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Remainders of locally Čech-complete spaces and homogeneity

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Abstract

We study remainders of locally Čech-complete spaces. In particular, it is established that if X is a locally Čech-complete non-Čech-complete space, then no remainder of X is homogeneous (Theorem 3.1). We also show that if Y is a remainder of a locally Čech-complete space X, and every $y \in Y$ is a G_δ-point in Y, then the cardinality of Y doesn't exceed 2^ω. Several other results are obtained.

Keywords: Remainder, Compactification, G_δ-point, Homogeneous, Point-countable base, Lindelöf Σ-space, Charming space, Countable type, Čech-complete.

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1. Introduction

All spaces considered in this article are assumed to be Tychonoff. Symbols X, Y, Z always stand for topological spaces. In terminology and notation we follow [7]. We say that a space X *has a topological property* $\mathcal{P}$ *locally*, if for each $x \in X$ there exists an open neighbourhood V of x such that the closure of V in X has the property $\mathcal{P}$.

A *compactification* of a space X is any compact space bX such that X is a dense subspace of bX. A *remainder* Y of a space X is the subspace $Y = bX \setminus X$ of a compactification bX of X.

A space X is of *countable type* if every compact subspace P of X is contained in a compact subspace $F \subset X$ with a countable base of open neighbourhoods in X [1]. Metrizable spaces, locally compact spaces, Čech-complete spaces, and Moore spaces are of countable type [1]. A remarkable classical result in the theory of compactifications is the following theorem of M. Henriksen and J. Isbell [8]:

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1.1. Theorem. A space X is of countable type if and only if the remainder of X in any (in some) compactification of X is Lindelöf.

A Lindelöf p-space is a preimage of a separable metrizable space under a perfect mapping [1]. K. Nagami has defined the class of Σ-spaces [9]. They can be characterized as continuous images of Lindelöf p-spaces.

It is proved below that if X is a locally Čech-complete space with a homogeneous remainder Y, then X is Čech-complete (Theorem 3.1). We show that a remainder of a locally Čech-complete space needn't have a dense σ-compact subspace and needn't be a Lindelöf Σ-space (Example 2.4). We also give a characterization of remainders of locally Čech-complete spaces (Theorem 4.1) and obtain some corollaries from it.

2. Two examples

A space X is *Čech-complete* if it is a G_δ-subspace of some (of any) of its compactifications. One of the simplest duality theorems involving remainders is the next statement: a space X is Čech-complete if and only if some (every) remainder of X is σ-compact. Locally compact spaces constitute an important subclass of the class of Čech-complete spaces. In this case, we have the classical theorem of P.S. Alexandroff: a space X is locally compact non-compact if and only if some remainder of X consists of exactly one point.

However, the next natural question remained unanswered: how to characterize in intrinsic terms the remainders of locally Čech-complete spaces? We answer this question in this article.

2.1. Theorem. Every remainder of any locally Čech-complete space X is Lindelöf.

Proof. By Henriksen-Isbell Theorem, it is enough to show that X is a space of countable type. Since every Čech-complete space is a space of countable type, we see that X is locally of countable type. It remains to use the next easy to verify assertion: if a space X is locally of countable type, then X is of countable type. □

How close are remainders of locally Čech-complete spaces to remainders of Čech-complete spaces? This question, Theorem 2.1, - and the obvious fact that every remainder of a Čech-complete space is σ-compact, - motivate the next question: is every remainder of any locally Čech-complete space a Lindelöf Σ-space? The answer is in the negative.

2.2. Example. Let B be the usual space of ordinal numbers not exceeding the first uncountable ordinal ω_1, and Z be the subspace of B consisting of all non-isolated points of B. Furthermore, let Y_0 be the subspace of Z consisting of all isolated points of Z. Finally, put $p = \omega_1$, $Y = Y_0 \cup \{p\}$, and $X = B \setminus Y$.

Clearly, B is a compactification bX of X, and Y is the remainder $bX \setminus X$ of X in bX. It is also easy to see that all points of Y_0 are isolated in Y and p is a non-isolated P-point in Y. Observe that every open neighbourhood of p contains all but countably many points of the set Y. Hence, Y is a Lindelöf P-space, and the space X is locally Čech-complete. It follows that every compact subspace of Y is finite, and that Y is not a Lindelöf Σ-space. Therefore, Y does not have a dense σ-compact subspace and hence, X is not Čech-complete. In particular, it is not true that every remainder of any locally Čech-complete space has a dense σ-compact subspace. It is also easy to see that the space Y is not locally σ-compact. In this connection, see the last section.

