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On the Distribution of Claims for a Process Terminating on a Run of Critical Events

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ABSTRACT

In modern actuarial risk management, the temporal clustering of severe claims is as critical as their cumulative financial magnitude. This study investigates a stochastic risk process that terminates upon the occurrence of a run of k consecutive claims exceeding a predefined critical threshold. First, using recursive conditioning techniques, we derive the exact moment generating function for the total severity of exceedances accumulated prior to termination, providing an explicit probability density function for the exponential case when $k = 2$. Second, we determine the exact cumulative distribution and probability mass functions for the maximum number of consecutive non-exceedances observed between two critical claims via first-order linear difference equations. The derived analytical expressions bridge the theory of runs and practical risk management, offering direct tools for dynamic solvency monitoring and operational stress testing without relying on asymptotic approximations.

Keywords: *Claim clustering, Risk management, Run-based stopping rule, Runs, Stopping time*

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1. Introduction

Actuarial science has a long tradition of modeling insurance liabilities using aggregate risk models [1, 2]. In classical risk theory, assessing the vulnerability of a system to catastrophic events is a fundamental objective. Traditional models often evaluate risk by examining the aggregate claim amount over a fixed horizon or by determining the probability of ultimate ruin when a surplus process drops below zero. However, in modern risk management, the temporal clustering and structural patterns of severe claims can be as destructive as their cumulative financial magnitude. A sequence of large losses occurring in rapid succession can severely strain an insurer's liquidity, exhaust the reinstatement limits of an Excess of Loss (XL) reinsurance treaty, and lead to systemic operational failure. Consequently, risk models that incorporate run-based stopping rules where a process terminates upon the occurrence of a predefined number of consecutive extreme events have gained significant traction in actuarial science and applied probability.

This study examines a complex claim recovery process, where claim severities (X_i) are modeled as random variables. A critical threshold c is defined for claim severities, and a retention rule based on k consecutive claim occurrences exceeding this threshold is used to define the termination condition of the process. Such scan statistics, generalized to event sequences and patterns, are thoroughly discussed in [3, 4]. A key component of effective risk management is the development of a proactive program to address the disruption caused by a critical event in a process [5]. The indicator function

$$I_i = \begin{cases} 1, & X_i > c, \\ 0, & X_i \leq c. \end{cases} \quad (1)$$

formally captures whether an individual claim X_i surpasses this critical level c. An illustrative diagram depicting such a claims development process for the case $k = 2$-including claim magnitudes (X_i), inter-arrival times (Y_i), and the corresponding indicator values I_i-is provided in Figure 1.

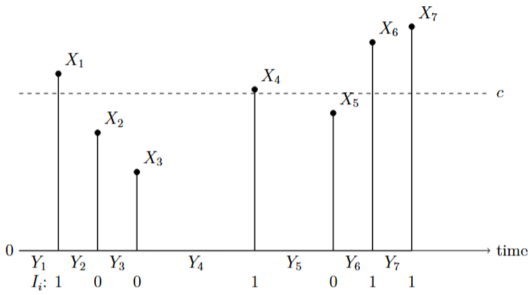

Figure 1. An illustrative example of the claims process terminating at $N = 7$ for a stopping rule of $k = 2$ consecutive critical claims

The total number of claims observed until the process stops, denoted by N, is a stopping time. In addition, some characteristics of the claim sequence up to this point such as $\min(X_1, X_2, \dots, X_N)$ and $\max(X_1, X_2, \dots, X_N)$, have also been studied in the literature [6, 7 and 8]. While the distribution of the stopping time itself has been extensively studied within the literature of runs and patterns [9-14], evaluating the behavior of the accumulated risks and the structural gaps prior to this absorbing state provides deeper insights into the stress the system endures before ultimate failure.

To this end, we investigate the exact distributional properties of two critical random variables realized during the process. The first variable, denoted by S, represents the total claim amount exceeding the threshold c accumulated until the process terminates. Unlike standard aggregate loss models, S isolates the financial burden of severe shocks, capturing the cumulative severity of exceedances under the strict constraint that the system has not yet experienced the terminating run of k consecutive shocks. The distribution of S, derived via its moment generating function using recursive techniques, provides actuaries with a precise measure of the total catastrophic load transferred to a reinsurance layer before a systemic collapse occurs.

