1.
Conversely, let (X, δ_τ, E) is FSR_1 and $\overline{x} \neq \overline{y} \Longrightarrow$ there exists $x \in X$ such that $x \in \overline{x}$ and $x \notin \overline{y} \Longrightarrow x_1^e \not\sqsubseteq \overline{y_1^e} \Longrightarrow x_1^e \tilde{q} \overline{y_1^e}$, then there exist $O_{x_1^e},\ O_{y_1^e} \in \delta_\tau$ such that $O_{x_1^e} \tilde{q} O_{y_1^e}$ and so, there exist $O_x,\ O_y \in \tau$ such that $O_{x_1^e} = \widetilde{\chi}_{O_x}$ and $O_{y_1^e} = \widetilde{\chi}_{O_y}$, then $\widetilde{\chi}_{O_x} \sqsubseteq \widetilde{\chi}^c_{O_y} \Longrightarrow O_x \subseteq O_y^c \Longrightarrow O_x \cap O_y = \emptyset$. Hence the result holds.

Theorem 3.9. Let (X, δ, E) be a fuzzy soft topological space. If (X, δ, E) is a FSR_0-space, then (X, τ_δ) is a R_0-space.

Proof. It is similar to that of the necessity part of the above theorem.

Note. An R_i-space (X, τ_δ) need not imply (X, δ, E) FSR_i-space, $i = 0, 1$, this fact can be shown by the following examples.

Examples 3.10. 1) Let $X = \{x, y, z\}$ and $E = \{e_1, e_2\}$, then the family $\delta = \{\tilde{0}_E, \tilde{1}_E, f_E = \{(e_1, (x_1, y_{0.5})), (e_2, \underline{1})\},\ g_E = \{(e_1, x_{0.5})\}\}$ is a fuzzy soft topology on X and ${}_\delta = \{\emptyset, X\}$ is a topology on X which is induced by δ. It is easy to check that $(X,_\delta)$ *is* R_0, but the fuzzy soft topological space (X, δ, E) is not FSR_0 . Indeed, for $x_{0.5}^{e_1} \in FSP(X, E)$, $\overline{x_{0.5}^{e_1}} = \{(e_1, (x_{0.5}, y_1,\ z_1)), (e_2, \underline{1})\}$, but there exists $O_{x_{0.5}^{e_1}} = \{(e_1, (\ x_1, y_{0.5})), (e_2, \underline{1})\}$ such that $\overline{x_{0.5}^{e_1}} \not\sqsubseteq O_{x_{0.5}^{e_1}}$.

2) Let $X = \{a, b, c\}$ and $E = \{e_1, e_2\}$. Then the family $\delta = \{\tilde{0}_E, \tilde{1}_E, f_E\}$, where $f_E = \{(e_1, (a_{0.3}\ , c_{0.5})), (e_2, (a_{0.3}, c_{0.5}))\}$ is a fuzzy soft topology on X and $\tau_\delta = \{\emptyset, X\}$ is a topology on X which is induced by δ. It is clear that (X, τ_δ) is R_1, but (X, δ, E) is not FSR_1, because for $a_{0.3}^{e_2},\ b_1^{e_2} \in FSP(X, E)$ with $a_{0.3}^{e_2} \tilde{q} \overline{b_1^{e_2}} \Longrightarrow O_{a_{0.3}^{e_2}} q O_{b_1^{e_2}}$ for all $O_{a_{0.3}^{e_2}},\ O_{b_1^{e_2}} \in \delta$.

Definition 3.11. A soft topological space (X, τ^*, E) is said to be:

i. Soft R_0(for short, SR_0) iff for every pair of soft points $x^e,\ y^e(x \neq y) \in SP(X, E)$ with $x^e \in \overline{y^e}$ implies $y^e \in \overline{x^e}$.

ii. Soft R_1(for short, SR_1) iff for every pair of soft points $x^e,\ y^e(x \neq y) \in SP(X, E)$ with $\overline{x^e} \neq \overline{y^e}$ implies there exist two soft open sets $F_E,\ G_E$ contains x^e and y^e respectively, such that $F_E \widetilde{\cap} G_E = \emptyset_E$.

Theorem 3.12. Let (X, τ^*, E) be a soft topological space, then we have:

i. (X, δ_τ^*, E) is FSR_0 if and only if (X, τ^*, E) is SR_0.

ii. If (X, τ^*, E) is SR_1, then $(X, \delta_\tau{}^*, E)$ is FSR_1.