2.3. Corollary. There exists a locally Čech-complete space X with a remainder Y such that no dense subspace of Y is a Lindelöf Σ-space.

Proof. Let us take X and Y constructed in the preceding example. We also use the notation introduced there. We have seen in Example 2.4 that X is locally Čech-complete. Assume that Y_1 is an arbitrary dense subspace of Y. Then Y_1 contains all isolated points of Y, and hence, either $Y_1 = Y \setminus \{p\}$, or $Y_1 = Y$. In the first case, Y_1 is discrete and uncountable, and therefore, is not Lindelöf. In the second case, Y_1 is not a Lindelöf Σ-space, since it has been shown above that Y is not a Lindelöf Σ-space. □

Alexandroff's Theorem on remainders of locally compact spaces leads to the next question: is it true that every Čech-complete space X has a remainder Y such that $|Y| \leq 2^\omega$? The answer is "no".

2.4. Example. a) Let X be any nowhere locally compact space metrizable by a complete metric and satisfying the condition: the weight of X is greater than 2^ω. To construct such a space, we can fix any cardinal number τ such that $\tau > 2^\omega$ and take X to be the countable power of a discrete space of the cardinality τ. Let bX be any compactification of X. Then the remainder Y of X in bX is dense in bX, since the spacee X is nowhere locally compact. Since the Souslin number of X is greater than 2^ω, it follows that $|Y| > 2^\omega$. Notice that the space X is, clearly, Čech-complete. Its additional nice feature is that it is metrizable. On the other hand, the Souslin number of X is quite large, and we have made a good use of this fact in our argument above. The next example serves the same purpose but the Souslin number of the space constructed in it is countable.

b) Let G be the countable power of the usual space R of real numbers, and B be a compact topological group such that $w(B) > 2^c$, where $c = 2^\omega$. Now define X as the topological product $G \times B$. Clearly, X is a Čech-complete nowhere locally compact topological group, and the Soulin number of X is countable. However, X is not metrizable. Let bX be any compactification of X. Then the remainder Y of X in bX is dense in bX, since the spacee X is nowhere locally compact. Assume that $|Y| \leq c = 2^\omega$. Then $w(bX) \leq 2^c$, since Y is dense in bX. Therefore, $w(bX) \leq 2^c$, a contradiction. It follows that $|Y| > 2^\omega$.

3. Homogeneous remainders

Recall that a space X is said to be *homogeneous* if for any $x, y \in X$ there exists a homeomorphism h of X onto X such that $h(x) = y$. In fact, we will use below a much weaker form of homogeneity. A space X will be called *meekly homogeneous* if for any $x, y \in X$ and any open neighbourhood Ox of x there exists an open neighbourhood Oy of y such that Oy is homeomorphic to some open subspace of Ox.

The remainder Y of the locally Čech-complete space X constructed in Example 2.4 is easily seen to be non-homogeneous. A natural question arises: can we construct a similar example, in which Y is, in addition, homogeneous, or at least, meekly homogeneous? Somewhat unexpectedly, the answer turns out to be in the negative.

3.1. Theorem. Suppose that X is a locally Čech-complete space with a meekly homogeneous remainder Y. Then X is Čech-complete, and Y is σ-compact.

This statement will be derived from the next slightly more general statement:

3.2. Proposition. Suppose that X is a space with an open Čech-complete non-locally compact subspace U. Then any remainder Y of X has a closed σ-compact subspace P such that the interior of P in Y is non-empty.

Proof. Fix $x \in U$ such that U is not locally compact at x. Clearly, we may assume that the closure of U in X is Čech-complete, - otherwise we can replace U with a non-empty open subset W of U such that $x \in W$ and the closure of W in X is contained in U. Let $Y = bX \setminus X$. We denote by F the closure of U in bX. Clearly, x is in the interior of F in bX, and x is in the closure of Y. Therefore, the interior of the set $P = F \cap Y$ in Y is non-empty as well.