The second variable of interest, denoted by W, shifts the analytical focus from financial severity to the structural sequence of the risk events. We define W as the maximum number of consecutive non-exceedances (ordinary claims) observed between two exceedances prior to termination. In a risk management and operational context, this maximum gap length represents the longest recovery window available to the insurer. It quantifies the most extended period of "breathing space" during which the insurer could potentially rebuild capital through regular premium collection without being interrupted by another critical shock. By modeling this maximum gap, we can evaluate the resilience and recovery capacity of the insurance portfolio under extreme stress.

2. Distribution of the Total Claim Amount Exceeding c

In this section, we focus on the distribution of the random variable S, defined as the sum of the claim amounts exceeding the threshold c until the process terminates. Mathematically, it is expressed as,

$$S = \sum_{j=1}^{N_1} X_j^* \tag{2}$$

where N_1 denotes the total number of claims exceeding c until the process stops [13], and $X^* = X|X > c$ represents the severity of a claim given that it exceeds the threshold c.

Since S only accumulates claims that exceed c, we first define the conditional distribution of these X^* claims. The probability density function of X^*, denoted by f_{X^*}, is given by,

$$f_{X^*}(x) = \begin{cases} \dfrac{f_X(x)}{p}, & x > c \\ 0, & x \le c \end{cases} \tag{3}$$

where $p = P(X > c)$. Consequently, the moment generating function (MGF) of X^*, $M_{X^*}(t)$, can be expressed as,

$$M_{X^*}(t) = E\left[e^{tX^*}\right] = \int_c^\infty e^{tx} \frac{f_X(x)}{p}\, dx \tag{4}$$

To derive the moment generating function of S, denoted as $\phi(t) = E[e^{tS}]$, let $\phi_i(t)$ represent the MGF of the total claim amount generated in the remainder of the process, given that we currently have a run of i consecutive large claims. When the process is in state i $(0 \le i < k)$, the next claim will either transition the process to state $i + 1$ with probability p (adding a new claim severity to the total), or transition it back to state 0 with probability $q = 1 - p$ (without adding any amount to the sum). Thus, the following recursive equation can be written,

$$\phi_i(t) = pM_{X^*}(t)\phi_{i+1}(t) + q\phi_0(t), \quad 0 \le i < k. \tag{5}$$

The boundary condition is $\phi_k(t) = 1$. This is because when we have k consecutive large claims, the process terminates, meaning the sum of large claims generated thereafter will be 0.

To simplify the solution of the recursive equation given in Eq.(5), let us define $Z(t) = pM_{X^*}(t)$. Under this substitution, Eq.(5) becomes,

$$\phi_i(t) = Z(t)\phi_{i+1}(t) + q\phi_0(t). \tag{6}$$

By expanding ϕ_0 recursively, we obtain,

$$\phi_0(t) = q\phi_0(t)(1 + Z(t) + Z(t)^2 + \cdots + Z(t)^{k-1}) + Z(t)^k\phi_k(t). \tag{7}$$

Utilizing the boundary condition $\phi_k(t) = 1$, the equation for $\phi_0(t)$ yields,

$$\phi_0(t) = q\phi_0(t)\frac{1 - Z(t)^k}{1 - Z(t)} + Z(t)^k. \tag{8}$$

Solving for $\phi_0(t)$, which is essentially the moment generating function $M_S(t)$ of the random variable S, we get,

$$M_S(t) = \phi_0(t) = \frac{Z(t)^k(1 - Z(t))}{p - Z(t) + qZ(t)^k}. \tag{9}$$

Finally, substituting $Z(t) = pM_{X^*}(t)$ back into Equation Eq.(9), the moment generating function of S is obtained as,

$$M_S(t) = \frac{[pM_{X^*}(t)]^k(1 - pM_{X^*}(t))}{p - pM_{X^*}(t) + q[pM_{X^*}(t)]^k}. \tag{10}$$

Using this function, properties such as the expected value and variance of the random variable S can be derived. Furthermore, the probability density function of S can be obtained via the inverse Laplace-Fourier transform of the moment generating function given in Eq.(10).