Proof. i.) Let (X, δ_{τ^*}, E) be FSR_0 and $x^e \in \overline{y^e}$, then $x_1^e q \overline{y_1^e}$ (by Lemma 3.7). Since (X, δ_{τ^*}, E) is FSR_0, then $x_1^e q O_{y_1^e} \Longrightarrow y_1^e q \overline{x_1^e}$ (by ii. of Lemma 2.14)$\Longrightarrow y_1^e \not\sqsubseteq \overline{x_1^e}^c \Longrightarrow y_1^e \sqsubseteq \overline{x_1^e}$. Thus $y^e \in \overline{x^e}$(by Lemma 3.7). Hence (X, τ^*, E) is SR_0.
Conversely, let (X, τ^*, E) be SR_0 and $x_\alpha^e \in FSP(X, E)$. Since $x_\alpha^e \in \delta_{\tau^*}^c$ $\forall \alpha \in I - \{0, 1\}$, then $\overline{x_\alpha^e} = x_\alpha^e \sqsubseteq O_{x_\alpha^e}$ $\forall O_{x_\alpha^e}$. When $\alpha = 1$, then clearly $\overline{x_1^e} = O_{x_1^e}$. Hence we obtain the result.

ii.) Let (X, τ^*, E) be SR_1 and x_α^e, $y_\beta^e \in FSP(X, E)$ with $x_\alpha^e \tilde{q} \overline{y_\beta^e} \Longrightarrow x_\alpha^e \tilde{q} y_\beta^e$. Then we have, either $x \neq y$ or ($x = y$ and $\alpha + \beta \leq 1$) (by 10. of Proposition 3.13).
Case I. If $x \neq y$, then $x^e \neq y^e \Longrightarrow (\overline{x^e} \neq \overline{y^e}$ *or* $\overline{x^e} = \overline{y^e})$. Now we have:

a. If $\overline{x^e} \neq \overline{y^e}$, then there exsit $O_{x^e}, O_{y^e} \in \tau^*$ such that $O_{x^e} \tilde{\sqcap} O_{y^e} = \emptyset_E$. Take $O_{x_\alpha^e} = \widetilde{\chi}_{O_{x^e}} \in \delta_{\tau^*}$ and $O_{y_\beta^e} = \widetilde{\chi}_{O_{y^e}} \in \delta_{\tau^*}$, then $O_{x_\alpha^e} \tilde{q} O_{y_\beta^e}$. Hence (X, δ_{τ^*}, E) is a FSR_1-space.

b. If $\overline{x^e} = \overline{y^e}$, then this case is excluded (since (X, τ^*, E) is SR_1).

Case II. If ($x = y$ and $\alpha + \beta \leq 1$). Take $O_{x_\alpha^e} = \underline{\alpha_E} \in \delta_{\tau^*}$, $O_{y_\beta^e} = \underline{\beta_E} \in \delta_{\tau^*}$, then $O_{x_\alpha^e} = \underline{\alpha_E} \tilde{q} O_{y_\beta^e} = \underline{\beta_E}$. Hence (X, δ_{τ^*}, E) is a FSR_1-space.

Note. A soft R_i-space (X, τ_δ^*, E) need not imply (X, δ, E) FSR_i , $i = 0, 1$, this fact can be shown by the following example.

Example 3.13. Let $X = \{a, b\}$ and $E = \{e_1, e_2\}$. The family $\delta = \{\tilde{0}_E, \tilde{1}_E, f_E, g_E, h_E\}$, where $f_E = \{(e_1, a_{0.6}), (e_2, a_{0.6})\}$, $g_E = \{(e_1, b_{0.9}), (e_2, b_{0.9})\}$, $h_E = \{(e_1, (a_{0.6}, b_{0.9})),$ $(e_2, (a_{0.6}, b_{0.9}))\}$ is a fuzzy soft topology on X and $\tau_\delta^* = \{\emptyset_E$, X_E, $F_E = \{(e_1, \{a\}),$ $(e_2, \{a\})\}$, $G_E = \{(e_1, \{b\}), (e_2, \{b\})\}\}$ is a soft topology on X which is induced by δ. It is clear that (X, τ_δ^*, E) is soft R_1, but (X, δ, E) is not FSR_0, because for $a_{0.3}^{e_1}$, $b_1^{e_1} \in FSP(X, E)$ with $a_{0.3}^{e_1} \tilde{q} \overline{b_1^{e_1}}$, but $b_1^{e_1} q \overline{a_{0.3}^{e_1}}$. Also, it is not FSR_1, because for $a_{0.3}^{e_1}$, $b_1^{e_1} \in FSP(X, E)$ with $a_{0.3}^{e_1} \tilde{q} \overline{b_1^{e_1}} \Longrightarrow O_{a_{0.3}^{e_1}} q O_{b_1^{e_1}}$ for all $O_{a_{0.3}^{e_1}}$, $O_{b_1^{e_1}} \in \delta$.

4 Conclusion

In this paper, we defined and studied some new axioms are called the R_0 and R_1 properties in fuzzy soft topological spaces and some of its properties. Also, the relationships of these properties are studied. We hope these basic results will help the researchers to enhance and promote the research on fuzzy soft theory and its applications. In the next work, by the same manner, we defined and study a new set of separation axioms on fuzzy soft spaces.

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