Obviously, P is the remainder of Z in F, where Z is the closure of U in X. It follows that the subspace P of Y is σ-compact, since Z is Čech-complete. $\square$

3.1. Lemma. *Suppose that Y is a meekly homogeneous space with a closed σ-compact subspace P such that the interior of P in Y is non-empty. Then Y is locally σ-compact.*

Proof. Fix x in the interior of P, and consider the interior of P as a neighbourhood Ox of x. Now take an arbitrary $y \in Y$. Since Y is meekly homogeneous, we can find an open neighbourhood Oy of y and an open subset V of Ox such that Oy is homeomorphic to V. Since P is σ-compact, and V is an open subspace of P, the space V is locally σ-compact. Hence, the space Oy is also locally σ-compact. Since Y is regular and Oy is open in Y, it follows that Y is locally σ-compact at y. $\square$

Proof. (of Theorem 3.1). If X is locally compact, then we have nothing to prove. If X is not locally compact, then there exists an open Čech-complete non-locally compact subspace U of X. By Proposition 3.2, there exists a closed σ-compact subspace P of Y such that the interior V of P in Y is non-empty. Since Y is meekly homogeneous, it follows from Lemma 3.1 that Y is locally σ-compact. Therefore, there exists an open covering γ of the space Y such that the closure of any member of γ in Y is σ-compact.

Notice that by Theorem 3.1 the space Y is Lindelöf. Therefore, γ has a countable subcovering η of Y. Since the closure in Y of each member of η is σ-compact, we conclude that Y is σ-compact. It follows that X is Čech-complete. $\square$

4. A characterization of remainders of locally Čech-complete spaces

It is not true that a space is locally Čech-complete if and only if its remainders are locally σ-compact. We have seen this in Example 2.4. In this section, we will characterize locally Čech-complete spaces by a somewhat unusual, but still natural and easy to use property of its remainders. Some corollaries are derived from this characterization.

First, let us introduce a piece of terminology. Suppose that Y is a space and F is a subspace of Y. We will say that Y *has a topological property* $\mathcal{P}$ *outside of* F if every closed subspace Z of Y such that $Z \cap F = \varnothing$ has $\mathcal{P}$.

4.1. Theorem. A space X is locally Čech-complete if and only if every (some) remainder Y of X is σ-compact outside of some compact subspace F of Y.

Proof. Necessity. Suppose that X is locally Čech-complete. Take any compactification bX of X, and let $Y = bX \setminus X$. Fix also an open covering γ of X such that the closure $X(V)$ of every $V \in \gamma$ in X is Čech-complete. For $V \in \gamma$, we denote by $U(V)$ an open subset of bX such that $U(V) \cap X = V$. Put $E = \cup\{U(V) : V \in \gamma\}$ and $F = bX \setminus E$. Clearly, E is open in bX, F is compact, $X \subset E$, and $F \subset Y$.

Let A be a closed subset of Y such that $A \cap F = \varnothing$, and B be the closure of A in bX. Clearly, $F \cap B = \varnothing$. Therefore, $B \subset E = \cup\{U(V) : V \in \gamma\}$. Since B is compact, and each $U(V)$ is open in bX, there exists a finite collection $V_1, ..., V_k$ of members of γ such that $B \subset \cup\{U(V_i) : i = 1, ..., k\}$.

Put $X_i = X(V_i)$ and let B_i be the closure of V_i in bX. By the definitions above, the space X_i is Čech-complete. Hence, the subspace $P_i = B_i \setminus X_i$ is σ-compact. Therefore, the subpace $P = \cup\{P_i : i = 1, ..., k\}$ is also σ-compact. Clearly, $A \cap X_i = \varnothing$ for $i = 1, ..., k$, since $A \subset Y$. Since $A \subset B \subset \cup\{\overline{X_i} : i = 1, ..., k\}$, it follows that $A \subset P$.

The set X_i is closed in X, and $bX = X \cup Y$. Therefore, $P_i \subset Y$, so that $P \subset Y$. Now we can conclude that A is a closed subspace of P. Finally, it follows that A is σ-compact, since P is σ-compact.