2.1 Exponential Case for k=2

Assume that the claim size distribution is exponential with parameter $\lambda > 0$, i.e.,

$$X \sim Exp(\lambda),\ f_X(x) = \lambda e^{-\lambda x},\ x > 0$$

then

$$p = P(X > c) = e^{-\lambda c},\ q = 1 - p.$$

Using the memoryless property of the exponential distribution, the conditional excess claim size satisfies

$$M_{X^*}(t) = e^{ct}\frac{\lambda}{\lambda - t},\ t < \lambda. \tag{11}$$

For the special case k = 2, the moment generating function given in Eq.(10) reduces to

$$M_S(t) = \frac{[pM_{X^*}(t)]^2(1 - pM_{X^*}(t))}{p - pM_{X^*}(t) + q[pM_{X^*}(t)]^2} \tag{12}$$

Substituting Eq.(11) and simplifying, we obtain

$$M_S(t) = \frac{e^{-\lambda c}\left(e^{ct}\frac{\lambda}{\lambda - t}\right)^2}{1 - (1 - e^{-\lambda c})e^{ct}\frac{\lambda}{\lambda - t}} \tag{13}$$

or equivalently, the Laplace transform of S is

$$\mathcal{L}_S(s) = M_S(-s) = \frac{e^{-\lambda c}\left(e^{-cs}\frac{\lambda}{\lambda + s}\right)^2}{1 - (1 - e^{-\lambda c})e^{-cs}\frac{\lambda}{\lambda + s}}. \tag{14}$$

Expanding the denominator as a geometric series yields

$$\mathcal{L}_S(s) = \sum_{n=2}^{\infty} e^{-\lambda c}\left(1 - e^{-\lambda c}\right)^{n-2} * e^{-ncs}\left(\frac{\lambda}{\lambda + s}\right)^n. \tag{15}$$

By inverse Laplace transform, the density of S is obtained as

$$f_S(s) = \sum_{n=2}^{\infty} e^{-\lambda c}(1 - e^{-\lambda c})^{n-2} * \frac{\lambda^n(s - nc)^{n-1}e^{-\lambda(s - nc)}}{(n - 1)!}\ \mathbb{1}\{s > nc\} \tag{16}$$

or equivalently,

$$f_S(s) = \sum_{n=2}^{\lfloor s/c \rfloor} e^{-\lambda c}(1 - e^{-\lambda c})^{n-2} * \frac{\lambda^n(s - nc)^{n-1}e^{-\lambda(s - nc)}}{(n - 1)!},\ s > 2c \tag{17}$$

and $f_S(s) = 0$ for $s \leq 2c$. Thus, for exponential claim sizes and k = 2, the distribution of S is an infinite mixture of shifted Gamma densities.

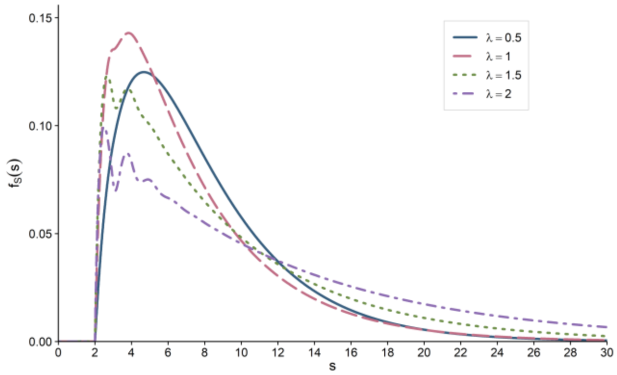

Figure 2. Probability density function of S for selected values of λ and $k = 2$.

Figure 2 demonstrates that the effect of the exponential rate parameter λ on the density of the aggregate exceedance amount S is not monotonic across the support. The curves intersect, indicating that changes in λ do not produce a simple uniform shift of the density. Instead, λ influences both the local shape and the tail behavior of the distribution. In particular, the impact of increasing λ differs across the lower, intermediate, and upper ranges of S. While larger values of λ are associated with smaller individual claim severities on average, their effect on the density of S is structurally more complex, as reflected by the changes in peak height, local irregularities, and tail decay observed in the figure.

3. Distribution of the Maximum Number of Non-Exceedances Between Two Exceedances

In this section, we aim to find the distribution of the random variable W, defined as the maximum number of consecutive claims not exceeding c (maximum gap length) observed between two claims exceeding c, prior to the termination of the process by a run of k consecutive exceedances.

Let $F_W(w) = P(W \leq w)$ be the cumulative distribution function of W. This probability represents the likelihood of achieving a sequence of k consecutive successes (i.e., k exceedances with a gap $G = 0$ between them) before encountering a failure sequence (gap) greater than w $(G > w)$.

Let i $(1 \leq i < k)$ denote the accumulated number of consecutive successes since the last observed success at any point in the process. Let V_i be the probability of reaching the target k without encountering a failure sequence greater than w, given that we currently have i consecutive successes. In this state, if the next claim exceeds c, the number of consecutive successes becomes $i+1$. If it does not exceed c, a failure sequence begins. If the length of this failure sequence exceeds w $(G > w)$, the condition is violated (probability 0). The probability of this event is,

$$\beta = P(G > w) = \sum_{j=w+1}^{\infty} pq^j = q^{w+1}, \tag{18}$$

where $p = P(X > c)$ and $q = 1 - p$. If the failure sequence does not exceed w $(1 \leq G \leq w)$, the process resets without terminating. Once the failure sequence ends-meaning a new success arrives-we are left with only that latest success. Thus, the process returns to state V_1. The probability of this safe reset is,

$$\alpha = P(1 \leq G \leq w) = \sum_{j=1}^{w} pq^j = q(1 - q^w). \tag{19}$$

Based on this logic, the following equation can be written,

$$V_i = pV_{i+1} + \alpha V_1, \; V_k = 1, \; 1 \leq i < k. \tag{20}$$

This can be rearranged into a recursive relation,

$$V_{i+1} = \frac{1}{p}V_i - \frac{\alpha}{p}V_1. \tag{21}$$

This is a first-order linear difference equation with constant coefficients, and its solution is of the form $V_i = Ap^i + B$. By substituting this back into the equation and solving for the coefficients, V_i is obtained as,

$$V_i = V_1(q^w p^{1-i} + (1 - q^w)). \tag{22}$$

We now need to determine V_1. Using the boundary condition $V_k = 1$, the fundamental equation for $i = k - 1$ can be written as,

$$V_{k-1} = p + q(1 - q^w)V_1. \tag{23}$$

Additionally, substituting $i = k - 1$ into the general solution yields,

$$V_{k-1} = V_1(q^w p^{2-k} + 1 - q^w). \tag{24}$$

Equating Eq.(23) and Eq.(24) for V_{k-1}, we get,

$$V_1(q^w p^{2-k} + 1 - q^w) = p + (q - q^{w+1})\, V_1. \tag{25}$$

After necessary algebraic rearrangements, this yields,

$$V_1\left[1 + q^w\left(p^{-(k-1)} - 1\right)\right] = 1. \tag{26}$$

Failures occurring before the first success are not counted as "failures between two successes". Since the probability $F_W(w)$ is defined from the moment of the first success, its mathematical equivalent is exactly V_1. Therefore, V_1, which is our desired $F_W(w)$, is obtained as,

$$F_W(w) = P(\mathrm{W} \leq w) = \frac{p^{k-1}}{p^{k-1} + (1 - p^{k-1})q^w}, \; w = 0, 1, 2, \dots. \tag{27}$$

From this, the probability mass function of the random variable W, given by the difference $P(W = w) = F_W(w) - F_W(w-1)$, is found for $w = 0$ as,

$$P(W = 0) = p^{k-1}, \tag{28}$$

and for $w = 1,2, \dots$ as,

$$P(W = w) = \frac{p^{k-1}}{p^{k-1} + (1 - p^{k-1})q^w} - \frac{p^{k-1}}{p^{k-1} + (1 - p^{k-1})q^{w-1}}. \tag{29}$$

The probability mass function is illustrated in Figure 3.