Sufficiency. Suppose that some remainder Y of X is σ-compact outside of some compact subspace F of Y. Fix a compactification bX of X such that $Y = bX \setminus X$. Take any $x \in X$. The set F is closed in bX, since F is compact. We also have: $x \notin F$. Hence, we can find an open neighbourhood U of x in bX such that the closure of U in bX doesn't intersect F. Since Y is σ-compact outside of F, it follows that the closed subspace $P = Y \cap \overline{U}$ of Y is σ-compact. Since $\overline{U}$ is compact, it follows that the subspace $X \cap \overline{U}$ is Čech-complete. Note that $U \cap X$ is an open subspace of $X \cap \overline{U}$. Therefore, the set $Ox = U \cap X$ is an open Čech-complete neighbourhood of x in X. Thus, the space X is locally Čech-complete. $\square$

We present now a few applications of the last theorem. The concept of a charming space has been introduced in [2]. A space Y is *charming* if there exists a subspace Z of Y such that Z is a Lindelöf Σ-space and $Y \setminus U$ is a Lindelöf Σ-space, for every open neighbourhood U of Z in Y.

The next statement immediately follows from Theorem 4.1.

4.2. Corollary. Every remainder of a locally Čech-complete space is a charming space.

According to Theorem 2.1, every remainder of a locally Čech-complete space is Lindelöf. It is still unknown whether there exists in ZFC a Lindelöf space Y such that every $y \in Y$ is a G_δ-point in Y and $|Y| > 2^\omega$. Let us show that it is impossible to find a space of this kind among remainders of locally Čech-complete spaces.

4.3. Theorem. Suppose that Y is a remainder of a locally Čech-complete space X such that every $y \in Y$ is a G_δ-point in Y. Then $|Y| \leq 2^\omega$.

Proof. By Corollary 4.2, Y is a charming space. Since the cardinality of every charming space of countable pseudocharacter does not exceed 2^ω [Theorem 3.7 in [2]], it follows that $|Y| \leq 2^\omega$. $\square$

The following easy to prove statement tells us that non-trivial locally Čech-complete spaces are never remainder-wise dual to locally σ-compact spaces.

4.4. Theorem. A space X is Čech-complete if and only if X is locally Čech-complete and every (some) remainder Y of X is locally σ-compact.

Proof. The necessity is clear.

Sufficiency. The space Y is Lindelöf, since X is locally Čech-complete. Since Y is also locally σ-compact, it follows that Y is σ-compact. Hence, X is Čech-complete. $\square$

4.5. Theorem. Suppose that X is a locally Čech-complete space with a remainder Y in a compactification bX. Furthermore, suppose that Y has a point-countable base. Then Y is separable, metrizable, and σ-compact, and X is Čech-complete.

Proof. By Theorem 2.1, Y is Lindelöf. Theorem 4.1 implies that Y is σ-compact outside of some compact subspace F of Y which we now fix. Clearly, F is separable metrizable, by the well-known Theorem of A.S. Mischenko [7]. Let us also fix a point-countable base $\mathcal{B}$ for Y. Since F is separable, the family η of members V of $\mathcal{B}$ such that $V \cap F \neq \varnothing$ is countable. Therefore, F has a countable base for open neighbourhoods in Y. Hence F is a G_δ-set in Y. Since Y is Lindelöf, it follows that $Y \setminus F$ is Lindelöf. Since $Y \setminus F$ is, obviously, locally σ-compact, it follows that $Y \setminus F$ and Y are σ-compact. Hence, X is Čech-complete. Using again Mischenko's Theorem, we conclude that $Y \setminus F$ is separable. Therefore, the family ξ of members V of $\mathcal{B}$ such that $V \cap (Y \setminus F) \neq \varnothing$ is countable. Since, clearly, $\mathcal{B} = \xi \cup \eta$, we conclude that the base $\mathcal{B}$ is countable. Hence, Y is separable metrizable. $\square$

In connection with the last theorem, recall that, under the Continuum Hypothesis CH, there exists a non-metrizable Lindelöf space with a point-countable base [6].

4.6. Theorem. Suppose that X is a locally Čech-complete space with a remainder Y in a compactification bX. Furthermore, suppose that Y is symmetrizable. Then Y is σ-compact, has a countable network, and is submetrizable, and X is Čech-complete.

Proof. By Theorem 2.1, Y is Lindelöf. Now it follows from a theorem of S.J. Nedev in [10] that Y is hereditarily Lindelöf.