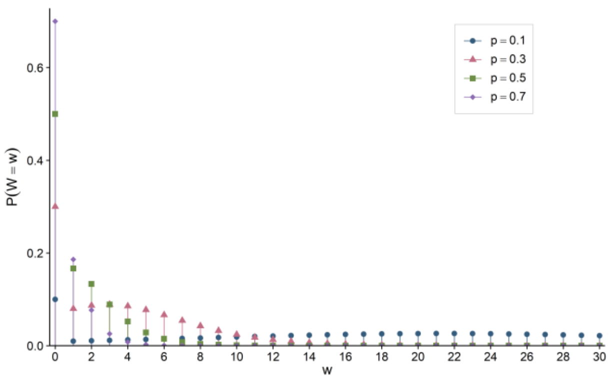

Figure 3. Probability mass function of W for selected values of p and $k = 2$.

Figure 3 depicts the probability mass function of the random variable W, which represents the maximum number of consecutive non-exceedances observed between two exceedances before the process terminates. For all parameter settings, the distribution is concentrated on relatively small values of w and decreases monotonically as w increases. This pattern indicates that long non-exceedance runs are progressively less likely, whereas short interruption lengths are substantially more probable. In particular, the largest probability mass is attained at $w = 0$, corresponding to the case in which no non-exceedance occurs between exceedances prior to the terminating run.

The figure further reveals a pronounced sensitivity of the distribution to the exceedance probability p. As p increases, the probability mass becomes increasingly concentrated at lower values of w, especially at $w = 0$. This reflects the fact that, when exceedances occur more frequently, the process is more likely to terminate before long stretches of non-exceedances can develop. Conversely, for smaller values of p, the distribution becomes more dispersed and allocates relatively greater probability to larger values of w, indicating a higher likelihood of longer recovery intervals between exceedances. From a risk-management perspective, these findings show that the parameter p plays a central role in determining the recovery structure of the process, as it directly governs the distribution of the longest non-critical gap observed before termination.

4. Conclusion

In this paper, we investigated the dynamics of a stochastic risk process that terminates upon the occurrence of k consecutive claims exceeding a critical threshold. Shifting the analytical focus from traditional ultimate ruin probabilities to the trajectory of the surviving path, we derived the exact distributions of two key random variables that characterize the financial and structural stress on an insurance portfolio.

First, by utilizing recursive conditioning techniques, we obtained the exact moment generating function for the total severity of claims exceeding the threshold accumulated prior to termination. From an actuarial perspective, this closed-form result provides a precise mathematical tool to quantify the catastrophic load that an Excess of Loss reinsurance layer must absorb before a systemic failure such as the exhaustion of reinstatement limits occurs.

Second, we derived the exact cumulative distribution and probability mass functions for the maximum number of consecutive non-exceedances observed between two critical claims. Formulated through first-order linear difference equations, this maximum gap length serves as a measure for operational resilience. It represents the longest recovery window available to the insurer, during which capital can be rebuilt through regular premium collection without the interruption of another major shock.

Ultimately, the analytical expressions presented in this study bridge the theory of runs and practical risk management. By eliminating the need for computationally expensive simulations or asymptotic approximations, these exact formulas offer risk managers direct tools for dynamic solvency monitoring and operational stress testing. Future research may naturally extend this framework by incorporating multiple critical thresholds to model layered reinsurance structures or by relaxing the assumption of independence in highly volatile economic environments.

Conflict of Interest

There are no conflicts of interest in this work.

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Declaration of Generative AI

The authors used ChatGPT for language editing during the preparation of this manuscript. All AI-assisted outputs were carefully reviewed and verified by the authors to ensure scientific integrity and accuracy. Consequently, the authors take full responsibility for the final content and conclusions of the study.

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