By Theorem 4.1, Y is σ-compact outside of some compact subspace F of Y. Clearly, $Y \setminus F$ is Lindelöf. Since $Y \setminus F$ is, obviously, locally σ-compact, it follows that $Y \setminus F$ is σ-compact. Hence, Y is σ-compact as well. Therefore, X is Čech-complete.

Clearly, every compact subspace of a symmetrizable space is symmetrizable. It is well-known that each symmetrizable compact space is metrizable and hence, has a countable base. Now we can conclude that Y has a countable network. $\square$

It is not difficult to see that the remainder Y, under the assumptions in the last statement, needn't have a countable base, and that the space X needn't be metrizable or paracompact.

4.7. Problem. Suppose that X is a locally Čech-complete space with a remainder Y in a compactification bX such that every $y \in Y$ is a G_δ-point in Y. Does it follow that X is Čech-complete?

5. Remainders of locally σ-compact spaces

5.1. Theorem. If a space X is locally σ-compact, then for every remainder Y of X there exists a compact subspace F of Y such that Y is Čech-complete outside of F.

Proof. Take any compactification bX of X, and put $Y = bX \setminus X$. Fix also an open covering γ of X such that the closure $X(V)$ in X of any $V \in \gamma$ is σ-compact. For $V \in \gamma$, we denote by $U(V)$ an open subset of bX such that $U(V) \cap X = V$. Put $E = \cup\{U(V) : V \in \gamma\}$ and $F = bX \setminus E$. Clearly, E is open in bX, F is compact, $X \subset E$, and $F \subset Y$.

Let A be a closed subset of Y such that $A \cap F = \varnothing$, and B be the closure of A in bX. Clearly, $F \cap B = \varnothing$. Therefore, $B \subset E = \cup\{U(V) : V \in \gamma\}$. Since B is compact, and each $U(V)$ is open in bX, there exists a finite collection $V_1, ..., V_k$ of members of γ such that $B \subset \cup\{U(V_i) : i = 1, ..., k\}$.

Put $X_i = X(V_i)$, and let H_i be the closure of V_i in bX. Clearly, X_i is σ-compact and closed in X. Hence, the subspace $P = \cup\{X_i : i = 1, ..., k\}$ is also closed in X and σ-compact. The set P is dense in the closure H of P in bX, and $H \cap Y$ is the remainder of P in H. It follows that $H \cap Y$ is Čech-complete. Clearly, $B \subset H$, so that $A = B \cap Y \subset H \cap Y$. Since A is closed in $H \cap Y$, it follows that A is Čech-complete. □

5.2. Theorem. Suppose that X is a locally σ-compact space with a homogeneous remainder Y. Then Y is Čech-complete.

Proof. By Theorem 5.1, there exists a compact subspace F of Y such that Y is Čech-complete outside of F. If $Y = F$, then we are done.

Assume now that $Y \setminus F \neq \varnothing$. Fix $y \in Y \setminus F$. Clearly, Y is locally Čech-complete at y. Since Y is homogeneous, it follows that Y is locally Čech-complete at every point. Therefore, we can find a finite collection $U_1, ..., U_n$ of open subsets of Y such that $F \subset U_1 \cup ... \cup U_n$ and the closure H_i of U_i in Y is Čech-complete, for each $i = 1, ..., n$. The subspace $P = Y \setminus (U_1 \cup ... \cup U_n)$ is a closed Čech-complete subspace of Y, since Y is Čech-complete outside of F. Clearly, $Y = (H_1 \cup ... \cup H_n) \cup P$, that is, Y is the union of a finite collection of closed Čech-complete subspace of Y. Hence, Y is also Čech-complete. □

5.3. Corollary. Suppose that X is a locally σ-compact space with a homogeneous remainder Y. Then $X = S \cup L$, where S is a closed σ-compact subspace of X, and L is an open locally compact subspace of X.

Proof. Fix a compactification bX of X such that $Y = bX \setminus X$, and let bY be the closure of Y in bX. Put $S = X \cap bY$. Then S is a closed subspace of X, and Y is

Čech-complete, by Theorem 5.2. Since $S = bY \setminus Y$, it follows that S is σ-compact. The subspace $L = (bX) \setminus bY$ is, clearly, locally compact and open in bX and in X. We also have $S \cup L = X$. $\square$

5.4. Corollary. Suppose that X is a locally σ-compact nowhere locally compact space with a homogeneous remainder Y. Then X is σ-compact, and Y is Čech-complete.

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